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Introducing a new tangent distribution

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Abstract

To date, only a limited number of basic unit distributions based on the tangent function have been considered. In this article, we expand this range by introducing a new candidate distribution and examining its key probabilistic properties, such as the cumulative distribution function, probability density function, quantile function, and moments. Particular emphasis is placed on analytical tractability. While potential applications fall outside the scope of this study, they could be investigated by specialists in relevant fields. Additionally, we highlight an open problem regarding the rigorous derivation of the second-order moment, which could stimulate further theoretical research.

Keywords: *Unit distributions, Cumulative distribution function, Probability density function, Quantile function, Moments, Open problem.*
2010 Mathematics Subject Classification: 62E99.

1 Introduction

Unit (continuous probability) distributions are essential for modelling variables that are constrained to the interval $(0, 1)$, such as proportions and probabilities. In recent years, numerous unit distributions have been proposed. The work [3] provides a comprehensive and up-to-date review of over one hundred unit distributions. This review serves as a valuable reference for further research in this area, including the research presented in this article.

Some unit distributions are defined using trigonometric functions. In particular, the standard tangent distribution, say of type I, was introduced in [4]. It is defined by the following cumulative distribution function:

$$F_0(x) = \tan\left(\frac{\pi}{4}x\right), \quad x \in (0,1),$$

and is completed by setting $F_0(x) = 0$ for any $x \leq 0$ and $F_0(x) = 1$ for any $x \geq 1$. We recall that $\tan(x) = \sin(x)/\cos(x)$. In [4], the type I tangent distribution forms the basis of a more general family of tangent distributions. Its strong performance in statistical applications underscores its significance. Another tangent distribution, say of type II, was proposed in [1]. It is defined by the following cumulative distribution function:

$$F_{00}(x) = 1 - \tan\left(\frac{\pi}{4}(1-x)\right), \quad x \in (0,1),$$

and is completed by setting $F_{00}(x) = 0$ for any $x \leq 0$ and $F_{00}(x) = 1$ for any $x \geq 1$. In [1], the type II tangent distribution forms the basis for a more general family of lifetime distributions, including a tangent modification of the Weibull distribution. This has been validated through extensive statistical applications.

To the best of our knowledge, these are the only two fundamental unit distributions constructed using the tangent function. In this article, we introduce a new candidate within this class and examine its core probabilistic properties, including the cumulative distribution function, probability density function, quantile function, and moments. Potential applications are left for future exploration by specialists in relevant fields. Notably, the study also reveals an open problem concerning the rigorous derivation of the second-order moment, which may encourage further theoretical investigation.

The remainder of the article is organized as follows: The new candidate tangent distribution is introduced and studied in Section 2. Section 3 presents an open problem. Concluding remarks are provided in Section 4.

2 New tangent distribution

2.1 Definition

The theorem below constitutes the central result of this article, introducing an original cumulative distribution function for a distribution supported on $(0,1)$ and based on the tangent function.

Theorem 2.1 *The following function is a valid cumulative distribution function of a distribution with support* $(0,1)$*:*

$$F(x) = \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)\right), \quad x \in (0,1),$$

and is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Proof. The function F is clearly continuous on $\mathbb{R}\backslash\{0,1\}$. Let us study its continuity at 0 and 1. We have

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)\right)$$
$$= \frac{1}{2}\left(1 - \tan\left(\frac{\pi}{4}\right)\right) = 0 = F(0)$$

and

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)\right)$$
$$= \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{4}\right)\right) = 1 = F(1).$$

Therefore, F is continuous on $\mathbb{R}$. Let us prove that F is increasing. Using standard derivation manipulations, we get

$$F'(x) = \frac{\pi}{2}\left(\frac{1}{1 + \sin(\pi x)}\right).$$

Since $\sin(\pi x) \in (-1, 1)$ for any $x \in (0, 1)$, it is evident that $F'(x) \geq 0$, implying that F is increasing. Based on this, it is also obvious that $F(x) \in [0, 1]$ for any $x \in \mathbb{R}$. Therefore, F is a valid cumulative distribution function. This concludes the proof. □

We refer to the distribution defined by the cumulative distribution function F in Theorem 2.1 as the new tangent (NT) distribution. From a functional perspective, it differs substantially from the type I and type II tangent distributions described in the introduction. As a mathematical remark, another possible expression of the cumulative distribution function F is as follows:

$$F(x) = \frac{1}{2}\left(1 + \frac{\sin\left(\pi(x - 1/2)\right)}{1 + \cos\left(\pi(x - 1/2)\right)}\right), \quad x \in (0, 1),$$

and is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

For illustrative purposes, Figure 1 presents a plot of F.

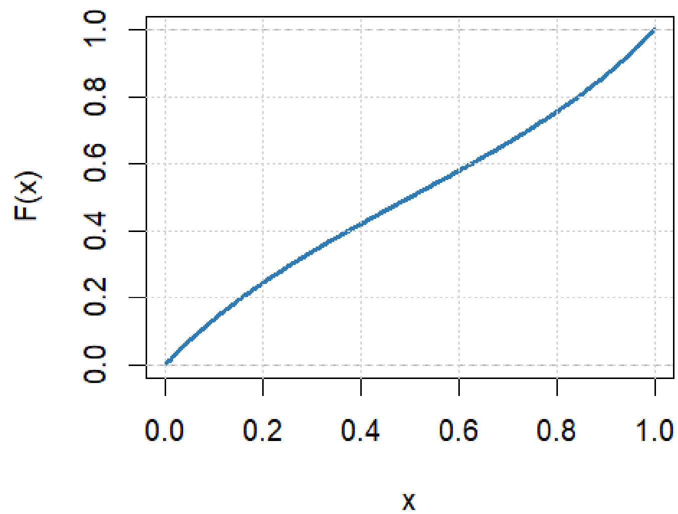

Figure 1: Plot of the cumulative distribution function of the NT distribution.

The probability density function associated with the NT distribution is specified in the theorem below.

Theorem 2.2 *The probability density function associated with the NT distribution is given by*

$$f(x) = \frac{\pi}{2}\left(\frac{1}{1+\sin(\pi x)}\right), \quad x \in (0,1),$$

and is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

Proof. Based on the cumulative distribution function F defined in Theorem 2.1, using standard derivation manipulations, we get

$$f(x) = F'(x) = \frac{\pi}{2}\left(\frac{1}{1+\sin(\pi x)}\right).$$

This completes the proof. □

As a mathematical remark, another possible expression of the probability density function f is as follows:

$$f(x) = \frac{\pi}{4}\left(\frac{1}{\left(\sin\left((\pi/4)(3-2x)\right)\right)^2}\right), \quad x \in (0,1),$$

and is completed by setting $f(x) = 0$ for any $x \notin (0, 1)$.

For illustrative purposes, Figure 2 displays a plot of f, which exhibits a bathtub-shaped form.

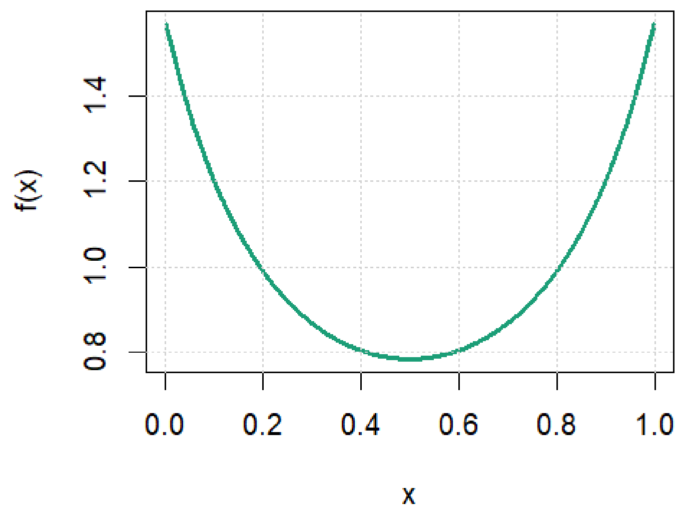

Figure 2: Plot of the probability density function of the NT distribution.

From the probability density and cumulative distribution functions, one may also derive the hazard rate function, defined as $h = f/(1 - F)$. However, we do not pursue its analysis further here.

One advantage of the NT distribution lies in the simplicity of its defining functions. In particular, the tractability of the cumulative distribution function makes it possible to derive the quantile function, as shown in the theorem below.

Theorem 2.3 *The quantile function associated with the NT distribution is given by*

$$Q(u) = \frac{2}{\pi} \arctan(2u - 1) + \frac{1}{2}, \quad u \in (0, 1).$$

Proof. By definition, the quantile function is the inverse of the corresponding cumulative distribution function. Based on the expression of the cumulative distribution function F defined in Theorem 2.1, we solve

$$F(x) = u \Leftrightarrow \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)\right) = u$$
$$\Leftrightarrow \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right) = 2u - 1$$
$$\Leftrightarrow \frac{\pi}{2}\left(x - \frac{1}{2}\right) = \arctan(2u - 1)$$
$$\Leftrightarrow x - \frac{1}{2} = \frac{2}{\pi}\arctan(2u - 1)$$
$$\Leftrightarrow x = \frac{2}{\pi}\arctan(2u - 1) + \frac{1}{2}.$$

Therefore, we have

$$Q(u) = \frac{2}{\pi}\arctan(2u - 1) + \frac{1}{2}, \quad u \in (0, 1).$$

This completes the proof. □

For illustrative purposes, Figure 3 displays the plot of Q.

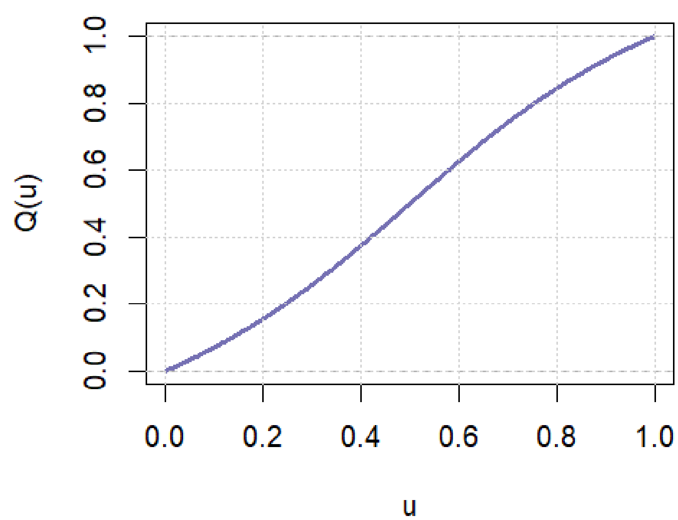

Figure 3: Plot of the quantile function of the NT distribution.

Using the quantile function, we can determine the quartiles of the NT distribution, as stated in the proposition below.

Proposition 2.1 *The first, second and third quartiles of the NT distribution are respectively given by*

$$Q_1 = -\frac{2}{\pi} \arctan\left(\frac{1}{2}\right) + \frac{1}{2} \approx 0.2048, \quad Q_2 = \frac{1}{2}$$

and

$$Q_3 = \frac{2}{\pi} \arctan\left(\frac{1}{2}\right) + \frac{1}{2} \approx 0.7951.$$

Proof. By the definition of the quartiles and Theorem 2.3, we have

$$\begin{aligned} Q_1 &= Q\left(\frac{1}{4}\right) = \frac{2}{\pi} \arctan\left(2 \times \frac{1}{4} - 1\right) + \frac{1}{2} \\ &= -\frac{2}{\pi} \arctan\left(\frac{1}{2}\right) + \frac{1}{2} \approx 0.2048, \end{aligned}$$

$$Q_2 = Q\left(\frac{1}{2}\right) = \frac{2}{\pi} \arctan\left(2 \times \frac{1}{2} - 1\right) + \frac{1}{2} = \frac{1}{2}$$

and

$$\begin{aligned} Q_3 &= Q\left(\frac{3}{4}\right) = \frac{2}{\pi} \arctan\left(2 \times \frac{3}{4} - 1\right) + \frac{1}{2} \\ &= \frac{2}{\pi} \arctan\left(\frac{1}{2}\right) + \frac{1}{2} \approx 0.7951. \end{aligned}$$

This completes the proof. □

In order word, the median of the NT distribution is

$$Q_2 = \frac{1}{2}.$$

2.2 Distribution results

A simple distribution result for the NT distribution is stated in the theorem below.

Theorem 2.4 *Let X be a random variable that follows the NT distribution. Then the random variable*

$$Y = 1 - X$$

also follows the NT distribution.

Proof. Let F_Y be the cumulative distribution function of Y and F be the cumulative distribution function of the NT distribution as defined in Theorem 2.1. Since X is with support $(0,1)$, it is clear that Y too, which implies that $F_Y(x) = 0$ for any $x \leq 0$ and $F_Y(x) = 1$ for any $x \geq 1$, so that $F_Y(x) = F(x)$ for any $x \notin (0,1)$. For any $x \in (0,1)$, we have

$$\begin{aligned}
&F_Y(x) = P(Y \leq x) = P(1 - X \leq x) = P(X \geq 1 - x) = 1 - P(X < 1 - x) \\
&= 1 - F(1-x) = 1 - \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(1 - x - \frac{1}{2}\right)\right)\right) \\
&= \frac{1}{2}\left(1 - \tan\left(\frac{\pi}{2}\left(\frac{1}{2} - x\right)\right)\right) = \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x - \frac{1}{2}\right)\right)\right) \\
&= F(x).
\end{aligned}$$

Therefore, Y and X follow the same NT distribution. This concludes the proof. □

Another simple distribution result for the NT distribution is stated in the theorem below.

Theorem 2.5 *Let U be a random variable that follows the uniform distribution with support $(0,1)$. Then the random variable*

$$Z = \frac{2}{\pi}\arctan(2U - 1) + \frac{1}{2}$$

follows the NT distribution.

Proof. It is sufficient to notice that

$$Z = Q(U),$$

where Q denotes the quantile function of the NT distribution, as described in Theorem 2.3.

In more detail, let F_Z be the cumulative distribution function of Z and F be the cumulative distribution function of the NT distribution as defined in Theorem 2.1. Since the NT distribution is with support $(0,1)$, it is clear that Z too, which implies that $F_Z(x) = 0$ for any $x \leq 0$ and $F_Z(x) = 1$ for any $x \geq 1$, so that $F_Z(x) = F(x)$ for any $x \notin (0,1)$. For any $x \in (0,1)$, we have

$$F_Z(x) = P(Z \leq x) = P(Q(U) \leq x) = P(U \leq F(x)) = F(x).$$

Therefore, Z follows the NT distribution. This concludes the proof. □

Based on this result and proof, the quantile function can be used to generate random values from the NT distribution, as described below. Given n values $u_1, \ldots, u_n$ drawn from a uniform distribution with support $(0,1)$, the corresponding values $Q(u_1), \ldots, Q(u_n)$ provide a sample from a random variable that follows the NT distribution.

2.3 Moments

The mean of a random variable X that follows the NT distribution is computed in the theorem below, with two distinct proofs provided.

Theorem 2.6 *The mean of a random variable X that follows the NT distribution is given by*

$$E(X) = \frac{1}{2}.$$

Proof. Two distinct proofs are proposed.
Proof 1. Using the integral expression of $E(X)$, the probability density function f as described in Theorem 2.2, and standard primitives, we obtain

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f(x) dx = \int_0^1 x \times \frac{\pi}{2} \left(\frac{1}{1+\sin(\pi x)} \right) dx \\ &= \frac{\pi}{2} \int_0^1 \frac{x}{1+\sin(\pi x)} dx \\ &= \frac{1}{2} \left[\frac{1}{\pi} \log(1+\sin(\pi x)) + x \tan\left(\frac{\pi}{2} \left(x - \frac{1}{2} \right) \right) \right]_{x\to 0}^{x\to 1} \\ &= \frac{1}{2} \tan\left(\frac{\pi}{4} \right) = \frac{1}{2}. \end{aligned}$$

Proof 2. By Theorem 2.4, the random variables X and $1-X$ follow the NT distribution. Based on this, we have

$$E(X) = E(1-X) \;\Leftrightarrow\; E(X) = 1 - E(X) \;\Leftrightarrow\; 2E(X) = 1 \;\Leftrightarrow\; E(X) = \frac{1}{2}.$$

This completes the proof. □

At this point, it is worth noting that the NT distribution shares several notable properties with the uniform distribution on $(0,1)$: both have a mean of $1/2$, a median of $1/2$, and the result presented in Theorem 2.4 applies to both distributions. However, the NT distribution has a richer functionality and probabilistic structure.

Another simple moment result is presented in the theorem below.

Theorem 2.7 *The mean of* $\sin(\pi X)$, *where X is a random variable that follows the NT distribution, is given by*

$$E(\sin(\pi X)) = \frac{\pi}{2} - 1.$$

Proof. Using the law of the unconscious statistician and the probability density function f as described in Theorem 2.2, we have

$$E(1+\sin(\pi X)) = \int_{-\infty}^{+\infty} (1+\sin(\pi x)) f(x) dx$$
$$= \int_0^1 (1+\sin(\pi x)) \times \frac{\pi}{2}\left(\frac{1}{1+\sin(\pi x)}\right) dx$$
$$= \frac{\pi}{2}\int_0^1 dx = \frac{\pi}{2}.$$

Therefore, we have

$$E(\sin(\pi X)) = E(1+\sin(\pi X)) - 1 = \frac{\pi}{2} - 1.$$

This concludes the proof. □

Another moment result is stated in the theorem below.

Theorem 2.8 *The mean of* $(1+\sin(\pi X))^2$, *where* X *is a random variable that follows the NT distribution, is given by*

$$E\left((1+\sin(\pi X))^2\right) = \frac{\pi}{2} + 1.$$

Proof. Using the law of the unconscious statistician and the probability density function f as described in Theorem 2.2, we have

$$E\left((1+\sin(\pi X))^2\right) = \int_{-\infty}^{+\infty} (1+\sin(\pi x))^2 f(x) dx$$
$$= \int_0^1 (1+\sin(\pi x))^2 \times \frac{\pi}{2}\left(\frac{1}{1+\sin(\pi x)}\right) dx$$
$$= \frac{\pi}{2}\int_0^1 (1+\sin(\pi x)) dx = \frac{\pi}{2}\left[x - \frac{1}{\pi}\cos(\pi x)\right]_{x\to 0}^{x\to 1}$$
$$= \frac{\pi}{2}\left(1+\frac{2}{\pi}\right) = \frac{\pi}{2} + 1.$$

This ends the proof. □

Another moment result is stated in the theorem below.

Theorem 2.9 *The mean of* $(\sin(\pi X))^2$, *where* X *is a random variable that follows the NT distribution, is given by*

$$E\left((\sin(\pi X))^2\right) = 2 - \frac{\pi}{2}.$$

Proof. From Theorems 2.7 and 2.8, we derive

$$
\begin{aligned}
&E\left((\sin(\pi X))^2\right) = E\left((1+\sin(\pi X))^2\right) - 2E\left(\sin(\pi X)\right) - 1\\
&= \frac{\pi}{2} + 1 - 2\left(\frac{\pi}{2} - 1\right) - 1\\
&= 2 - \frac{\pi}{2}.
\end{aligned}
$$

This completes the proof. □

2.4 NT family of distributions

Since the NT distribution is with support $(0,1)$, it can be used as a generator of distributions with distinct supports. A comprehensive treatment of this approach can be found in [2]. The theorem below formalizes the result.

Theorem 2.10 *Let G be a cumulative distribution function of a continuous distribution. Based on G and the cumulative distribution function F of the NT distribution, the following function is a valid cumulative distribution function:*

$$
\begin{aligned}
&F_\dagger(x) = F(G(x))\\
&= \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(G(x) - \frac{1}{2}\right)\right)\right), \quad x \in \mathbb{R}.
\end{aligned}
$$

Proof. Since $G(x) \in [0,1]$ and F is the cumulative distribution function of a distribution with support $(0,1)$, the composition $F_\dagger(x) = F(G(x))$ is well-defined. Moreover, the composition of two continuous functions is a continuous function, the composition of two increasing functions is an increasing function and clearly $F_\dagger(x) \in [0,1]$ because $F(x) \in [0,1]$. Therefore, $F_\dagger$ is a valid cumulative distribution function. This completes the proof. □

We call the family of distributions defined with the cumulative distribution function $F_\dagger$ in Theorem 2.10 the NT family of distributions.

Three notable members of this family with distinct supports are described below.

First member: We define the NT power distribution as the distribution defined by the following cumulative distribution function:

$$
F_\vee(x) = \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(x^\alpha - \frac{1}{2}\right)\right)\right), \quad x \in (0,1),
$$

with $\alpha > 0$, and is completed by setting $F_\vee(x) = 0$ for any $x \leq 0$ and $F_\vee(x) = 1$ for any $x \geq 1$. In this case, Theorem 2.10 was applied to the cumulative distribution function of the power distribution with support $(0,1)$ and parameter α given by $G(x) = x^\alpha$ for any $x \in (0,1)$, and is completed by setting $G(x) = 0$ for any $x \leq 0$ and $G(x) = 1$ for any $x \geq 1$.

Second member: We define the NT Weibull distribution as the distribution defined by the following cumulative distribution function:

$$F_\triangle(x) = \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left((1 - e^{-(x/\beta)^\alpha}) - \frac{1}{2}\right)\right)\right), \quad x > 0,$$

with $\alpha > 0$ and $\beta > 0$, and is completed by setting $F_\triangle(x) = 0$ for any $x \leq 0$. In this case, Theorem 2.10 was applied to the cumulative distribution function of the Weibull distribution with the parameters α and β given by $G(x) = 1 - e^{-(x/\beta)^\alpha}$ for any $x > 0$, and is completed by setting $G(x) = 0$ for any $x \leq 0$.

Third member: We define the NT logistic distribution as the distribution defined by the following cumulative distribution function:

$$F_\ddagger(x) = \frac{1}{2}\left(1 + \tan\left(\frac{\pi}{2}\left(\frac{1}{1 + e^{-(x-\mu)/\sigma}} - \frac{1}{2}\right)\right)\right), \quad x \in \mathbb{R},$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. In this case, Theorem 2.10 was applied to the cumulative distribution function of the logistic distribution with the parameters μ and σ given by $G(x) = 1/(1 + e^{-(x-\mu)/\sigma})$ for any $x \in \mathbb{R}$.

3 Open problem

We claim that the moment of order 2 of a random variable X that follows the NT distribution is given by

$$E(X^2) = \frac{1}{2\pi^2}\left(\pi(\pi + \log(4)) - 8C\right) \approx 0.349409,$$

where C denotes the Catalan constant, or, equivalently in terms of integral,

$$\int_0^1 x^2 \times \frac{\pi}{2}\left(\frac{1}{1 + \sin(\pi x)}\right) dx$$
$$= \frac{1}{2\pi^2}\left(\pi(\pi + \log(4)) - 8C\right) \approx 0.349409.$$

This expression can be verified using various symbolic software packages, including Mathematica. However, a simple, fully rigorous proof remains an open problem.

4 Conclusion

In this article, we introduced a new unit distribution based on the tangent function and established its key probabilistic properties, thereby expanding the current limited range. However, the identification of an open problem regarding the rigorous derivation of the second-order moment indicates that important theoretical aspects still need to be clarified. Future research could focus on resolving this issue, exploring inferential procedures and investigating potential applications of the proposed distribution in different fields.

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