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Uniqueness results on meromorphic functions with q-shift difference polynomials

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Abstract

In this article, we use weighted sharing values to investigate the uniqueness theorems for meromorphic functions of specific types of q-shift difference polynomials of zero order. The results in this article extend and improve certain previous results due to [10]

Keywords: *Meromorphic functions, q-shift, Weighted sharing, Uniqueness*

2010 Mathematics Subject Classification: 30D35.

1 Introduction and Main Results

Nevanlinna theory is a field of complex analysis that studies the value distribution of meromorphic functions. It was established by Rolf Nevanlinna in the early twentieth century, and Nevanlinna theory is a useful framework for investigating the value distribution of meromorphic functions, providing significant insights into their genesis and behavior. Nevanlinna theory provides a thorough and rigorous framework for comprehending the complex behavior of meromorphic functions, making it a key field of study in complex analysis.

For the elementary definitions and standard notations of the Nevanlinna value distribution theory such as $T(r, f)$, $N(r, f)$, $N\left(r, \frac{1}{f}\right)$, $m(r, f)$ *etc* see Hayman [6]. The uniqueness theory of meromorphic functions focuses on the criteria that allow for the existence of essentially only one function that meets these conditions. It demonstrated that any non-constant meromorphic function may be uniquely defined by five values, i.e., if two non-constant meromorphic functions f and g take the same five values at the same locations, then $f \equiv g$.

Let f and g be two non-constant meromorphic functions defined in the open complex plane.The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r, f)$ and $S(r, f)$ is any quantity satisfying $S(r, f) = o(T(r, f)$ where $r \to \infty \in \mathbb{R}^+ \backslash E$, where measure of E is finite. We have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non-negative. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (Counting Multiplicity) and we have $E(a, f) = E(a, g)$. Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (Ignoring Multiplicity) and we will have $\overline{E}(a, f) = \overline{E}(a, g)$. We denote by $E_{k)}(a, f)$ the set of all a- points of f with multiplicities not exceeding k, where an a- point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a- points of f with multiplicities not greater than k.

Definition 1.1 *[8] Let f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*
Clearly, $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$, where $\overline{N}_L(r, a; f)$ denotes the counting function of those 1-points of f and g, when two meromorphic functions f and g share the value 1 IM and z_0 is a 1-point of f of order p, and a 1-point of g of order q, such that $q < p$.

Definition 1.2 *[7] For a complex number $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. For a complex number $a \in \mathbb{C} \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.*

The definition implies that if f, g share the value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with wieght k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 1996, R. Bruck [2] posed the following conjecture.
Conjecture.[2] Let f be a non-constant entire function. Suppose that $\rho_1(f)$

is not positive integer or infinite, if f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a}=c, \tag{1}$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f)=limsup_{r\to\infty}\frac{loglogT(r,f)}{logr}. \tag{2}$$

In 1998, Gundersen and Yang [5] proved that the conjecture is true if f is of finite order and in 1999, Yang [14] generalized their result to the k^{th} derivatives. In 2004, Chen and Shon [4] proved that the conjecture is true for entire functions of first iterated order $\rho_1(f)<\frac{1}{2}$.

In 2008, Yang and Zhang [15] considered the uniqueness problems on meromorphic function f^n sharing value with its first derivative. One of their results can be stated as follows.

Theorem A.[15] Let $f(z)$ be a non-constant meromorphic function and $n\geq 12$ be an integer. Let $F=f^n$. If F and F' share 1 CM, then $F=F'$, and f assumes the form $f(z)=ce^{\frac{1}{n}z}$.

The difference Nevanlinna theory and its application to the uniqueness theory have recently increased interest among researchers.

In 2012, Chen, Chen and Li obtained following results.

Theorem B.[3] Let $f(z)$ be a non-constant meromorphic functionof finite order and $n\geq 9$ be an integer. Let $F(z)=f(z)^n$. If $F(z)$ and Δ_cF share $(1,\infty)$ CM, then $F(z)=\Delta_cF$.

In 2019, Meng and Li[11] proved the following results.

Theorem C.[11] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, d, k be positive integers with $n>2k+\frac{3k+9}{d}, d\geq 2$, and $S=\{a\in\mathbb{C}:a^d=1\}$. If $E_{(f^n)^{(k)}}(S,1)=E_{(g^n)^{(k)}}(S,1)$, then one of the following two cases holds:

1. $f(z)=c_1e^{cz}, g(z)=c_2e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd}=1$;

2. $f=tg$ with $t^{nd}=1$, $t\in\mathbb{C}$.

Theorem D.[11] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, d, k be positive integers with $n>2k+\frac{8k+14}{d}, d\geq 2$ and $S=\{a\in\mathbb{C}:a^d=1\}$. If $E_{(f^n)^{(k)}}(S,0)=E_{(g^n)^{(k)}}(S,0)$, then one of the following two cases holds:

1. $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$;

2. $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.

In 2019, Meng and Liu [10] obtained the following results by considering q-shift $f(qz + c)$ by replacing $F^{'}$.

Theorem E.[10] Let f be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 7$. If $f^n(z)$ and $f^n(qz+\eta)$ share $(1, 2)$, $f(z)$ and $f(qz+\eta)$ share (∞, ∞), then $f(z) = tf(qz + \eta)$, where t is a constant and $t^n = 1$.

Corollary 1.3 *[10] Let f be a non-constant entire function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 5$. If $f^n(z)$ and $f^n(qz + \eta)$ share $(1, 2)$, then $f(z) = tf(qz + \eta)$, where t is a constant and $t^n = 1$.*

Theorem F.[10] Let f be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 8$. If $f^n(z)$ and $f^n(qz + \eta)$ share $(1, 2)$, $f(z)$ and $f(qz + \eta)$ share $(\infty, 0)$, then $f(z) = tf(qz + \eta)$, where t is a constant and $t^n = 1$.

Theorem G.[10] Let f be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 7$. If $f(z)$ and $f(qz + \eta)$ share (∞, ∞) and $E_{3)}(1, f^n(z)) = E_{3)}(1, f^n(qz + \eta))$ then $f(z) = tf(qz + \eta)$, where t is a constant and $t^n = 1$.

Theorem H.[10] Let f be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $\eta \in \mathbb{C}$ and n is an integer satisfying $n \geq 8$. If $f(z)$ and $f(qz + \eta)$ share $(\infty, 0)$ and $E_{3)}(1, f^n(z)) = E_{3)}(1, f^n(qz + \eta))$ then $f(z) = tf(qz + \eta)$, where t is a constant and $t^n = 1$.

Here, we used the idea of weighted sharing values to extend the above results.
Where $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a non-zero polynomial, where $a_0, a_1, ..., a_n(\neq 0)$ are complex constants and m be the number of distinct zeros of $P(z)$.
Now, it will be interesting to study what happens to Theorems E - H when we consider a more generalized q- shift form $f^n(z)P(f(z))$ and $f^n(qz+c)P(f(qz+c))$ and obtained the following results.

Theorem 1.4 *Let f be a non-constant meromorphic function of zero-order. Suppose that q is a non-zero complex constant, $c \in \mathbb{C}$ and n is an integer* *satisfying* $n \geq m+6$. *If* $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* $(1,2)$, $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* (∞,∞), *then* $f^n(z)P(f(z)) \equiv f^n(qz+c)P(f(qz+c))$.

Corollary 1.5 *Let* f *be a non-constant entire function of zero-order. Suppose that* q *is a non-zero complex constant,* $c \in \mathbb{C}$ *and* n *is an integer satisfying* $n \geq m+4$. *If* $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* $(1,2)$, *then the conclusion of Theorem 1.4 holds.*

Theorem 1.6 *Let* f *be a non-constant meromorphic function of zero-order. Suppose that* q *is a non-zero complex constant,* $c \in \mathbb{C}$ *and* n *is an integer satisfying* $n \geq m+7$. *If* $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* $(1,2)$, $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* $(\infty,0)$, *then then the conclusion of Theorem 1.4 holds.*

Theorem 1.7 *Let* f *be a non-constant meromorphic function of zero-order. Suppose that* q *is a non-zero complex constant,* $c \in \mathbb{C}$ *and* n *is an integer satisfying* $n \geq m+6$. *If* $f^n(z)P(f(z))$ *and* $f^n(qz+c)P(f(qz+c))$ *share* (∞,∞) *and* $E_{3)}(1, f^n(z)P(f(z)) = E_{3)}(1, f^n(qz+c)P(f(qz+c))$ *then the conclusion of Theorem 1.4 holds.*

Theorem 1.8 *Let* f *be a non-constant meromorphic function of zero-order. Suppose that* q *is a non-zero complex constant,* $c \in \mathbb{C}$ *and* n *is an integer satisfying* $n \geq m+7$. *If* $f(z)$ *and* $f(qz+c)$ *share* $(\infty,0)$ *and* $E_{3)}(1, f^n(z)P(f(z))) = E_{3)}(1, f^n(qz+c)P(f(qz+c)))$ *then the conclusion of Theorem 1.4 holds.*

2 Preliminaries

In this section we provide all the necessary lemmas required to prove our theorems.
Let us define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{3}$$

Lemma 2.1 *[12, 9] Let* $f(z)$ *be a non-constant meromorphic function of zero-order. Suppose that* q *is a non-zero complex constant,* $\eta \in \mathbb{C}$. *Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S(r, f).$$

where $S(r,f) = o(T(r,f))$ *for all* r *on a set of logarithmic density 1.*

Lemma 2.2 *[12] Let* $f(z)$ *be a non-constant meromorphic function of finite order and* $c \in \mathbb{C}$. *Then*

$$N(r,\infty; f(qz+c)) \leq N(r,\infty; f) + S(r,f),$$
$$N(r,0; f(qz+c)) \leq N(r,0; f) + S(r,f).$$

Lemma 2.3 *[1] Let F and G be two non-constant meromorphic functions. If F and G share $(1,2)$ and (∞,k), where $0 \le k \le \infty$. If $H \not\equiv 0$, then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G),$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those a- points of F whose multiplicities differ from the multiplicities of the corresponding a-points of G.

Lemma 2.4 *[1] Let F and G be two non-constant meromorphic functions. If F and G share (∞,k) and $E_{3)}(1;F) = E_{3)}(1;G)$, where $0 \le k \le \infty$. If $H \not\equiv 0$, then*

$$T(r,F) + T(r,G) \le 2N_2(r,0;F) + 2N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G)$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those a-points of F whose multiplicities differ from the multiplicities of the corresponding a-points of G.

Lemma 2.5 *[13] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z)$, $a_1(z)$,...,$a_n(z)(\not\equiv 0)$ be small functions with respect to f. Then*
$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f)$.

3 Proof of the Main Results

Proof of Theorem 1.4.

$$F = f^n(z)P(f(z)), G = f^n(qz+c)P(f(qz+c)). \tag{4}$$

Then it is easy to verify that F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.3 that

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G). \tag{5}$$

According to Lemma 2.5 we have

$$T(r,F) = (n+m)T(r,f) + S(r,f). \tag{6}$$

It is obvious that

$$N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{f^n(z)P(f(z))}\right) \tag{7}$$

$$\overline{N}(r,F)=\overline{N}(r,G)=\overline{N}(r,f) \tag{8}$$

$$\overline{N}_*(r,\infty;F,G)=0 \tag{9}$$

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{f^n(qz+c)P(f(qz+c))}\right) \tag{10}$$

By combining equations (5) to (10), we deduce,

$$(n-m-6)T(r,f)\leq S(r,f), \tag{11}$$

which contradicts that $n\geq m+6$.
Thus, we have $H\equiv 0$ and hence,

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)=\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

By integrating twice ,we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B. \tag{12}$$

where $A\neq 0$ and B are constants, From (12) we have,

$$G=\frac{(B-A)F+(A-B-1)}{BF-(B+1)} \tag{13}$$

Now, we have the following three subcases:
Subcase 1.4.1. Suppose that $B\neq 0,-1$. Then from (13), we have,

$$\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)=\overline{N}(r,G). \tag{14}$$

From the Second Fundamental Theorem, Lemma 2.5 and (6), we have,

$$\begin{aligned}(n+m)T(r,f)&=T(r,F)+S(r,f)\\&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right)+S(r,f)\\&\leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{(f^nP(f(z))}\right)+S(r,f),\end{aligned} \tag{15}$$

which contradicts $n\geq m+6$.
Subcase 1.4.2. Suppose that $B=-1$. From (13) we have

$$G=\frac{(A+1)F-A}{F} \tag{16}$$

i) If $A \neq -1$, we obtain from (16), we get,

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{17}$$

From the Second Fundamental Theorem, Lemma 2.5, we have

$$\begin{aligned}(n+m)T(r,f) = T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) + S(r,f),\\ &\leq \overline{N}(r, f^n P(f(z))) + \overline{N}\left(r, \frac{1}{f^n P(f(z))}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{f^n(qz+c)P(f(qz+c))}\right) + S(r,f),\end{aligned}$$

which contradicts $n \geq m + 6$.

ii) If $A = -1$ and from (16), we get $FG = 1$, that is $[f^n P(f(z))][f^n(qz + c)P(f(qz+c))] = 1$, from above it is clear that the function f can't have any zero and poles. Therefore $\overline{N}(r, \frac{1}{f}) = S(r,f) = \overline{N}(r,f)$. which is a contradiction .

Subcase 1.4.3. Suppose that $B = 0$. From (13)

$$G = AF - (A-1) \tag{18}$$

If $A \neq 1$, from (18) we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right) \tag{19}$$

Then from the Second Fundamental Theorem and Lemma 2.5

$$\begin{aligned}(n+m)T(r,f) = T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r,f),\\ &\leq \overline{N}(r, f^n P(f(z))) + \overline{N}\left(r, \frac{1}{f^n P(f(z))}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{f^n(qz+c)P(f(qz+c))}\right) + S(r,f),\end{aligned}$$

which contradicts $n \geq m + 6$.

Hence $A = 1$. From (18) we have $F = G$, i.e

$$[f^n P(f(z))] \equiv [f^n(qz+c)P(f(qz+c))]$$

This completes the proof of Theorem 1.4.

Proof of Theorem 1.6.

$$F = f^n(z)P(f(z)), G = f^n(qz + c)P(f(qz + c)). \tag{20}$$

Then it is easy to verify that F and G share $(1, 2)$ and $(\infty, 0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.3 that

$$\begin{aligned} T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) \\ + S(r, F) + S(r, G). \end{aligned} \tag{21}$$

According to Lemma 2.5 we have

$$T(r, F) = (n + m)T(r, f) + S(r, f). \tag{22}$$

It is obvious that

$$N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f^n(z)P(f(z))}\right) \tag{23}$$

$$\overline{N}(r, F) = \overline{N}(r, G) = \overline{N}(r, f) \tag{24}$$

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, f) \tag{25}$$

$$N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{f^n(qz + c)P(f(qz + c))}\right) \tag{26}$$

By combining (21) to (26), we deduce,

$$(n - m - 7)T(r, f) \leq S(r, f), \tag{27}$$

which contradicts that $n \geq m + 7$.
Thus, we have $H \equiv 0$ and similar arguments as in Theorem 1.4, we see that Theorem 1.6 holds.
This completes the proof of Theorem 1.6.

Proof of Theorem 1.7.

$$F = f^n(z)P(f(z)), G = f^n(qz + c)P(f(qz + c)). \tag{28}$$

Then it is easy to verify that F and G share $E_{3)}(1, F) = E_{3)}(1, G)$ and (∞, ∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.4 that

$$\begin{aligned} T(r, F) + T(r, G) \leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, F) + 2\overline{N}(r, G) \\ + 2\overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G). \end{aligned} \tag{29}$$

According to Lemma 2.5 we have

$$T(r,F) = (n+m)T(r,f) + S(r,f). \tag{30}$$

By Lemmas 2.1 and 2.5 we have

$$T(r,G) = (n+m)T(r,f) + S(r,f). \tag{31}$$

It is obvious that

$$N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{f^n(z)P(f(z))}\right) \tag{32}$$

$$\overline{N}(r,F) = \overline{N}(r,G) = \overline{N}(r,f) \tag{33}$$

$$\overline{N}_*(r,\infty;F,G) = 0 \tag{34}$$

$$N_2\left(r,\frac{1}{G}\right) = 2\overline{N}\left(r,\frac{1}{f^n(qz+c)P(f(qz+c))}\right) \tag{35}$$

By combining (29) to (35), we deduce,

$$(2n-2m-12)T(r,f) \le S(r,f), \tag{36}$$

which contradicts that $n \ge m+6$.
Thus, we have $H \equiv 0$ and similar arguments as in Theorem 1.4, we see that Theorem 1.7 holds.
This completes the proof of Theorem 1.7.

Proof of Theorem 1.8.

$$F = f^n(z)P(f(z)), G = f^n(qz+c)P(f(qz+c)). \tag{37}$$

Then it is easy to verify that F and G share $E_{3)}(1,F) = E_{3)}(1,G)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.4 that

$$\begin{aligned} T(r,F) + T(r,G) \le 2N_2\left(r,\frac{1}{F}\right) + 2N_2\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) + 2\overline{N}(r,G) \\ + 2\overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{38}$$

According to Lemma 2.5 we have

$$T(r,F) = (n+m)T(r,f) + S(r,f). \tag{39}$$

By Lemmas 2.1 and 2.5 we have

$$T(r,G)=(n+m)T(r,f)+S(r,f). \tag{40}$$

It is obvious that

$$N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f^n(z)P(f(z))}\right) \tag{41}$$

$$\overline{N}(r,F)=\overline{N}(r,G)=\overline{N}(r,f) \tag{42}$$

$$\overline{N}_*(r,\infty;F,G)\leq \overline{N}(r,f) \tag{43}$$

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{f^n(qz+c)P(f(qz+c))}\right) \tag{44}$$

By combining (38) to (44), we deduce,

$$(2n-2m-14)T(r,f)\leq S(r,f), \tag{45}$$

which contradicts that $n\geq m+7$.
Thus, we have $H\equiv 0$ and similar arguments as in Theorem 1.4, we see that Theorem 1.8 holds.
This completes the proof of Theorem 1.8.

4 Conclusion

By considering the q- shift difference polynomial in the functions of the form $f^nP(f(z))$ and $f^n(qz+c)P(f(qz+c))$, along with weighted sharing concept in Theorem 1.4 to Theorem 1.8, we prove important analogous results for transcendental meromorphic functions of zero order.

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5 Open Problem

1. Can the condition for the lower bound n in Theorems 1.4 to 1.8 be reduced any further?
2.Can the difference polynomials in Theorems 1.4 to 1.8 be replaced by the difference polynomials of form $f^nP(f)\prod_{j=1}^{d}f(z+c_j)^{v_j}\prod_{j=1}^{s}f^{(i)}(z)$?

References

[1] A. Banerjee, *Uniqueness of meromorphic functions that share two sets*, Southeast Asian Bull.Math., 31, (2007), 7-17.

[2] R. Bruck, *On entire functions which share one value CM with their first derivative*, Results Math., 30, (1996), 21-24.

[3] B. Q. Chen, Z. X. Chen and S. Li, *Uniqueness of difference operators of meromorphic functions*, J. Ineq. Appl., (2012), Art 48.

[4] Z. X. Chen and K. H. Shon, *On conjecture of R. Bruck concering the entire function sharing one value CM with its derivatives*, Taiwanese J. Math., 8, (2004), 235-244.

[5] G. G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., 223, (1998), 88-95.

[6] W. K. Hayman, Meromorphic functions, *Oxford Mathematical Monographs*, Clarendon Press, Oxford, 1964.

[7] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables and Elliptic Equations, 46(3), (2001), 241-253.

[8] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J, 161, (2001), 193-206.

[9] K. Liu, X. Liu and T. Cao, *Some results on zeros and uniqueness of difference-differential polynomials*, Appl. Math. J. Chinese Univ. Ser. B , 27(1), (2012), 94-104.

[10] C. Meng and G. Liu, *Weighted sharing and uniqueness for the shifts of meromorphic functions*, Tbilisi Mathematical Journal 12(2), (2019), 17-27.

[11] C. Meng and Xu. Li, *On unicity of meromorphic functions and their derivatives*, The Journal of Analysis, 28(3), (2019), 879-894.

[12] J. Xu and X. Zhang, *The zeros of q-shift difference polynomials of meromorphic functions*, Adv. Difference Equ, (2012), 200, 10 pp.

[13] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions, Mathematics and its Applications*, 557, Kluwer Academic Publishers Group, Dordrecht, 2003.

[14] L. Z. Yang, *Solution of a differential equation and its applications*, Kodai Math. J. 22. (1999), 458-464.

[15] L. Z. Yang and J. L. Zhang, *Non-existence of meromorphic solution of a Fermat type functional equation*, Aequationes Math. 76, (2008), 140-150.