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On Bishop-Phelps-Bollobas Property in Banach Spaces

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Abstract

In this paper, we establish Bishop-Phelps-Bollobas Property (BPBp) for finite rank operators (fro) between Banach spaces (BS). We prove that $BPBp$ ***for*** fro ***holds in several settings including when a Banach space*** X ***is of finite dimension or uniformly convex. We also extend these results and show that this property also holds on*** BS ***with geometrical properties. Moreover, we characterize the numerical radius (nr) of*** fro ***via the*** $BPBp$***. We establish the extent to which*** fro ***satisfy*** $BPBp$ ***with respect to*** nr***. We show that this property holds in*** BS ***settings which include when*** X ***is reflexive.***

Keywords: *Bishop-Phelps-Bollobas Property, Banach space, norm-attainability, Finite rank operator.*
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1 Introduction

Several studies have been done on the Bishop and Phelps (BP) property [1] and it was shown that one can be able to estimate the distance between continuous linear functionals. In [2] it was shown that given two norm one elements, the distance between the two elements or the total distance of the two elements is less that or equal to epsilon.
In [3] the authors considered estimation of linear functionals and not linear

mappings and hence in our study we consider mappings of finite rank and check whether the distance between them can be easily estimated as suggested by [4].

Lindenstrauss [43] extended this study of closeness of norm achieving functionals to linear mappings between Banach spaces. In this case is was proved that it is not possible in general to obtain the distance between mappings on Banach spaces. However, the author obtained some positive results by characterizing Banach spaces as shown in [5].
Some results in [6] that were obtained was that, one can be able to estimate distance between linear mappings on Banach spaces given then the spaces are of finite dimension and given that the spaces are characterized by uniform convexity.
Bollobas [10] showed that apart from estimating functionals, one can also be able measure the distance between involved points. Hence, it was emphasized in [7] that the distance between norm achieving functionals is less than epsilon, and the distance between points at which those functionals achieve their norms is also less that epsilon.
In Theorem 3.0 of [8] it was shown that the distance between involved functionals and points can be estimated but in our study we shall extend this study to linear mappings of finite rank. More researchers (see [9]-[15] and the references therein) have also shown interest in the study of Bishop-Phelps theorem.

Jerry [34] considered norm attainability and RNP on Banach spaces. Denseness of linear mappings on Lebesgue spaces was studied and Radon-Nikodym property and strict convexity property was characterized on Banach spaces. The author of [16] showed that norm achieving mappings from the Lebesgue space $L^1[0, 1]$ into a complete normed vector space Z can be estimated by those that achieve their norms provided that Z has RNP and given that Z is characterized by strict convexity. The main aim of the author was to show that the closeness of linear mappings hold s in strictly convex Banach spaces as discussed in [18], [19] and [20].
In [21] it was shown that mappings on the Lebesgue space $L^1(\mu)$ can be estimated by those that achieve their norms given that Z has RNP but in our study we shall consider $BPBp$ for finite rank mappings. In [22] the work characterized linear mappings in terms of norm attainability from a Lebesgue space $L^1(\mu)$ to a Banach space Y but in our study we extend this result to denseness of norm achieving mappings especially those mappings of finite rank as suggested in [23].
Also, in [24] a result comparable to Theorem 1 of [34] was obtained for the closeness of norm achieving mappings but the author considered mappings from a Lebesgue space over the interval $[0, 1]$ into a strict convex Banach space Y. Our study considered this result for mapping with finite dimensional range as recommended in [25]. Finally, in the same spirit, the author of [26]

came up with a question asking if it was possible for norm achieving mappings from a Lebesgue space $L^1[0,1]$ to itself can estimated by norm attaining ones. This was given a consideration by authors of [27]-[30].

Anzelm [4] showed that norm achieving mappings can be estimated by those that achieve their norms in Lebesgue spaces. This was emphasized in Theorem 2 of [31] it was shown that norm achieving mappings is dense in $B(L^1(m_1), L^1(m_2))$ but in our study we shall extend this result to mappings with finite dimensional range.

Baker [12] showed that completely continuous mappings attain their norms. The author showed that weakly compact operators also attain their norms on Banach spaces provided that Banach spaces have Dunford-Pettis property. It was shown in [32] that a proper subset of finite rank transformations consists of transformations which achieve their norms.

From [33], compactness of linear mapping was characterized but in our study we shall extend this result to finite rank operators. In [34] denseness of linear mappings was characterized but in our study we shall consider finite rank operators and characterize their denseness.

Schachermayer [54] characterized linear mappings that achieve their norms on Banach spaces. The author characterized the denseness of linear mapping by showing that there exist linear mappings on $(L^1[0,1], C[0,1])$ which can not be estimated by those that achieve their norms. Later, the work of [35] obtained positive results by proving that weakly completely continuous mappings on $C(K)$ space can be estimated by those mappings that achieve their norms.

In [36] it was shown that norm achieving mappings on $(L[0,1], C[0,1])$ can not be estimated by those that achieve their norms but in our study we shall consider mappings with finite dimensional range and investigate whether the estimation property holds.

Also the results of [37] showed that completely continuous mappings achieve their norm but the study was limited to characterization of finite rank mappings.

The work of [38] characterized the denseness of numerical radius of linear mappings on CL-spaces. The author investigated the relationship between denseness of numerical radius achieving mappings and norm achieving mappings and it was proved also in [39] that in both cases the density and the results were proved to be similar.

In c_o, l_1 and $C(K)$ spaces it was shown that a mapping attains is norm if it also attains it numerical radius and this was extended to CL-spaces where the same results were proved in [40] to hold. Furthermore, in [41] it was established that the nr of a mapping is equivalent to its norm but the result was limited to finite rank mappings.

Sevilla [55] characterized the denseness of functionals which do not achieve their norms in Banach spaces. These results were extended to separable case in [42] and reflexivity of Banach spaces was characterized in [43]. In [44] reflexivity of Banach spaces was characterized and it was shown in [45] that a complete normed vector space is reflexive if its dual contains a norm achieving mapping but in this note we characterize finite rank operators on Banach spaces as recommended in [46].

In [47] the author studied bilinear mappings that achieve their norms on $C(K)$ spaces. It was examined that continuous bilinear mappings can be estimated by those that achieve their norms. The author extended this study to multilinear mapping that achieve their norms on $C(K)$ spaces and it was also shown in [48] that the closeness between multilinear mappings can e determined. Finally, the numerical radius of multilinear mappings was characterized on the spaces of continuous functions by [49].

The work of [50] studied $BPBp$ for Asplund operators on Banach spaces. It was shown that given a norm one Asplund mapping on a Banach space attaining it's norm at some point, there exist a new norm one mapping that achieves it's norm at some other point, then the distance between the new Asplund mapping and the original one can be determined and it was further shown in [51] that the involved points are close to each other. The study was further extended to weakly compact mappings and it was shown in [52] that distance between weakly compact mappings can be determined.

In [53] the authors investigated the notion of closeness of norm achieving mappings in complex normed inner spaces using spectral integral. Their main interest was to prove that for any complete inner product space H, the space (H, H) admits the $BPBp$ implying that linear mappings on Hilbert spaces can be estimated by those that achieve their norms.

Chica et al [18] introduced two BPB moduli of a complete normed vector space which show the appropriate theorem for denseness of linear transformations on certain Banach spaces. It was shown that there is the most precise upper estimate for these moduli for all Banach spaces. The author defined this moduli as being continuous and an inequality related to duality was investigated. On complete inner product spaces the two moduli were solved and some examples such as compact spaces $C(K)$ and sequence space $L_1(\mu)$ were later presented where the moduli had the greatest achievable value.

Cascalesa *et al* [16] studied $BPBp$ for Asplund mappings on uniform Algebras, in this case it was shown that operators that are Asplund can be estimated by those that achieve their norms. It was also shown that Asplund mappings can be estimated by those that achieve their norms he result was limited to finite rank mappings.

In [39] Kim *et al* characterized continuous retractions on Banach spaces, it was shown that linear transformations on complete vector spaces admit the BPB theorem implying that they can be approximated by those that achieve their norms. It was further shown that the closeness holds given that X^* admits continuous retractions that are uniformly simultaneous.

Cho [22] considered the notion of closeness of linear mappings on bounded closed convex subsets on Banach spaces. It was shown that all Banach spaces with finite dimension admit the closeness property. The result was further extended to Aspund spaces where the closeness was also proved to hold. Finally, $BPBp$ was also characterized on l_1-sum of Banach spaces.

Kim and Lee [38] studied estimation property for linear mappings on the pair $(C(K), X)$ on BS. This was further emphasized in Theorem 2.2 where the $BPBp$ was examined to hold for the pair $(C(K), X)$, implying that the distance between linear mappings can be determined. In our study we shall extend this result to mappings with finite dimensional range.

Martin [47] showed that there are norm achieving mappings that can not be estimated by those that achieve their norms, it was then shown that completely continuous mappings are among those mappings that can not be estimated by mappings that achieve their norms thereby answering the vital question raised by [8] in the negative sense.

It was further shown that if a Banach space that is characterized by uniform convexity fails to admit the estimation property then it serves as the codomain space. Later, the author showed that there exist domain spaces with the schauder system and known conditions ensuring the closeness norm achieving mappings were also presented.

Proposition 6 in [47] shows that there exist completely continuous mappings on Banach spaces with strict convexity that can not be estimated by completely continuous mappings that achieve their norms, provided that the given Banach space does not posses estimation property (BPBp), in our study we shall consider mappings of finite rank and investigate whether they can be estimated by those that achieve their norms.

In Theorem 7 of [54] it was proved that completely continuous mappings can not be estimated by those that achieve their norms hence, they do not admit the approximation property, in our study we shall prove this in the positive sense and show that it is possible to estimate a mapping by a norm achieving mapping, we shall further consider mappings of finite rank and we shall show that they admit estimation property.

Later the authors of [55] identified the following problems: Is it true that mappings on complete normed vector spaces with finite dimension admit the estimation property? Lastly, the authors of [56] gave conditions on the domain space ensuring that completely continuous mappings that achieve their norms can be estimated by those that achieve their norms. One of he conditions they got was that the estimation property holds provided that Banach spaces have Radon-Nikodym property.

Aron *et al* [5] investigated the denseness of norm achieving mappings for Lindenstrauss properties A and B. The authors characterize domain spaces for the closeness to hold from the domain space Q to an arbitrary Banach space Z. It this case, the authors gave some assumptions for the property to hold for the space Q.

One of those conditions was that if the space Q admits the property for any equivalent norm, then it is of one dimension. Later, BPB property was studied from any Banach space Q to the codomain space Z. Finally, to achieve these results c_o-, l_1- and l_∞- sums of Banach spaces were characterized to admit BPB property.

Talponen [57] studied estimation property($BPBp$) for linear mappings on Banach spaces. The author characterized local $BPBp$ in strictly convex spaces for completely continuous mappings. It was shown that local property implies that the space has strongly convexity properties. The author discussed property 1 which states that the epsilon that appears on the estimation property does not only depend on epsilon but also on a fixed operator with a norm of one.

Sain [56] studied norm attainment of linear transformations between Banach spaces and through this the authors were able to characterize smooth points in the space of completely continuous mappings. The relationship between $sBPBp$ and $\epsilon - BPB$ approximation was studied in relation to $BPBp$. This was emphasized in [57] whereby smoothness of bounded linear mappings was characterized in Banach spaces but in our study we shall consider $BPBp$ for finite rank operators.

Cascales *et al* [23] studied the set of norm achieving Lipschitz mappings that achieve their Lipschitz norms. It was showed that norm achievig Lipschitz mappings are dense in the space of Lipschitz mappings implying that they can be estimated by those that achieve their Lipschitz norms. In [58] Lipschitz functionals and mappings were characterized in terms of weak convergence but the study was limited to characterization of $BPBp$ for finite rank mappings.

Choi et al [17] characterized the closeness of norm achieving mappings i.e $BPBp$ for absolute addends between complete normed vector spaces. It was shown that for pair (Q, Z) of BS that satisties $BPBp$, (Q, Z_1) admits $BPBp$ provided that Z_1 is an absolute addend of Z of type 1 or ∞.

However, this result was extended to those mappings that are characterized by compactness and in this case it was shown that the property holds for such mappings hence, one could determine their closeness between Banach spaces. Later, the author considered the numerical radius of linear mappings and proved that the property holds for numerical radius for absolute addends of type 1 or ∞.

In addition, the authors gave comparable results for operators that achieve their numerical radius and for the $BPBp$ for numerical radius for completely continuous transformations. In [36] the authors characterized $BPBp$ for operators with respect to absolute summands on Banach spaces but in our study

we shall characterize numerical radius for finite rank operators via $BPBp$.
The result was extended to where $BPBp$ for linear mappings was also characterized with respect to absolute norms but the result was limited to finite rank mappings.
Dantas *et al* [27] studied $BPBpp$ and $BPBop$ for multilinear mappings and gave exampls of Banach spaces with such property. The relation between the local and uniform types of $BPBp$ was studied for multilinear mappings. The authors gave the definitions of n-linear mappings on Banach spaces and later the relationship between BPB type properties and subdifferentiability of the norm was investigated.
Bala et al [13] characterized BPB theorem for bounded analytic functions by showing that bounded analytic functions can be estimated by those that achieve their norms. It was further proved that the $BPBp$ holds for operator ideal of complete normed vector spaces of linear mappings.
In [12] it was shown that BPB theorem holds for bounded analytic functions but the result was limited to mappings with finite dimensional range. In [15] it was shown that BPB property holds for operator ideals of Banach spaces but in our study we shall consider BPB property for finite rank mappings.

2 Literature review

Lee [45] characterized denseness of Holomorphic functions that achieve their numerical radius on Banach spaces using Lindenstrauss method. Conditions were given for the denseness of Holomorphic functions and one of the conditions was that the numerical radius of Holomorphic functions can be estimated given that Banach spaces are characterized by uniform convexity. In Theorem 2.1 [45] it was discovered that the distance between holomorphic functions that achieve their numerical radii can be estimated but the result was limited to mappings with finite rank In Corollary 2.3 [45] it was proved that numerical radius achieving elements in the space of holomorphic functions can be estimated by those that achieve their numerical radii but the result was limited to mappings with finite dimensional range.

Corollary 2.1 *[[4], Corollary 2] Let G and H be separable Lebesgue spaces. Then the mappings that achieve their norms are dense in $B(G, H)$.*

Corollary 2.1 characterizes denseness of operators that attain norms in Lebesgue spaces but we shall consider denseness of mappings with finite dimensional range in sequence spaces.
Aviles, Guirao and Rodriguez [3] studied $BPBp$ for numerical radius for linear transformations on $C(K)$ spaces. The authors showed that $C(K)$ has $BPBp$ given K is metrizable. In Theorem 2.2 [3] it was proved the space $C(K)$

admits $BPBp$ provided K has local compensation but the study was limited to finite rank mappings. In Theorem 4.7 [3] $BPBp$ for numerical radius was characterized for operators on $C(K)$ spaces, in our study we shall consider other spaces like sequence spaces. Also in Corollary 4.8 [3] $BPBp$ for numerical radius was investigated for operators but the result was limited to finite rank operators.

Garcia *et al* [33] proved that compact operators can be estimated by those that achieve their norms. It was shown that $C_o(L)$ spaces satisfy $BPBp$-nu for completely continuous mappings. Approximation property was characterized on $C_o(L)$ spaces. Finally, it was shown that completely continuous mappings on real Hilbert spaces and on the sequence space l_1 admit $BPBp - nr$. From Lemma 2.1 [33] $BPB - nr$ was characterized for completely continuous mappings but the study was limited to mappings with finite dimension. Also in Corollary 2.3 [33] it was shown that $BPBp$ holds for compact mappings on $c_o(X)$ and $l^n_\infty(X)$ whenever X has a finite dimension but in our study we shall characterize the numerical radius of of finite rank mappings in relation to $BPBp$. Finally, in Corollary 2.6 [33] it was discussed completely continuous mappings satisfy $BPBp$ on BS provided that X^* is isometrically isomorphic but the result was limited to finite rank mappings.

Chakraborty [14] studied mappings that satisfy $BPBp$ for norm and numerical radius. The authors considered properties Lo, o and $BPBop$ and their weak versions on Banach spaces. Conditions were given on Banach spaces for the properties to hold, one of the conditions was that the properties hold provided that Banach spaces are reflexive and provided that they are uniformly convex. This was emphasized in the following results: In Theorem 2.3 [14] Lo, o was characterized for linear mappings but the study was limited to finite rank mappings. This was extended to Theorem 2.4 [14] whereby it was shown that reflexivity of Banach spaces implies that $BPBp$ holds for linear mappings but the study was limited to characterization of $BPBp$ for finite rank operators. In Theorem 4.3 [14] Lo, o was characterized for numerical radius for linear mappings but the study was limited to finite rank mappings. Finally, in Theorem 4.4 [14] the connection between the norm and the nr of linear mappings was characterized but not for finite rank mappings, hence in our study we shall extend this result to mappings with finite rank.

Lemma 2.2 *[[47], Lemma 2] Suppose W and Z are complete normed spaces where $W = c_o$ and Z is strictly convex. Then $NA(W, Z) \subseteq F(W, Z)$.*

In Lemma 2.2 it was shown that the norm achieving mappings in sequence spaces can be estimated by the ones that achieve their norms but the result was limited to mappings with finite dimensional range.

3 Preliminaries

This section provides basic concepts and definitions that will be useful throughout the study.

Definition 3.1 ([36]) *Let Q and W be Banach spaces and $P : Q \to W$ be a linear transformation. Then P is a* fro *if its range is of finite dimension.*

Definition 3.2 ([10]) *Let Q be a BS, then if $\epsilon > 0$ and $r^* \in Q^*$ there is $r^*_o \in Q^*$ such that $\|r^*_o(r_o)\| = \|r^*_o\|$ and $\|r^*_o - r^*\| < \epsilon$.*

Definition 3.3 ([6]) *A real normed Riesz space Ω has the Hereditary norm attaining property if for $(q, q^\star) \in S_\Omega \times S_{\Omega^\star}$, $q^\star(q) = \|q^\star\|\|q\| \Rightarrow q^\star(q^+) = \|q^{\star+}\|\|q^+\|$ and $q^{\star-}(q^-) = \|q^{\star-}\|\|q^-\|$.*

Definition 3.4 ([40]) *Let Ω be a Banach space, $w \in S_\Omega$ is said to be an extreme point of $L(\Omega)$ if $w = \gamma u + (1-\gamma)v$, for $u, v \in B(\Omega)$ and $\gamma \in (0,1)$ implying $u = v$. This is denoted as $w \in extL(\Omega)$.*

4 Main results

This section provides results for characterization of nr of fro in terms of the $BPBp$. We begin by examining the nr-attainment of fro.

Proposition 4.1 *If X is a BS, $a^*_o \in S_{X^*}$, $b_o \in extB_X$ and $V(a^*_o \otimes b_o) = 1$. Then, the operator $(a^*_o \otimes b_o)$ attains its numerical radius if $|a^*_o(b_o)| = 1$.*

Proof. Suppose that $a^*_o \otimes b_o$ attains its numerical radius, then $|a^*_o(b_o)| = 1$. By assumption there exist $(y, y^*) \in S_X \times S_{X^*}$ such that $1 = V(a^*_o \otimes b_o) = |y^*(a^*_o \otimes b_o)y| = y^*(b_o) \cdot a^*_o(y)$. So $|y^*(b_o)| = |a^*_o(y)| = 1$. Now take $\alpha \in \mathcal{K}$, $|\alpha| = 1$ with $y^*(\alpha b_o) = 1$. It follows that $\|\alpha b_o + y\| \geq |y^*(\alpha b_o + y)| = 2$. So $\|\alpha b_o + y\| = 2$. By assumption $y = \alpha b_o$ and hence $|a^*_o b_o| = |a^*_o(y)| = 1$.

Lemma 4.2 *Let X be a BS. Assume that a sequence $(u_m, u^*_m) \in S_X \times S_{X^*}$ exists such that $u_m \to u$ in norm for each $u \in X$, $|u^*_m Q u_m| \to V(Q)$. Then $Q \in NRA(X)$.*

Proof. Let u^* be any functional that can be approximated by $\{u^*_m\}$. If $u_m \to u$ in norm, then for each $u \in X$ we show that there exist $(u, u^*) \in S_X \times S_{X^*}$ such that Q attains its numerical radius. Therefore,

$$\begin{aligned} |1 - u^*(u)| = |u^*_m(u_m) - u^*(u)| &\leq |u^*_m(u_m) - u^*_m(u)| + |u^*_m(u) - u^*(u)| \\ &\leq |u^*_m||u_m - u| + |u^*_m(u) - u^*(u)| \end{aligned}$$

So $\|u^*_m\||u_m - u| = |u_m - u| \to 0$ and $|u^*_m(u) - u^*(u)| \to 0$ since $u^*_m \to u^*$ in weak $*$ topology.

We now consider $BPBp - nr$ for fro in reflexive spaces in the next result.

Theorem 4.3 *Every Reflexive Banach space X with Radon Riez property satisfies $BPBp - nr$ for finite rank operators.*

Proof. Assume that the property fails. Then given $\varepsilon \in (0,1)$ and $R \in F(X)$ with $V(R) = \|R\| = 1$, there is $(z_n, z_n^*) \in S_X \times S_{X^*}$ such that for each $n \in \mathcal{N}$

$$1 \geq |z_n^* Rzn| \geq 1 - \frac{1}{n} \tag{1}$$

and for every $(y, y^*) \in S_X \times S_{X^*}$ satisfying $\|y - z_n\| < \varepsilon$, $\|y^* - z_n^*\| < \varepsilon$ and $\|S - R\| < \varepsilon$ we have $|\langle y^* S(y)\rangle| < 1$. Since X is reflexive there exist a subsequence of z_n denoted again by z_n and $y_o \in B_X$ such that $z_n \to y_o$ and $Rz_n \to Ry_o$ in norm. From this and since $1 = V(R) = \|R\| \geq \|Rz_n\| \geq |\langle zn^* Rzn\rangle| \to 1$, we have $\|Ry_o\| = 1$ and hence $y_o \in S_X$.By Radon Riez Property we have $z_n \to y_o$ in norm and for all $n \in \mathcal{N}$ we have $1 \geq |\langle z_n^* Ry_o\rangle| \geq |z_n^* Ry_o| \geq |z_n^* Rz_n| - \|Rz_n - Ry_o\|$. Since $z_n \to y_o$ in norm and using equation 1 we have $|\langle z_n^* Ry_o\rangle| \to 1$. Thus, the subsequence z_n^* of z_n^* exists such that $|\langle z_n^* Ry_o\rangle| \to e^{i\Theta}$ for some $\theta \in [0, 2\pi)$. Define $R_o \in F(X)$ by $R_o = e^{-i\Theta} R$. Then $R_o y_o \in S_X$ and $\langle z_n^* R_o y_o\rangle \to 1$. By Smulian lemma there exist $y_o^* \in B_X^*$ for which $z_n^* \to y_o^*$. Since $(z_n^*, z_n) = 1$ for all $n \in \mathcal{N}$ we have $\langle y_o^*, y_o\rangle = 1$. So $y_o^* \in S_{X^*}$ and then $(y_o, y_o^*) \in S_X \times S_{X^*}$. Define $S = R_o + (1 - y_o^* R_o y_o) y \otimes y_o \in F(X)$. Then $y_o^* S y_o = 1$. Thus, from Lemma 4.2 $V(S) = 1$ and hence $\|S - R_o + R_o - R\| < \varepsilon$ which is a contradiction.

In the next result, we extend the above result to dual spaces.

Corollary 4.4 *Suppose X is reflexive, if X satisfies $BPBp - nr$ for finite rank operators so does X^*.*

Proof. Since X is reflexive, we show that $\eta_{X^*}(\varepsilon) = \eta_X(\varepsilon) > 0$. Let $P \in NRA(X) \cap F(X)$, then $P^* \in F(X^*)$ also achieves it nr. Let $\varepsilon > 0$ be given and $(v_o, v_o^{**}) \in S_{X^*} \times S_{X^{**}}$. By reflexivity there is $v_o \in S_X$ such that $v_o = v_o^{**}$. Assume $P^* \in F(X^*)$ and $(v_o^*, v_o) \in S_{X^*} \times S_{X^{**}}$ for which $V(P^*) = 1$, $|v_o P^* v_o^*| > 1 - \eta(\varepsilon)$. Given $P \in F(X)$, we can find $Q \in F(X)$ and $(z_o, z_o^*) \in S_X \times S_{X^*}$ in which $|z_o^* Q z_o| = V(Q) = 1$, $\|z_o - v_o\| < \varepsilon$, $\|z_o^* - v_o^*\| < \varepsilon$ and $\|Q - P\| < \varepsilon$. Then, from Theorem 4.3 for $Q^* \in F(X^*)$ and $(z_o^*, z_o) \in S_X \times S_{X^{**}}$ we have $|z_o Q^* z_o^*| = V(Q^*) = 1$, $\|z_o - v_o\| < \varepsilon$, $\|z_o^* - v_o^*\| < \varepsilon$ and $\|Q^* - P^*\| < \varepsilon$. Thus, X^* satisfies $BPBp - nr$ as desired.

As a consequence of Corollary 4.4 we have the following result.

Proposition 4.5 *Suppose X is reflexive then $\overline{NRA(X) \cap F(X)} = F(X)$ iff $\overline{NRA(X^*) \cap F(X^*)} = F(X^*)$.*

Proof. $\Rightarrow$ Assume that $\overline{NRA(X) \cap F(X)} = F(X)$, then $\overline{NRA(X^*) \cap F(X^*)} = F(X^*)$. Let $R^* \in F(X^*)$ and $\varepsilon > 0$. Since $R \in F(X)$ by assumption there exist $Q \in F(X)$ with $\|R - Q\| < \varepsilon$. Consider $(u_o, u_o^*) \in S_X \times S_{X^*}$ such

that $V(R) = u_o^* R u_o$. Define $u_o^{**} \in X^{**}$ as the canonical image of u_o. Then $\|u_o\| = \|u_o^{**}\| = 1$ and $(u_o^*, u_o^{**}) \in S_{X^*} \times S_{X^{**}}$. Since $V(R) = V(R^*)$ we have $|u_{o^*} R u_o| = |u_o^{**}(R^* u_o^*)|$ and thus $R^* \in F(X^*)$. From Corollary 4.4, it follows that $\|R - Q\| = \|R^* - Q^*\| < \varepsilon$. Hence $\overline{NRA(X^*) \cap F(X^*)} = F(X^*)$.
$\Leftarrow$ If $\overline{NRA(X^*) \cap F(X^*)} = F(X^*)$, then $\overline{NRA(X) \cap F(X)} = F(X)$. Let $R^* \in F(X^*)$ and $\varepsilon > 0$. Then by assumption there exist $Q^* \in F(X^*)$ with $\|R^* - Q^*\| < \varepsilon$. Since $V(R^*) = V(R^{**})$. It follows that $R^{**} \in F(X^{**})$ and $Q^{**} \in F(X^{**})$, $\|R^* - Q^*\| = \|R^{**} - Q^{**}\| = \|R - Q\| < \varepsilon$. Hence $\overline{NRA(X) \cap F(X)} = F(X)$.
In the following results, we give a characterization of nr of an operator showing that it equals to its norm.

Lemma 4.6 *Let $X = c_o^n$, then for each $Q \in F(X)$, $V(Q) = \|Q\|$*

Proof. We show that $V(Q) = \|Q\|$. The equality $V(Q) \leq \|Q\|$ always holds, thus we prove the reverse inequality. Let $\varepsilon > 0$ and $y_o \in S_X$ such that $\|Qy_o\| > \|Q\| - \varepsilon$. Since $Q \in F(X)$ consider $j \in \mathcal{N}$ such that $\|(Qy_o)_j\| = \|Qy_o\| > \|Q\| - \varepsilon$. Define α_t^* as $\alpha_t^* = sgn((Qy_o)_j) y_j$ for $y \in c_o$. Then $\alpha_{t^*} \in S_{X^*}$ and from Lemma 4.2 it follows that, $V(Q) = \alpha_{t^*} Q y_o = \|Qy_o\| > \|Q\| - \varepsilon$. Since ε is arbitrary $V(Q) \geq \|Q\|$. Therefore, $V(Q) = \|Q\|$.

Theorem 4.7 *Suppose that $X = l_1^n$. Then for every $R \in F(X)$, $V(R) = \|R\|$*

Proof. Let $R \in F(X)$ and $\varepsilon > 0$ be given. Then from Proposition 4.5, there is $S \in F(X)$ such that $\|R - S\| < \varepsilon$ and $\|S\| = \|Sz_o\|$ for some $z_o \in S_X$. Since $Rz_o \in F(X)$, there exist $z_o^* \in S_{X^*}$ such that $z_o^*(Rz_o) = \|Rz_o\|$. But $sgn(z_o, z_o^*) = \|z_o^*\| = \|z_o\| = 1$. Then $V(R) \geq sgn|\langle z_o^* R z_o \rangle| = \|Rz_o\|$. Indeed $V(R) \geq sgn|z_o^* R z_o| = \|Rz_o\| \geq \|R\| - \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $V(R) = \|R\|$.

Corollary 4.8 *Let $X = l_1$ and $T \in F(X)$, then $V(T) \geq sup\{\|T_{n_j}\| : j \in \mathcal{N}\}$*

Proof. Let $T \in F(X)$ and $\varepsilon > 0$ be given. Let $s = sup\{\|Tn_j\|_{j \in \mathcal{N}}\}$. Then, there exist j_o for which $\|Tn_{j_o}\| \geq s - \varepsilon$. Define $z_o^* \in l_\infty$ by $z_o^* = (sgn(n_i^* \circ Tn_{j_o}))$ for $i \in \mathcal{N}$. Then, from Theorem 4.7, $\|z_o^*\| = 1$, and $V(T) = |z_o^* T(n_{(j_o)}| = \|Tn_{j_o}\| > s - \varepsilon$. Since ε is arbitrary $V(T) \geq s$.

Proposition 4.9 *Every finite dimensional Banach space satisfies $BPBp-nr$ for fro.*

Proof. Consider the set $G = \{Q \in F(X) : V(Q) = 0\}$. Since X if of finite dimension, $\frac{F(X)}{G}$ is also of finite dimension with norms

$$\begin{aligned} \|[R]\| &= inf\{\|R - Q\|\} : Q \in G \\ V([R]) &= inf\{V(R - Q) : Q \in G = V(R)\}. \end{aligned}$$

Hence, $K\|[R]\| \leq V(R) = \|R\|$ for a constant $K > 0$. Now assume that $BPBp - nr$ fails to hold in $F(X)$. Then, there exist $\varepsilon > 0$ and $R_n \in F(X)$ with $V(R_n) = 1$ and $(y_n, y_{n^*}) \in S_X \times S_{X^*}$ for which $|y_n^* R_n y_n| > 1 - \eta(\varepsilon)$, but for each $S \in F(X)$ with $V(S) = 1$ and $(y, y^*) \in S_X \times S_{X^*}$, $\max\{\|R_n - S\|, \|y_n - y\|, \|y_n^* - y^*\|\} \geq \varepsilon$. By compactness assume that $R_n \to R_o$ in norm for each $R_o \in F(X)$ and $V(R_o) = 1$. Then, there exist a sequence $\{Q_n\}_n \in G$ for which $R_n \to (R_o + Q_n)$ in norm. Again by compactness assume that $(y_n, y_n^*) \to (y_o, y_o^*) \in X \times X^*$. Then $(y_o, y_o^*) \in S_X \times S_{X^*}$ and $|y_o^*(R_o + Q_n)y_o| = V(R_o + Q_n) = 1$. Define $U_n = R_o + Q_n$. Then from Theorem 4.3, $|y_o^*(R_o + Q_n)y_o| = |y_o^* U_n y_o| = 1$, $\|R_n - U_n\| = \|R_n - (R_o + Q_n)\| \to 0$ as $y_n \to y_o$ and $y_n^* \to y_o^*$. Thus, $\max\{\|R_n - U_n\|, \|y_n - y_o\|, \|y_n^* - y_o^*\|\} \leq \varepsilon$ which is a contradiction.

In the above Lemma, we have shown that $BPBp - nr$ holds in Banach spaces with finite dimension. As a consequence of this lemma, the following results follow.

Lemma 4.10 *Let $X = c_o^n$. If X satisfies $BPBp - nu$, then $\overline{NRA(X)} = F(X)$.*

Proof. Suppose $R \in F(X)$ with $\|R\| = 1$ and let $\varepsilon > 0$ be given. Since X has property β, then from Theorem 4.3, there exist $S \in F(X)$ such that $\|R - S\| < \varepsilon$ and $\|S\| = \|Sz_o|$ for some $z_o \in S_{X_i^*}$ where $X_i \subset X$. Let $X_i \subset X$, $R_o = R|_{X_i}$. Then $R_o \in F(X_i, X)$. Define $V(R_o) = sup\{|z_o^* R_o z_o|\}$ with $z_o \in S_X$ and $z_o^* \in S_{X^*}$, $z_o^*(z_o) = 1$. Since X_i is finite dimensional, R_o attains its numerical radius but $\|R\| = V(R)$ by Theorem 4.7, $V(R_o) \leq \|R_o\| = \|R\| = V(R)$. Thus if we prove that $V(R_o) = \|R_o\|$, then R attains its numerical radius. Let $R_o = z_o^* R_o$ then $\|R_o z_o\| = \|R_o\|$. From Theorem 4.7, $V(R_o) \geq sgn|z_o^* R_o z_o| = |z_o^* R_o z_o| = \|R_o z_o\| = \|R_o\|$. Hence R attains its numerical radius.

Corollary 4.11 *Let $X = l_1^n$, then $\overline{NRA(X)} = F(X)$.*

Proof. Let $Q \in F(X)$ with $\|Q\| = 1$ and let $\varepsilon > 0$ be given. Let $r_o \in S_X$ such that $\|Qr_o\| > \|Q\| - \varepsilon$. Suppose $P \in F(X)$. Then from Lemma 4.10, $\|Q - P\| < \varepsilon$ and $\|P\| = \|Pr_o\|$. Let $r_o^* \in S_X$, then $\|r_o^*\| = 1$ and $V(P) = |r_o^* P r_o| = \|P(r_o)\| = \|P\|$. Hence, $P \in NRA(X)$.

In the next result, we extend the $BPBp - nr$ for fro to measure spaces.

Theorem 4.12 *For every measure μ, the space $L_1(\mu)$ satisfies $BPBp-nr$ for finite rank operators. In particular, given $\varepsilon > 0$ and $\eta(\varepsilon) > 0$, for every $R \in F(L_1(\mu))$ with $V(R) = 1$ ad $(u_t, u_t^*) \in S_{L_1(\mu)} \times S_{L_\infty(\mu)}$ satisfying $|\langle u_t^*, Ru_t\rangle| > 1-\eta(\varepsilon)$ there exist $S \in F(L_1(\mu))$ for which $|\langle v_t^*, Sv_t\rangle| = V(S) = 1$, $\|v_t-u_t\| < \varepsilon$ and $\|S - R\| < \varepsilon$.*

Proof. Let $R \in F(L_1(\mu))$ with $V(R) = 1$ and $|\langle u_{t^*} Ru_t\rangle| > 1 - \eta(\varepsilon)$. Let $Y = Span\chi i_{Ai}...\chi i_{Ai} \subset L_1(\mu)$. Then $u_o = \sum ai\chi Ai \in Y$. Take $u_o \in Y$ such that $\|u_o - u_t\| < \varepsilon$. Then, Y is isometrically isomorphic to l_1^n. Consider $R_o = R|_Y \in Y$, $u_o = u_t|_Y \in S_Y$, $u_o^* = u_t^* \in S_{Y^*}$ such that $(u_o, u_o^*) \in S_Y \times S_Y^*$ and $|\langle u_o^* R_o u_o\rangle| = \langle u_t^* Ru_t\rangle| > 1 - \eta(\varepsilon)$. Then, from Proposition 4.9 there exist $S_o \in F(Y)$ and $(v_o, v_o^*) \in S_Y \times S_Y^*$ such that $|v_o^* S_o v_o| = V(S_o), \|S_o - R_o\| < \varepsilon$, $\|v_o - u_o\| < \varepsilon$ and $\|v_o^* - u_o^*\| < \varepsilon$ Define S on $L_1(\mu) = Y \oplus W$ by $S(y, w) = (S_o y, 0) + R(0, w)$ $(y, w) \in L_1(\mu)$. Then $\|S\| = V(S) = 1$. Clearly, $\|S(y, w)\| = \|(S_o y, 0)\| + \|R(0, w)\| \leq \|y\| + \|w\| = \|y, w\|$ for every $(y, w) \in L_1(\mu)$. $\|S_o(v_o, 0)\| = \|(S_o v_o, 0)\| = \|S_o v_o\| = 1$. Let $v_t = (v_o, 0)$ and $v_t^* = (v_o^*, u_t|_W)$. Then $(v_t, v_t^*) \in S_{L_1(\mu)} \times S_{L_\infty(\mu)}$. In addition we have $\|v_t^* Sv_t\| = \|v_o^* S_o v_o\| = 1 = V(S)$, $\|v_t^* - u_t^*\| = \|v_o^* - u_o^*\| < \varepsilon$, $\|v_t - u_t\| = \|v_o - u_o\| < \varepsilon$ and $\|S - R\| = sup_{\|y\|+\|w\|\leq 1}\|S(y, w) - R(y, w)\| = sup_{\|y\|\leq 1}\|S_o y - R_o y\| = \|S_o - R_o\| < \varepsilon$.
We now discuss the $BPBp - nr$ for fro in microtranstive spaces.

Proposition 4.13 *Suppose X is microtransitive and $\eta_F(X) > 0$. Then X satisfies $BPBp - nr$ for fro.*

Proof. Define $K(X) = \{G \in F(X) : V(G) = 0\}$. Suppose that X is microtranstive. Then, from [**??**] we have that X satisfies $BPBpp - nr$. That is, for $P \in L(X)$ with $V(P) = 1$ and given $(r_o, r_o^*) \in S_X \times S_{X^*}$ such that $|r_o^* P(ro)| = |r_o^*(P + G)r_o| > 1 - \eta(\varepsilon)$ for every $G \in K(X)$. Then, there exist $Q_1 \in L(X)$ with $V(Q_1) = 1$ for which $|r_o^* Q_1(r_o)| = 1$ and $\|Q_1 - (P + G)\| < \varepsilon$. Now, since K is microtranstive, for $r_o, z_o \in S_X$ there is a linear isometry $U \in F(X)$ such that $Ur_o = z_o$ and $\|U - Id_X\| < \varepsilon$, $r_o^*(r_o) = 1$, $z_{o^*}(z_o) = (U^{-1})^* r_o^*(Ur_o) = r_o^* r_o = 1$. Define $Q = U^{-1}(Q_1)U$. From Theorem 4.12, it follows that $z_{o^*} Qz_o = r_o^*(Q_1)r_o = V(Q) = V(Q_1) = 1$. Thus Q attains its norm at (z_o, z_o^*) and $(z_o, z_o^*) \in S_X \times S_{X^*}$. Thus $\|Q_1 - (P + G)\| = \|Q - P\| < \varepsilon$ and $\|(z_o, r_o)(z_o^*, r_o^*)\| < \varepsilon$.
As a consequence of the above result, we present the following result we gives equivalent conditions on denseness of fro.

Lemma 4.14 *For a BS X, the following assertions are equivalent:*

(i). The set $NRA(X) \cap F(X)$ is dense in $F(X)$

(ii). If $Q \in F(X)$ with $V(Q) = 1$, there is a sequence $\{Q_n\} \subset NRA(X) \cap F(X)$.

(iii). If $Q \in F(X)$ with $V(Q) = 1$ and each $\varepsilon > 0$, there is $P \in F(X)$ with $V(P) = 1$ such $\|P - Q\| < \varepsilon$.

Proof. $(a) \Rightarrow (b)$ Assume that $NRA(X)$ is dense in $F(X)$. Let $Q \in F(X)$ with $V(Q) = 1$. By assumption a sequence $\{R_n\} \subset NRA(X)$ exists for which $v(R_n) = 1$, $R_n \to Q$ in norm. Since $NRA(X)$ is dense in $F(X)$, from Lemma 4.10, $R_n \in NRA(X)$ such that $R_n \to Q$ in norm. Define Q_n as $Q_n = \frac{1}{V(R_n)}V(R_n)$. Then $V(Q_n) = 1$. Since $R_n \to Q$ and $V(R_n) \to V(Q) = 1$, we have $Q_n \to Q$ in norm. This proves (b).
$(b) \Rightarrow (c)$ Assume for each $Q \in F(X)$ with $V(Q) = 1$, there is a sequence $Q_n \in NRA(X)$ with $V(Q_n) = 1$ such that $\|Q_n - Q\| < \varepsilon$, from $Q_n \to Q$, there is $N \in \mathcal{N}$ such that $\|Q_N - Q\| < \varepsilon$, since $V(Q_N) = 1$ and $Q_N \in NRA(X)$. We let $P = Q_N$ and this proves (c).
$(c) \Rightarrow (a)$ Suppose $Q \in F(X)$. If $V(Q) = 0$, then Q achieves its nr which completes the proof. If $V(Q) \neq 0$, we consider the operator $R = \frac{Q}{V(Q)}$ for which $V(R) = 1$. By assumption for any $\varepsilon > 0$, there exist $Q_n \in NRA(X)$ such that $V(Q_n) = 1$ and $P_n \to R$ in norm for each $n \in \mathcal{N}$. This gives that $V(Q)Q_n \to V(Q)R = Q$. Hence, $V(Q)Q_n \in NRA(X)$ for each $n \in \mathcal{N}$.
We now verify the validity of $BPBp - nr$ for fro in HS.

Theorem 4.15 *The complex Hilbert space H satisfies $BPBp-nr$ for finite rank operators.*

Proof. Let $\varepsilon > 0$ be given and let $\eta(\varepsilon) > 0$. Assume that H satisfies $BPBp-nu$, then $R \in F(H)$ with $V(R) = 1$ and $(x_t, x_t^*) \in S_H \times S_{H^*}$ such that $|\langle x_{t^*} R x_t \rangle| > 1 - \eta(\varepsilon)$, there exist $Q_o \in F(H)$ with $V(Q_o) = 1$ and $(y_t, y_{t^*}) \in S_H \times S_{H^*}$ $|y_t^* Q_o y_t| = 1$, $\|y_t^* - x_t^*\| < \varepsilon$, $\|y_t - x_t\| < \varepsilon$ and $\|Q_o - R\| < \varepsilon$. Let $Q = Q_o \in F(H)$ by hypothesis $Q \in F(H)$ and $\|Q_o\| \leq 2V(Q) = 2$. Since H is a HS, it follows that from Theorem 4.12 that, $|\langle y_t^* Q y_t \rangle| = 1 = V(Q) = V(Q_o)$, $\|Q_o - R\| \leq \|Q - R\| < \varepsilon$ and $\|y_t^* - x_t^*\| < \varepsilon$, and $\|y_t - x_t\| < \varepsilon$.

Corollary 4.16 *Assume that X satisfies $BPBp - nr$ for finite rank operators with $\eta_F(X) = 1$. Then X is uniformly convex.*

Proof. Take $\varepsilon \in (0, 1)$ and let $\eta = \eta(\frac{\varepsilon}{2})$ be given. To show that X is uniformly convex, we proof that for every $x_t \in S_X$ and $x_t^* \in S_X^*$ with $|x_t^*(x_t)| > 1 - \eta$ there exist $z_t^* \in S_X^*$ for which $z_t^*(x_t) = 1$ and $\|z_t^* - x_t^*\| < \varepsilon$. Let $(x_t, x_t^*) \in S_X \times S_X^*$ such that $|x_t^*(x_t)| > 1 - \eta(\varepsilon)$. Define $G \in F(X)$ by $G = x_t^* \otimes x_t$, $G(x) = x_t^*(x) x_t$. Since $\eta_F(X) = 1$, then $V(G) = \|G\| = 1$. Also $|x_{t^*} G(x_t)| = |x_t^*(x_t)| > 1 - \eta$. Since X has $BPBp - nr$ then, from Theorem 4.12 there exist $H \in F(X)$ with $V(H) = \|H\| = 1$ and $(y_t, y_t^*) \in S_X \times S_X^*$ such that $|y_t^* H y_t| = 1$, $\|G - H\| < \frac{\varepsilon}{2}$, $\|y_t - x_t\| < \varepsilon$, and $\|y_t^* - x_t^*\| < \varepsilon$,. Define $z_t^* = H^* y_t^* \in X^*$ then $\|z_t^*\| \leq 1$

and $\|z_t^*(x_t)\| = 1$. Therefore $\|z_t^*\| = 1$. Moreover, we have that $\|z_t^* - x_t^*\| < \varepsilon$. Indeed for every $x \in S_X$,

$$\begin{aligned} |z_t^*(x) - x_t^*(x)| &= |y_t^*(Hx) - x_t^*(x)x_t^*(x_t)| \\ &\leq |y_t^*(Hx) - y_t^*(Gx)| + |y_t^*(Gx) - x_t^*(Gx)| \\ &\leq \|H - G\| + \|y_t^* - x_t^*\|\|G\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $\|z_t^* - x_t^*\| < \varepsilon|$ and hence X is uniformly smooth.
In the above result we have given a necessary condition for Banach spaces to satisfy $BPBp - nr$. We now present a result that compliments the above result.

Proposition 4.17 *Suppose that X is a uniformly smooth and uniformly convex BS. Let $\varepsilon > 0$ and $\eta(\varepsilon) > 0$ such that for $R_o \in F(X)$ with $V(R_o) = 1$ and $(u_t, u_t^*) \in S_X \times S_{X^*}$ satisfying $|u_t^* R_o u_t| > 1 - \eta(\varepsilon)$ there exist $Q \in F(X)$ and $(v_t, v_t^*) \in S_X \times S_{X^*}$ for which $V(Q) = |v_t^* Q v_t|$, $\|v_t - u_t\| < \varepsilon$, $\|v_t^* - u_t^*\| < \varepsilon$, and $\|Q - R_o\| < \varepsilon$.*

Proof. Let $\delta_X(\varepsilon)$ and $\delta_{X^*}(\varepsilon)$ be the moduli of convexity of X and X^* respectively. For $\varepsilon \in (0,1)$ consider $\eta(\varepsilon) = \frac{\varepsilon}{4} min\{\delta_X(\frac{\varepsilon}{4}), \delta_{X^*}(\frac{\varepsilon}{4})\} > 0$. Let $R_o \in F(X)$ with $V(R_o) = 1$ and $(u_t, u_t^*) \in S_X \times S_{X^*}$ for which $|u_t^* R_o u_t| > 1 - \eta(\varepsilon)$. Define $R_1 \in F(X)$ by $R_1 x = R_o x + \alpha \frac{\varepsilon}{4} u_t^*(x) u_t$ for each $x \in X$ where α is a scalar with $|\alpha| = 1$. Take $u_i \in S_X$ and $u_i^* \in S_X^*$ such that $|u_i^*(u_i)| = 1$, $u_i^*(u_t) = |u_i^*(u_t)|$ and $|u_i R_1 u_i| \geq V(R_1) - \eta(\frac{\varepsilon^2}{4^2})$. Now define a sequence $(u_n, u_n^* R_n) \in S_X \times S_{X^*} \times F(X)$ inductively. Let $R_{n+1}(x) = R_n x + \alpha_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} u_n^*(x) u_n$. Then choose $u_{n+1} \in S_X$ and $u_{n+1}^* \in S_{X^*}$ such that $|u_{n+1}^*(u_{n+1})| = 1$ and $u_{n+1}^*(u_n) = |u_{n+1}^*(u_n)|$ $|u_{n+1}^* R_{n+1} u_{n+1}| \geq V(R_{n+1}) - \eta(\frac{\varepsilon^{n+2}}{4^{n+1}})$. Notice that from Proposition 4.13 for all $n > 0$ we have $\|R_{n+1} - R_n\| \leq \frac{\varepsilon^{n+1}}{4^{n+1}}$ and $|V(R_{n+1}) - v(R_n)| \leq \frac{\varepsilon^{n+1}}{4^{n+1}}$. Thus R_n is a cauchy sequence and assume that it converges to $Q \in F(H)$. Then we have $lim_n R_n = Q$, $\|R_o - Q\| < \varepsilon$ and $lim_n |u_n^* R_n u_n| = lim_n V(R_n) = V(Q)$. We now show that both sequences u_n and u_n^* are cauchy. Since X is uniformly convex, we have $\|\frac{u_n + u_{n+1}}{2}\| \geq 1 - \delta_X(\frac{\varepsilon^{n+1}}{4^{n+2}})$ and $\|\frac{u_n^* + u_{n+1}^*}{2}\| \geq 1 - \delta_X(\frac{\varepsilon^{n+1}}{4^{n+2}})$. Thus, $\|u_n - u_{n+1}\| \leq \frac{\varepsilon^{n+1}}{4^{n+2}}$ and $\|u_n^* - u_{n+1}^*\| \leq \frac{\varepsilon^{n+1}}{4^{n+2}}$ for all n. So u_n and u_n^* are cauchy. Let $u_\infty = lim_n u_n$ and $u_\infty^* = lim_n u_n^*$. Then, we have $\|u_t - u_\infty\| < \frac{\varepsilon}{4}$ and $\|u_t^* - u_\infty^*\| < \frac{\varepsilon}{4}$. Hence, $|u_\infty^*(u_\infty)| = lim_n |u_n^* R_n u_n| = |u_\infty^* Q u_\infty|$. Let $v_t = u_\infty$ and $v_t^* = u_\infty^*$ then we have $v_t^*(v_t) = 1$, $V(Q) = |v_t^* Q v_t|$ and $\|v_t - u_t\| < \varepsilon$ and $\|v_t^* - u_t^*\| < \varepsilon$.
In the next result we give the connection between weak $BPBP - nr$ and $BPBp - nr$.

Lemma 4.18 *Suppose X is a BS with $\eta_F(X) > 0$. If X satisfies weak $BPBp - nr$ for finite rank operators, then X has $BPBp - nr$.*

Proof. Assume that X satisfies weak $BPBp-nr$, then for $P \in F(X)$ with $V(P)=1$ and $(k_t,k_t^*) \in S_X \times S_X^*$ for which $|k_t^*Pk_t| > 1-\eta(\varepsilon)$, there is $Q \in F(X)$ and $(u_t,u_t^*) \in S_X \times S_X^*$ such that $V(Q)=|u_t^*Qu_t|$, $\|Q-P\| < \varepsilon$, $\|u_t-k_t\| < \varepsilon$ and $\|u_t^*-k_t^*\| < \varepsilon$.We deduce that $1-V(Q) \leq V(P)-V(Q) \leq V(Q-P) \leq \|Q-P\| < \varepsilon$. So $V(Q) > 1-\varepsilon > 0$. Define $Q_o = \frac{Q}{V(Q)}$. Then we have $V(Q_o)=1$, $\|u_t^*Q_ou_t\|=1$, $\|u_t-k_t\| < \varepsilon$ and $\|u_t^*-k_t^*\| < \varepsilon$.It follows that

$$\begin{aligned}\|Q_o-P\| &\leq \|Q_o-Q\|+\|Q-P\| \\ &\leq \|\frac{Q}{V(Q)}-Q\|+\|Q-P\| \\ &\leq \|(\frac{1}{V(Q)}-1)\|\|Q\|+\|Q-P\| \\ &\leq (\frac{1-V(Q)}{V(Q)})\|Q\|+\|Q-P\|.\end{aligned}$$

Since $\eta(X)\|Q\| \leq V(Q)=\|Q\|$, $\|Q\|=\frac{V(R)}{\eta(X)}$. So,

$$\begin{aligned}\|Q_o-P\| &\leq \frac{1-V(Q)}{V(Q)}.\frac{V(Q)}{\eta(X)}+\varepsilon \\ &\leq \frac{1-V(Q)}{\eta(X)}+\varepsilon \\ &\leq \frac{\varepsilon}{\eta(X)}+\varepsilon \leq (\frac{1}{\eta(X)}+1)\varepsilon = (\frac{\eta(X)}{\eta(X)}+1)\varepsilon.\end{aligned}$$

A trivial change of operators completes the proof.

Theorem 4.19 *Let* $X=[\oplus_{K=1}^{\infty}]_\infty$. *If* X *has* $BPBp-nr$ *for finite rank operators, then so does* X_i *for all* $i \in \mathcal{N}$.

Proof. Let $F_i : X \to X_i$ and $F_i^* : X^* \to X_i^*$ be the canonocal projection and let $E_i : X_i \to X$ and $E_i^* : X_i^* \to X^*$ be the natural embeddings. Suppose that the operator $R_i \in F(X_i)$ and $(u_i,u_i^*) \in S_{X_i} \times S_{X_i^*}$. Define $R \in L(X)$ and $(u,u^*) \in S_X \times S_{X^*}$ by $R=E_i \circ R_i \circ F_i$ and $(u,u^*)=(E_iu_i,E_i^*u_i^*)$. Then , we have $|u^*Ru|=|u_iR_iu_i| > 1-\eta(\varepsilon)$. By hypothesis, there exist $Q \in F(X)$ and $(n,n^*) \in S_X \times S_{X^*}$ for which $|n^*Qn|=1=V(Q)$, $\|Q-R\| < \varepsilon$, $\|n-u\| < \varepsilon$ and $\|n^*-u^*\| < \varepsilon$. Define $Q_i=F_i \circ Q \circ E_i$ and $n_i=F_in$ and $n_i^*=F_{i^*}n^*$. Then, from Lemma 4.18 it follows that $\|Q_i-R_i\| < \varepsilon$, $\|n_i-u_i\| < \varepsilon$ and $\|n_{i^*}-u_i^*\| < \varepsilon$. Now we show that $|F_i^*n^*(F_i \circ Q \circ E_i)F_in|=1$. That is $n_i^*Q_in_i=1$. Since

$$\begin{aligned}1=n^*n &= \sum_{j\in\mathcal{N}} F_j^*n^*(F_jn) \leq \sum_{j\in\mathcal{N}} \|F_j^*n^*\|\|F_jn\| \\ &\leq \|F_i^*n^*\|+\varepsilon \sum_{j\in\mathcal{N},j\neq i} \|F_j^*n^*\| \leq \|n^*\|=1.\end{aligned}$$

This shows that $\|F_i^* n^*\| = 1$ and $\|F_j^* n^*\| = 0$ for every $j \neq i$ so $n^* = E_i^* F_i^* n^*$ and $F_i^* n^*(F_i n) = 1$. Then $\|n - E_i F_i n\| < \varepsilon$. Thus, $|\langle (E_i F_i n) + \frac{1}{\varepsilon}(n - E_i F_i n), E_i^* F_i^* n^* \rangle| = 1$. Therefore, $(E_i^* F_i^* n^*) Q(E_i F_i n + \frac{1}{\varepsilon}(n - E_i F_i n)) \leq V(Q) = 1$. Hence,

$$\begin{aligned} 1 &= |n^* Q n| = |(E_i^* F_i^* n^*) Q_n| \\ &= (1-\varepsilon) E_i^* F_i^* n^* Q(E_i F_i n) + \sum (E_i^* F_i^* n^* Q E_i F_i n + \frac{1}{\varepsilon}(n - E_i F_i n), \end{aligned}$$

and so we have $|F_i^* n^*(F_i \circ Q \circ E_i) F_i n| = |E_{i^*} F_i^* n^* Q(E_i F_i n)| = 1$.
In the above result we have established how $BPBp - nr$ can be extended from a direct sum space to its component space. We finish this section with the following result which gives a sufficient condition under which $BPBp - nr$ holds.

Corollary 4.20 *Let X be a BS and $F \subset S_X$ be dense in B_X. If $P \in F(X)$ and $\varepsilon > 0$, then $V(P) = sup\{x_o(Px_o)\} : (x_o, x_o^*) \in S_X \times S_{X^*}$, there exist $f_o \in F$ with $\|x_o - f_o\| < \varepsilon$. Moreover, if $P \in F(X)$ with $V(P) = 1$, then $BPBp - nr$ holds.*

Proof. By definition, since $F \subset S_X$ is dense in B_X, for every $\varepsilon > 0$ we have $V(P) = sup\{x_o(Px_o)\} : (x_o, x_o^*) \in S_X \times S_{X^*}$ with $\|x_o - f_o\| < \varepsilon$ for some $f_o \in F$. Now suppose $S \in F(X)$ with $V(S) = 1$. From Theorem 4.19, given $\varepsilon > 0$, there exist $(f_o, f_o^*) \in S_X \times S_{X^*}$ for which $|f_o^*(Sf_o)| = 1$, $\|S - P\| < \varepsilon$, $\|x_o - f_o\| < \varepsilon$ and $\|x_o^* - f_o^*\| < \varepsilon\|$ whenever $x_o^*(Px_o)$ is sufficiently close to 1.

5 Open Problems

In this work, we have established Bishop-Phelps-Bollobas Property for finite rank operators between Banach spaces. We proved that $BPBp$ for fro holds in several settings including when a Banach space X is of finite dimension or uniformly convex. We have also extended these results and show that this property also holds on Banach spaces with geometrical properties. Moreover, we characterized the numerical radius of fro via the $BPBp$. We establish the extent to which fro satisfy $BPBp$ with respect to nr. We showed that this property holds in BS settings which include when X is reflexive. This leads to natural questions which we state in the following form of two problem as follows. **Problem1 :** Do these results hold in noncommutatice Banach algebras? **Problem 2:** What are their possible applications in convex optimization?

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