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Solution to an Open Problem on Riemann Zeta Function

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Abstract

In this note, we provide two solutions to an open problem concerning the Riemann zeta function. The first solution relies on some properties of the Riemann zeta function whilst the second solution relies on the positivity of a certain function associated with the polygamma function.

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1 Introduction

It is well known that the Riemann zeta function is defined as

$$\begin{aligned}\zeta(s) &= \sum_{k=1}^{\infty} \frac{1}{k^s}, \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{y^{s-1}}{e^y - 1} dy,\end{aligned}$$

for $s > 1$, where $\Gamma(s)$ is the gamma function. The Riemann zeta function has interesting applications in areas such as analytic number theory, probability theory, mathematical physics and mathematical analysis. Due to its usefulness, many researchers have studied its properties along different directions. In recent times, researchers have shifted attention to inequalities involving the function. As a result, many remarkable inequalities involving the function have been established. See for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the related references therein.

In accordance with this direction of research, in October 2024, the following problem was posted on ResearchGate [12].

Problem 1.1. *Show whether or not the function*

$$K(n) = (n+2)\zeta(n+1)\zeta(n+3) - (n+1)(\zeta(n+2))^2 - \zeta(n+1)\zeta(n+2)$$

is positive for $n \in \mathbb{N}$.

In this short note, our main objective is to provide an answer to the problem. We achieve this by providing two solutions to the problem as outlined in Section 2 and Section 3. We shall require the following definitions in order to establish the results of Section 3.

The polygamma function is defined for $z > 0$ and $r \in \mathbb{N}$ as (see [13] and the references therein)

$$\begin{aligned} \psi^{(r)}(z) &= (-1)^{r+1} \int_0^\infty \frac{s^r e^{-zs}}{1-e^{-s}} ds, \\ &= (-1)^{r+1} \sum_{n=0}^{\infty} \frac{r!}{(n+z)^{r+1}}, \\ &= (-1)^{r+1} r! \zeta(r+1, z), \end{aligned} \tag{1}$$

where $\zeta(s, z)$ is the Hurwitz zeta function. In particular,

$$\zeta(s, 1) = \zeta(s). \tag{2}$$

2 First Solution

In this section, we provide our first solution to the problem.

Lemma 2.1. *For every* $n \in \mathbb{N}$, *we have*

$$\zeta^2(n+2) < \zeta(n+1). \tag{3}$$

Proof. From the definition of the Riemann zeta function, we have $\zeta(n+2) = \sum_{k=1}^{\infty} k^{-(n+2)}$. Thus,

$$\zeta^2(n+2) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{(k\ell)^{n+2}} = \sum_{m=1}^{\infty} \frac{d(m)}{m^{n+2}} = 1 + \sum_{m=2}^{\infty} \frac{d(m)}{m^{n+2}},$$

where $d(m)$ is the divisor function. For $m \geq 2$, $d(m) \leq m-1 < m$. Therefore,

$$\frac{d(m)}{m^{n+2}} < \frac{m}{m^{n+2}} = \frac{1}{m^{n+1}}.$$

Summing over m, we obtain $\zeta^2(n+2) < 1 + \sum_{m=2}^{\infty} m^{-(n+1)} = \zeta(n+1)$. □

Lemma 2.2. *For every $s > 1$, we have $\zeta(s) < 1 + \frac{1}{2^s-1}$.*

Proof. Note that for $s > 1$, $\zeta(s) - 1 = \sum_{n=2}^{\infty} n^{-s}$. Comparing this to a geometric series or using the integral test, we observe the bound $\sum_{n=2}^{\infty} n^{-s} < \sum_{k=1}^{\infty} (2^{-s})^k = \frac{1}{2^s-1}$. □

Lemma 2.3. *For every $n \in \mathbb{N}$, we have $\zeta(n) > 1 + \frac{1}{2^n}$.*

Proof. This follows immediately from $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots$ where all terms are positive. □

Theorem 2.4. *For every $n \in \mathbb{N}$, the quantity*

$$K(n) = (n+2)\zeta(n+1)\zeta(n+3) - (n+1)\zeta^2(n+2) - \zeta(n+1)\zeta(n+2) \tag{4}$$

is positive.

Proof. By Lemma 2.1, we have $-(n+1)\zeta^2(n+2) > -(n+1)\zeta(n+1)$. Substituting this into $K(n)$:

$$K(n) > \zeta(n+1)\left[(n+2)\zeta(n+3) - \zeta(n+2) - (n+1)\right].$$

Let $f(n) = (n+2)\zeta(n+3) - \zeta(n+2) - n - 1$. Using Lemma 2.2 for $\zeta(n+2)$:

$$f(n) > (n+2)\zeta(n+3) - \left(1 + \frac{1}{2^{n+2}-1}\right) - n - 1 = (n+2)\zeta(n+3) - n - 2 - \frac{1}{2^{n+2}-1}.$$

Applying Lemma 2.3 to $\zeta(n+3)$, we have $(n+2)\zeta(n+3) > (n+2)(1+2^{-(n+3)})$. Thus:

$$f(n) > (n+2) + \frac{n+2}{2^{n+3}} - n - 2 - \frac{1}{2^{n+2}-1} = \frac{n+2}{2^{n+3}} - \frac{1}{2^{n+2}-1}.$$

For $n \geq 1$, $\frac{n+2}{2^{n+3}} > \frac{1}{2^{n+2}-1}$ is equivalent to $(n+2)(2^{n+2}-1) > 2^{n+3}$, which simplifies to $n \cdot 2^{n+2} > n+2$. This holds for all $n \in \mathbb{N}$. Thus $f(n) > 0$, implying $K(n) > 0$. □

3 Second Solution

In this section, we provide our second solution to the problem.

Theorem 3.1. *For every $n \in \mathbb{N}$, we have*

$$K(n) = (n+2)\zeta(n+1)\zeta(n+3) - (n+1)(\zeta(n+2))^2 - \zeta(n+1)\zeta(n+2) > 0. \quad (5)$$

Proof. Let $A(n)$ be defined as

$$A(n) = \psi^{(n+2)}(1)\psi^{(n)}(1) - \left(\psi^{(n+1)}(1)\right)^2 + \psi^{(n)}(1)\psi^{(n+1)}(1)$$

for $n \in \mathbb{N}$. It is known from [14] and [15] that $A(n) > 0$. As a result of (1) and (2), we have

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1).$$

Then

$$\begin{aligned} A(n) &= n!(n+2)!\zeta(n+1)\zeta(n+3) - ((n+1)!\zeta(n+2))^2 - n!(n+1)!\zeta(n+1)\zeta(n+2) \\ &= n!(n+2)(n+1)!\zeta(n+1)\zeta(n+3) - (n+1)n!(n+1)!(\zeta(n+2))^2 \\ &\quad - n!(n+1)!\zeta(n+1)\zeta(n+2). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{A(n)}{n!(n+1)!} &= (n+2)\zeta(n+1)\zeta(n+3) - (n+1)(\zeta(n+2))^2 - \zeta(n+1)\zeta(n+2) \\ &= K(n). \end{aligned}$$

The conclusion then follows from the fact that $A(n) > 0$. □

4 Conclusion

In this paper, we have provided two solutions to an open problem involving the Riemann zeta function. It is our expectation that the present results will lay a good foundation for further research on the function.

5 Open Problem

In order to expand the domain of application of the open problem solved in this work, it will be interesting to extend the result to the set of positive real numbers. Based on this, we present the following problem.

Problem 5.1. *Show that*

$$F(z) = (z+2)\zeta(z+1)\zeta(z+3) - (z+1)(\zeta(z+2))^2 - \zeta(z+1)\zeta(z+2) > 0.$$

for $z \in (0, \infty)$.

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