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Two new extensions of Hardy-Hilbert-type integral inequalities

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Abstract

In this paper, we investigate two new Hardy-Hilbert-type integral inequalities that incorporate both the maximum and the sum of the variables within the integrands. The second inequality is presented as a three-dimensional analogue of the first. Our proofs rely on the Hölder integral inequality and various integral identities. Finally, the paper concludes by posing a related open problem.

Keywords: *Hardy-Hilbert integral inequality, maximum of variables, Hölder integral inequality, integral calculus.*
2010 Mathematics Subject Classification: 26D15.

1 Introduction

This paper is centered on the Hardy-Hilbert integral inequality, which is formally stated as follows. Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

Then, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy \right)^{1/q}.$$

See [5, 6]. Numerous refinements and modifications of this inequality have been proposed over the years. For a comprehensive survey of these developments, we refer the reader to [2].

Of particular interest are variants involving the maximum of the variables, $\max(x, y)$. This approach was notably developed in [6], yielding the following Hardy-Hilbert-type integral inequality:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\max(x,y)}dxdy \leq pq\left(\int_0^{+\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}.$$

This result has served as a catalyst for numerous extensions. A broad range of such developments can be found in [6, 1, 8, 7, 9, 10, 11, 3, 4].

The present study furthers this research direction by establishing two new Hardy-Hilbert-type integral inequalities. The first is centered on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y)\max(x,y)}dxdy.$$

Therefore, the denominator is defined as the product of the maximum $\max(x, y)$ and the sum $x + y$. Despite its apparent simplicity, this form appears to have remained unexplored in the literature. The second inequality serves as a three-dimensional analogue of the first, based on the following triple integral:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^2\max(x,y,z)}dxdydz,$$

where $h : (0, +\infty) \to (0, +\infty)$ denotes an additional function. Therefore, the denominator is defined as the product of the maximum $\max(x, y, z)$ and the polynomial term $(x + y + z)^2$. The resulting upper bounds are sharp, with the core inequality derived directly from the Hölder integral inequality. The paper concludes with the formulation of an open problem.

The remainder of the paper is organized as follows: Section 2 establishes two new Hardy-Hilbert integral inequalities, while Section 3 addresses a related open problem.

2 Theorems

Our first result is presented in the theorem below.

Theorem 2.1 *Let $p > 1$, $q = p/(p - 1)$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} f^p(x)\frac{1}{x}dx < +\infty, \quad \int_0^{+\infty} g^q(y)\frac{1}{y}dy < +\infty.$$

Then, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y)\max(x,y)}dxdy$$
$$\leq \log(4)\left(\int_0^{+\infty} f^p(x)\frac{1}{x}dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)\frac{1}{y}dy\right)^{1/q}.$$

Proof. Using $1/p+1/q=1$ and the Hölder integral inequality, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y)\max(x,y)}dxdy$$
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(x+y)^{1/p}\max(x,y)^{1/p}}\times\frac{g(y)}{(x+y)^{1/q}\max(x,y)^{1/q}}dxdy$$
$$\leq\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y)\max(x,y)}dxdy\right)^{1/p}$$
$$\times\left(\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{(x+y)\max(x,y)}dxdy\right)^{1/q}.$$

Let us evaluate each double integral within this upper bound individually.

By the Fubini-Tonelli integral theorem, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y)\max(x,y)}dxdy$$
$$=\int_0^{+\infty} f^p(x)\left(\int_0^{+\infty}\frac{1}{(x+y)\max(x,y)}dy\right)dx.$$

Let us calculate the integral inside the parentheses. Using the Chasles integral relation and some integral formulas, we get

$$\int_0^{+\infty}\frac{1}{(x+y)\max(x,y)}dy$$
$$=\int_0^{x}\frac{1}{(x+y)x}dy+\int_x^{+\infty}\frac{1}{(x+y)y}dy$$
$$=\frac{1}{x}\int_0^{x}\frac{1}{x+y}dy+\frac{1}{x}\int_x^{+\infty}\left(\frac{1}{y}-\frac{1}{x+y}\right)dy$$
$$=\frac{1}{x}\left([\log(x+y)]_{y=0}^{y=x}+[\log(y)-\log(x+y)]_{y=x}^{y=+\infty}\right)$$
$$=\frac{1}{x}\left(\log(2x)-\log(x)+\left[\log\left(\frac{y}{x+y}\right)\right]_{y=x}^{y=+\infty}\right)$$
$$=\frac{1}{x}\left(\log(2)+\log(1)-\log\left(\frac{x}{2x}\right)\right)$$
$$=\frac{1}{x}\left(\log(2)+\log(2)\right)=\frac{1}{x}\log(4).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y)\max(x,y)}dxdy = \log(4)\int_0^{+\infty} f^p(x)\frac{1}{x}dx.$$

Proceeding similarly, we find that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{(x+y)\max(x,y)}dxdy = \log(4)\int_0^{+\infty} g^q(y)\frac{1}{y}dy.$$

Finally, combining the above equations and using $1/p+1/q=1$, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y)\max(x,y)}dxdy\\ &\leq \left(\log(4)\int_0^{+\infty} f^p(x)\frac{1}{x}dx\right)^{1/p}\left(\log(4)\int_0^{+\infty} g^q(y)\frac{1}{y}dy\right)^{1/q}\\ &= \log(4)\left(\int_0^{+\infty} f^p(x)\frac{1}{x}dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)\frac{1}{y}dy\right)^{1/q}.\end{aligned}$$

This conclludes the proof of the theorem. □

Since the proof relies exclusively on the Hölder integral inequality, we assert the sharpness of the resulting inequality. Furthermore, we emphasize the simplicity of the constant factor, log(4), and the associated weighted integral norms of f and g.

The theorem below may be viewed as a three-dimensional analogue of the preceding result.

Theorem 2.2 *Let $p,q>1$, $r=pq/(pq-p-q)$, and $f,g,h:(0,+\infty)\to(0,+\infty)$ be three functions such that*

$$\int_0^{+\infty} f^p(x)\frac{1}{x}dx<+\infty,\quad \int_0^{+\infty} g^q(y)\frac{1}{y}dy<+\infty,\quad \int_0^{+\infty} h^r(z)\frac{1}{z}dz<+\infty.$$

Then, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^2\max(x,y,z)}dxdydz\\ &\leq 3\log\left(\frac{4}{3}\right)\left(\int_0^{+\infty} f^p(x)\frac{1}{x}dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)\frac{1}{y}dy\right)^{1/q}\left(\int_0^{+\infty} h^r(z)\frac{1}{z}dz\right)^{1/r}.\end{aligned}$$

Proof. Using $1/p+1/q+1/r=1$ and the generalized Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^2\max(x,y,z)}dxdydz\\
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)}{(x+y+z)^{2/p}\max(x,y,z)^{1/p}}\\
&\times\frac{g(y)}{(x+y+z)^{2/q}\max(x,y,z)^{1/q}}\\
&\times\frac{h(z)}{(x+y+z)^{2/r}\max(x,y,z)^{1/r}}dxdydz\\
&\leq\left(\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y+z)^2\max(x,y,z)}dxdydz\right)^{1/p}\\
&\times\left(\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{(x+y+z)^2\max(x,y,z)}dxdydz\right)^{1/q}\\
&\times\left(\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{h^r(z)}{(x+y+z)^2\max(x,y,z)}dxdydz\right)^{1/r}.
\end{aligned}$$

Let us evaluate each triple integral within this upper bound individually.

By the Fubini-Tonelli integral theorem, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y+z)^2\max(x,y,z)}dxdydz\\
&=\int_0^{+\infty}f^p(x)\left(\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz\right)dx.
\end{aligned}$$

Let us calculate the integral inside the parentheses. Dividing $(0,+\infty)^2$ into four non-overlapping subsets and using the Chasles integral relation, we can write

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz=A+B+C+D,$$

where

$$A=\int_0^{x}\int_0^{x}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz,$$

$$B=\int_x^{+\infty}\int_0^{x}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz,$$

$$C=\int_0^{x}\int_x^{+\infty}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz$$

and

$$D=\int_x^{+\infty}\int_x^{+\infty}\frac{1}{(x+y+z)^2\max(x,y,z)}dydz.$$

Let us evaluate each of these integrals individually. With some integral manipulations and formulas, we obtain

$$\begin{aligned}A&=\frac{1}{x}\int_0^x\int_0^x\frac{1}{(x+y+z)^2}dydz=\frac{1}{x}\int_0^x\left[-\frac{1}{x+y+z}\right]_{y=0}^{y=x}dz\\&=\frac{1}{x}\int_0^x\left(-\frac{1}{2x+z}+\frac{1}{x+z}\right)dz\\&=\frac{1}{x}\left[-\log(2x+z)+\log(x+z)\right]_{z=0}^{z=x}\\&=\frac{1}{x}\left(-\log(3x)+\log(2x)+\log(2x)-\log(x)\right)\\&=\frac{1}{x}\log\left(\frac{4}{3}\right).\end{aligned}$$

With some integral manipulations and formulas, we obtain

$$\begin{aligned}B&=\int_x^{+\infty}\int_0^x\frac{1}{(x+y+z)^2z}dydz\\&=\int_x^{+\infty}\frac{1}{z}\left[-\frac{1}{x+y+z}\right]_{y=0}^{y=x}dz\\&=\int_x^{+\infty}\frac{1}{z}\left(-\frac{1}{2x+z}+\frac{1}{x+z}\right)dz\\&=\int_x^{+\infty}\left(-\frac{1}{z(2x+z)}+\frac{1}{z(x+z)}\right)dz\\&=\int_x^{+\infty}\left(\frac{1}{2x}\left(-\frac{1}{z}+\frac{1}{2x+z}\right)+\frac{1}{x}\left(\frac{1}{z}-\frac{1}{x+z}\right)\right)dz\\&=\frac{1}{x}\left[\frac{1}{2}\left(-\log(z)+\log(2x+z)\right)+\log(z)-\log(x+z)\right]_{z=x}^{z=+\infty}\\&=\frac{1}{x}\left[\frac{1}{2}\log\left(\frac{2x+z}{z}\right)+\log\left(\frac{z}{x+z}\right)\right]_{z=x}^{z=+\infty}\\&=\frac{1}{x}\left(-\frac{1}{2}\log\left(\frac{3x}{x}\right)-\log\left(\frac{x}{2x}\right)\right)\\&=\frac{1}{2x}\log\left(\frac{4}{3}\right).\end{aligned}$$

By the Fubini-Tonelli integral theorem and noting the equality with B by exchanging the role of y and z, we have

$$C = \int_0^x \int_x^{+\infty} \frac{1}{(x+y+z)^2 y} dydz = \int_x^{+\infty} \int_0^x \frac{1}{(x+y+z)^2 y} dzdy = B$$
$$= \frac{1}{2x} \log\left(\frac{4}{3}\right).$$

Applying the Fubini-Tonelli integral theorem, exchanging the role of y and z, and using some integral manipulations and formulas, we get

$$\begin{aligned}
D &= \int_x^{+\infty} \int_x^{+\infty} \frac{1}{(x+y+z)^2 \max(y,z)} dydz \\
&= \int_x^{+\infty} \int_x^{y} \frac{1}{(x+y+z)^2 y} dzdy + \int_x^{+\infty} \int_y^{+\infty} \frac{1}{(x+y+z)^2 z} dzdy \\
&= \int_x^{+\infty} \int_x^{y} \frac{1}{(x+y+z)^2 y} dzdy + \int_x^{+\infty} \int_x^{z} \frac{1}{(x+y+z)^2 z} dydz \\
&= 2\int_x^{+\infty} \int_x^{y} \frac{1}{(x+y+z)^2 y} dzdy \\
&= 2\int_x^{+\infty} \frac{1}{y} \left[-\frac{1}{x+y+z}\right]_{z=x}^{z=y} dy \\
&= 2\int_x^{+\infty} \frac{1}{y} \left(-\frac{1}{x+2y} + \frac{1}{2x+y}\right) dy \\
&= 2\int_x^{+\infty} \left(-\frac{1}{y(x+2y)} + \frac{1}{y(2x+y)}\right) dy \\
&= 2\int_x^{+\infty} \left(\frac{1}{x}\left(-\frac{1}{y} + \frac{2}{x+2y}\right) + \frac{1}{2x}\left(\frac{1}{y} - \frac{1}{2x+y}\right)\right) dy \\
&= \frac{2}{x}\left[-\log(y) + \log(x+2y) + \frac{1}{2}\left(\log(y) - \log(2x+y)\right)\right]_{y=x}^{y=+\infty} \\
&= \frac{2}{x}\left[\log\left(\frac{x+2y}{y}\right) + \frac{1}{2}\log\left(\frac{y}{2x+y}\right)\right]_{y=x}^{y=+\infty} \\
&= \frac{2}{x}\left(\log(2) - \log\left(\frac{3x}{x}\right) - \frac{1}{2}\log\left(\frac{x}{3x}\right)\right) \\
&= \frac{1}{x}\log\left(\frac{4}{3}\right).
\end{aligned}$$

Combining the above equalities, we obtain

$$A+B+C+D$$
$$= \frac{1}{x}\log\left(\frac{4}{3}\right) + \frac{1}{2x}\log\left(\frac{4}{3}\right) + \frac{1}{2x}\log\left(\frac{4}{3}\right) + \frac{1}{x}\log\left(\frac{4}{3}\right) = \frac{3}{x}\log\left(\frac{4}{3}\right).$$

Therefore, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)}{(x+y+z)^2\max(x,y,z)}dxdydz$$
$$=3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{x}f^p(x)dx.$$

Proceeding similarly, we find that

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{g^q(y)}{(x+y+z)^2\max(x,y,z)}dxdydz$$
$$=3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{y}g^q(y)dy$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{h^r(z)}{(x+y+z)^2\max(x,y,z)}dxdydz$$
$$=3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{z}h^r(z)dz.$$

Finally, combining the above equations and using $1/p+1/q+1/r=1$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{(x+y+z)^2\max(x,y,z)}dxdydz$$
$$\leq\left(3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right)^{1/p}\left(3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right)^{1/q}$$
$$\times\left(3\log\left(\frac{4}{3}\right)\int_0^{+\infty}\frac{1}{z}h^r(z)dz\right)^{1/r}$$
$$=3\log\left(\frac{4}{3}\right)\left(\int_0^{+\infty}f^p(x)\frac{1}{x}dx\right)^{1/p}\left(\int_0^{+\infty}g^q(y)\frac{1}{y}dy\right)^{1/q}\left(\int_0^{+\infty}h^r(z)\frac{1}{z}dz\right)^{1/r}.$$

This concludes the proof of the theorem. □

Furthermore, we emphasize the explicit nature of the constant factor, $3\log(4/3)$, and its relationship to the weighted integral norms of f, g, and h.

3 Open problem

An open problem arises in the determination of a Hardy-Hilbert-type integral inequality based on the following four-dimensional integral:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)k(w)}{(x+y+z+w)^3\max(x,y,z,w)}dxdydzdw,$$

where $k : (0, +\infty) \to (0, +\infty)$ denotes an additional function. Furthermore, this result naturally invites generalization to a higher-dimensional setting.

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