Int. J. Open Problems Compt. Math., Vol. 19, No. 3, September 2026
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Sharp inequalities for sine and cosine on complex circles and applications

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Received 20 March 2026; Accepted 05 April, 2026

Abstract

We investigate sharp inequalities for the functions $|\sin z|$ ***and*** $|\cos z|$ ***along circles in the complex plane. Building on recent results of Qi, we establish precise bounds for the quantities***

$$|\sin(re^{i\theta})| - |\cos(re^{i\theta})| \quad \textbf{\textit{and}} \quad |\sin(re^{i\theta}) - \cos(re^{i\theta})|.$$

We show that their behavior undergoes a phase transition governed by a unique critical parameter r_0 ***defined by*** $\cos(2r_0) = 2r_0$***. As an application, we completely resolve several open problems posed by Bagul and Chesneau concerning double-sided inequalities for trigonometric and hyperbolic functions. In particular, we prove that***

$$\frac{\sin(kx)}{kx} + k\frac{\sin x}{x} > 1 + k\cos x$$

holds if and only if $k \in (0, 2)$***, and that***

$$\frac{\sinh(qx)}{qx} + q\frac{\sinh x}{x} > 1 + q\cosh x$$

holds if and only if $q \geq 2$***. We further obtain weighted extensions of these inequalities. Our approach combines complex-analytic techniques with sharp real-variable inequalities and reveals new connections between classical inequalities and the geometry of analytic functions on circles.***

Keywords: *bound, norm, sine, cosine, double inequality, circle, complex plane.*

2020 Mathematics Subject Classification: Primary: 33B10; Secondary: 30A10.

1 Introduction

The study of trigonometric inequalities on the real line has a long and rich history, with numerous applications in analysis, approximation theory, and mathematical inequalities. Classical results involving the functions $\sin x$, $\cos x$, and the sinc function $\frac{\sin x}{x}$ provide sharp bounds and play a fundamental role in both theoretical and applied contexts; see, for example, [5, 10, 12]. In recent years, considerable attention has been devoted to refining these inequalities and understanding their structural properties [7, 8, 9].

A natural direction of investigation is to extend these classical results beyond the real line and examine the behavior of trigonometric functions in the complex plane. In this setting, new phenomena emerge due to the interplay between oscillatory and exponential behavior. While $\sin x$ and $\cos x$ are bounded on the real axis, their complex counterparts $\sin z$ and $\cos z$ exhibit exponential growth along the imaginary direction. This leads to a richer and more intricate structure when these functions are evaluated on nontrivial subsets of the complex plane. This behavior is classical in complex analysis and follows from the exponential representations of trigonometric functions (see [1, 4]).

A particularly interesting and geometrically meaningful setting is provided by circles centered at the origin, parameterized as

$$z = re^{i\theta}, \quad r > 0, \ \theta \in [0, 2\pi).$$

Along such curves, the real and imaginary parts of z vary simultaneously, producing a mixture of oscillatory and hyperbolic effects. Consequently, the analysis of trigonometric functions along circles is both challenging and insightful.

Recently, Qi [8] established sharp bounds for the quantities $|\sin z|$ and $|\cos z|$ along complex circles, showing that

$$|\sin r| \le |\sin(re^{i\theta})| \le \sinh(r), \quad |\cos r| \le |\cos(re^{i\theta})| \le \cosh(r).$$

These results describe the extremal behavior of the modulus of sine and cosine individually. However, they do not address expressions that involve interactions between these functions, such as their differences or linear combinations. Understanding such quantities is a natural and nontrivial continuation of this line of research.

Motivated by this problem, we investigate the interaction terms

$$|\sin(re^{i\theta})| - |\cos(re^{i\theta})|, \quad \text{and} \quad |\sin(re^{i\theta}) - \cos(re^{i\theta})|,$$

which capture the combined oscillatory and exponential behavior of the trigonometric functions. Unlike the individual bounds, these expressions require a more delicate analysis to accurately describe their extremal properties.

Our study is also closely connected to classical inequalities on the real line. In particular, inequalities involving the sinc and hyperbolic sinc functions have been extensively investigated, and several open problems regarding their sharpness and admissible parameter ranges have been posed [3]. By analyzing trigonometric functions along complex circles, we provide a unified framework that not only resolves these open problems but also clarifies the underlying structure of the inequalities.

The main contributions of this paper are as follows. First, we establish sharp bounds for the above interaction terms and provide a complete description of their extremal behavior. Our analysis reveals a critical parameter r_0, defined by the equation $\cos(2r_0) = 2r_0$, which governs a qualitative change in behavior. Second, we apply these results to resolve open problems concerning double-sided inequalities for trigonometric and hyperbolic functions, determining the exact ranges of parameters for which such inequalities hold. We also derive weighted extensions that further generalize these results.

2 Main results

Lemma 2.1. *Let $r > 0$ and define*

$$f(t) = \frac{\cos t}{t} + \frac{\sinh\big(\sqrt{4r^2 - t^2}\big)}{\sqrt{4r^2 - t^2}}, \quad t \in (-2r, 2r) \setminus \{0\},$$

and

$$g(r) = 2r - \cos(2r), \quad r \geq 0.$$

Then the following hold:

1. *The function f is continuous on $(-2r, 2r) \setminus \{0\}$, strictly decreasing on $(-2r, 0)$, and satisfies*

$$f(t) > 0 \quad \text{for all } t \in (0, 2r).$$

Moreover,

$$\lim_{t \to 0^+} f(t) = +\infty, \qquad f(-2r) = \frac{g(r)}{2r}.$$

2. *The function g is strictly increasing on $(0,\infty)$, with*

$$g(0) = -1, \qquad g(1/2) = 1 - \cos(1) > 0.$$

Consequently, there exists a unique $r_0 \in (0,1/2)$ such that

$$g(r_0) = 0 \quad \textit{(equivalently, } \cos(2r_0) = 2r_0\textit{).}$$

Moreover,

$$g(r) < 0 \textit{ for } r \in (0,r_0), \qquad g(r) > 0 \textit{ for } r \in (r_0, 1/2).$$

3. *Let $r \in (0,1/2)$ and $t \in (-2r, 0)$.*

   - *If $r \in (0, r_0)$, then $f(t) < 0$ for all $t \in (-2r,0)$.*
   - *If $r \in (r_0, 1/2)$, then there exists a unique $t_r \in (-2r, 0)$ such that*

$$f(t_r) = 0,$$

and

$$f(t) > 0 \textit{ for } t \in (-2r, t_r), \qquad f(t) < 0 \textit{ for } t \in (t_r, 0).$$

4. *For $r \in (r_0, 1/2)$, there exists a unique $\theta_r \in (\pi/2, \pi)$ such that*

$$\frac{\cos(2r\cos\theta_r)}{2r\cos\theta_r} + \frac{\sinh(2r\sin\theta_r)}{2r\sin\theta_r} = 0.$$

Moreover,

$$f(2r\cos\theta) > 0 \textit{ for } \theta \in (\pi/2, \theta_r), \qquad f(2r\cos\theta) < 0 \textit{ for } \theta \in (\theta_r, \pi).$$

Proof. **(1)** The continuity of f is immediate.

Let $t \in (0, 2r)$. Since $\sinh(x) > x$ for all $x > 0$, we obtain

$$\frac{\sinh\big(\sqrt{4r^2 - t^2}\big)}{\sqrt{4r^2 - t^2}} > 1,$$

hence

$$f(t) > \frac{\cos t}{t} + 1 = \frac{\cos t + t}{t}.$$

The function $t \mapsto \cos t + t$ is strictly increasing on $(0, \infty)$ and $\cos t + t \geq 1$. Hence $f(t) > 0$ on $(0, 2r)$.

Moreover, as $t \to 0^+$, $\cos t / t \to +\infty$, hence $f(t) \to +\infty$.

For $t \in (-2r, 0)$, the function $t \mapsto \cos t/t$ is strictly decreasing. Also,

$$\frac{\sinh x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}.$$

Differentiating the power series term-by-term yields $\left(\frac{\sinh x}{x}\right)' > 0$ for $x > 0$, hence $\frac{\sinh x}{x}$ is strictly increasing on $(0, \infty)$, and the function $t \mapsto \sqrt{4r^2 - t^2}$ is strictly decreasing on $(-2r, 0)$, then the function

$$t \mapsto \frac{\sinh\left(\sqrt{4r^2 - t^2}\right)}{\sqrt{4r^2 - t^2}},$$

is strictly decreasing. Hence f is strictly decreasing on $(-2r, 0)$.

Finally,

$$f(-2r) = \frac{\cos(2r)}{-2r} + \lim_{x \to 0^+} \frac{\sinh x}{x} = \frac{2r - \cos(2r)}{2r}.$$

(2) We have

$$g'(r) = 2 + 2\sin(2r) > 0 \quad \text{for all } r > 0,$$

hence g is strictly increasing. Since $g(0) = -1$ and $g(1/2) = 1 - \cos(1) > 0$, there exists a unique $r_0 \in (0, 1/2)$ such that $g(r_0) = 0$. The sign of g follows from monotonicity.

(3) Let $t \in (-2r, 0)$. Since f is strictly decreasing on this interval, its sign is determined by the endpoint values.

If $r \in (0, r_0)$, then $g(r) < 0$, hence

$$f(-2r) = \frac{g(r)}{2r} < 0.$$

We obtain $f(t) < 0$ for all $t \in (-2r, 0)$.

If $r \in (r_0, 1/2)$, then $f(-2r) > 0$ while $\lim_{t \to 0^-} f(t) = -\infty$. By continuity and strict monotonicity, there exists a unique $t_r \in (-2r, 0)$ such that $f(t_r) = 0$, and the sign change follows.

(4) Let $\theta \in (\pi/2, \pi)$ and set $t = 2r\cos\theta \in (-2r, 0)$. The equation is equivalent to $f(t) = 0$. By (3), for $r \in (r_0, 1/2)$ there exists a unique solution $t_r \in (-2r, 0)$, hence a unique $\theta_r \in (\pi/2, \pi)$ such that $t_r = 2r\cos\theta_r$.

The sign properties follow directly from those of f. □

We show that the behavior of $\psi(r, \theta) = |\sin(re^{i\theta}) - \cos(re^{i\theta})|$ undergoes a transition at the critical value r_0 defined by $\cos(2r_0) = 2r_0$.

Theorem 2.2. *Let $r > 0$ and $\theta \in (-\pi, \pi]$. Define*

$$\varphi(r,\theta) = |\sin(re^{i\theta})| - |\cos(re^{i\theta})|, \quad \psi(r,\theta) = |\sin(re^{i\theta}) - \cos(re^{i\theta})|.$$

1. *For all $r > 0$ and $\theta \in (-\pi, \pi]$,*

$$-\frac{1}{\sqrt{\cosh(2r\sin\theta)}} \leq \varphi(r,\theta) \leq \frac{1}{\sqrt{\cosh(2r\sin\theta)}}.$$

   Equality holds if and only if $\theta = 0$ or $\theta = \pi$.

2. *Let $r > 0$. Then:*

   (i) For $\theta \in (0, \pi/2)$,

$$\sqrt{1-\sin(2r)} \leq \psi(r,\theta) \leq \sqrt{\cosh(2r)}.$$

   (ii) For $\theta \in (\pi/2, \pi)$:

   – *If $r > 1/2$,*

$$\sqrt{1+\sin(2r)} \leq \psi(r,\theta) \leq \sqrt{\cosh(2r)}.$$

   – *If $r \in (0, r_0)$,*

$$\sqrt{\cosh(2r)} \leq \psi(r,\theta) \leq \sqrt{1+\sin(2r)}.$$

   – *If $r \in (r_0, 1/2)$, there exists a unique $\theta_r \in (\pi/2, \pi)$ such that*

$$\frac{\cos(2r\cos\theta_r)}{2r\cos\theta_r} + \frac{\sinh(2r\sin\theta_r)}{2r\sin\theta_r} = 0,$$

   and

$$\sqrt{\cosh(2r)} \leq \psi(r,\theta) \leq \psi(r,\theta_r).$$

Here r_0 is the unique solution of $\cos(2r) = 2r$ in $(0, 1/2)$.

Proof. Since $\varphi(r,\theta+\pi) = \varphi(r,\theta)$ and $\psi(r,\theta+2\pi) = \psi(r,\theta)$, it suffices to consider $\theta \in (0,\pi)$.

Step 1. Bounds for $\varphi(r,\theta)$.
Let

$$h(r,\theta) = |\sin(re^{i\theta})| + |\cos(re^{i\theta})|.$$

A straightforward computation shows that

$$|\sin(re^{i\theta})|^2 = \sin^2(r\cos\theta) + \sinh^2(r\sin\theta),$$

$$|\cos(re^{i\theta})|^2 = \cos^2(r\cos\theta) + \sinh^2(r\sin\theta).$$

Hence

$$\varphi(r,\theta)\, h(r,\theta) = -\cos(2r\cos\theta),$$

and therefore

$$|\varphi(r,\theta)| \le \frac{1}{h(r,\theta)}.$$

Moreover,

$$h(r,\theta) \ge \sqrt{\sin^2(r\cos\theta) + \cos^2(r\cos\theta) + 2\sinh^2(r\sin\theta)} = \sqrt{\cosh(2r\sin\theta)}.$$

Thus

$$-\frac{1}{\sqrt{\cosh(2r\sin\theta)}} \le \varphi(r,\theta) \le \frac{1}{\sqrt{\cosh(2r\sin\theta)}}.$$

Equality holds if and only if $\theta = 0$ or $\theta = \pi$.

Step 2. Expression and derivative of $\psi(r,\theta)$.
A direct computation yields

$$(\psi(r,\theta))^2 = \cosh(2r\sin\theta) - \sin(2r\cos\theta).$$

Differentiating, we obtain

$$\frac{d}{d\theta}(\psi(r,\theta))^2 = 2r^2\, f(2r\cos\theta)\sin(2\theta),$$

where f is defined in Lemma 2.1.

Step 3. Case $\theta \in (0,\pi/2)$.
Here $\sin(2\theta) > 0$ and $2r\cos\theta \in (0,2r)$. By Lemma 2.1, $f(t) > 0$ on $(0,2r)$, hence $(\psi(r,\theta))^2$ is strictly increasing on $(0,\pi/2)$. Therefore,

$$\psi(r,0) = \sqrt{1-\sin(2r)} \le \psi(r,\theta) \le \psi(r,\pi/2) = \sqrt{\cosh(2r)}.$$

Step 4. Case $\theta \in (\pi/2,\pi)$.
In this case $t = 2r\cos\theta \in (-2r,0)$ and $\sin(2\theta) < 0$. The sign of the derivative is determined by the sign of $f(t)$.

- If $r > 1/2$, then by Lemma 2.1, $f(t)$ is strictly decreasing and $f(-2r) > (2r - \cos(2r))/2r > (1-\cos(2r))/2r \ge 0$, hence $(\psi(r,\theta))^2$ is strictly decreasing. Thus

$$\sqrt{1+\sin(2r)} \le \psi(r,\theta) \le \sqrt{\cosh(2r)}.$$

- If $r \in (0, r_0)$, then $f(t) < 0$ on $(-2r, 0)$ by Lemma 2.1, hence $(\psi(r,\theta))^2$ is strictly increasing. Therefore
$$\sqrt{\cosh(2r)} \le \psi(r,\theta) \le \sqrt{1+\sin(2r)}.$$
- If $r \in (r_0, 1/2)$, then by Lemma 2.1 there exists a unique $\theta_r \in (\pi/2, \pi)$ such that $f(2r\cos\theta_r) = 0$. Hence $(\psi(r,\theta))^2$ increases on $(\pi/2, \theta_r)$ and decreases on (θ_r, π), and
$$\sqrt{\cosh(2r)} \le \psi(r,\theta) \le \psi(r,\theta_r).$$

This completes the proof. □

Corollary 2.3. *For $r > 0$ and $\theta \in \mathbb{R}$, let*
$$g(r,\theta) = |\cos(re^{i\theta}) + \sin(re^{i\theta})|.$$

For $\theta \in (\pi/2, \pi)$ and $r > 0$,
$$\sqrt{1-\sin(2r)} \le g(r,\theta) \le \sqrt{\cosh(2r)},$$
for $\theta \in (0, \pi/2)$ and $r > 1/2$,
$$\sqrt{1+\sin(2r)} \le g(r,\theta) \le \sqrt{\cosh(2r)}.$$

For $r \in (0, r_0)$ and $\theta \in (0, \pi/2)$
$$\sqrt{\cosh(2r)} \le g(r,\theta) \le \sqrt{1+\sin(2r)},$$
for $r \in (r_0, 1/2)$ and $\theta \in (\pi - \theta_r, \pi/2)$
$$\sqrt{\cosh(2r)} \le g(r,\theta) \le g(r, \pi - \theta_r),$$
and for $\theta \in (0, \pi - \theta_r)$
$$\sqrt{1+\sin(2r)} \le g(r,\theta) \le g(r, \pi - \theta_r),$$
where r_0 and θ_r are the parameters of Theorem 2.2.

Proof. For $z \in \mathbb{C}$ and $z = x + iy$, we have
$$\sin(z) + \cos(z) = \cosh(y)(\cos x + \sin x) + i\sinh(y)(\cos x - \sin x),$$
and if we set $g(r,\theta) = |\sin(re^{i\theta}) + \cos(re^{i\theta})|$, we get
$$g(r,\theta) = \sqrt{\cosh(2r\sin\theta) + \sin(2r\cos\theta)}.$$
Therefore, for $r > 0$ and $\theta \in \mathbb{R}$
$$g(r, \pi - \theta) = \psi(r,\theta).$$
Applying the results of Theorem 2.2, we get the desired results. □

3 Auxiliary real and hyperbolic inequalities

Proposition 3.1 (Sharp trigonometric and hyperbolic inequalities)**.**

1. *For all* $x \in (0, \pi/2)$*,*

$$\frac{\sin(kx)}{kx} + k\frac{\sin(x)}{x} > 1 + k\cos(x) \quad \textit{if and only if} \quad k \in (0,2).$$

2. *For all* $x \in \mathbb{R}$*,*

$$\frac{\sinh(qx)}{qx} + q\frac{\sinh(x)}{x} > 1 + q\cosh(x) \quad \textit{if and only if} \quad q \geq 2.$$

3. *For all* $x \in \mathbb{R}$*,*

$$q + \cosh(x)^q > \frac{\sinh(qx)}{qx} + q\frac{\sinh(x)}{x} \quad \textit{if and only if} \quad q \in (1,2).$$

Proof. **(1)** Define

$$\varphi_k(x) = \sin(kx) + k^2\sin(x) - k^2 x\cos(x) - kx.$$

Then $\varphi_k(0) = 0$, and the inequality is equivalent to $\varphi_k(x) > 0$ for $x \in (0, \pi/2)$.

Differentiating, we obtain

$$\varphi_k'(x) = k\big(\cos(kx) + kx\sin(x) - 1\big) = k\big(kx\sin x - 2\sin^2(\tfrac{kx}{2})\big).$$

If $k \in (0,2)$ and $x \in (0, \pi/2)$, then $\frac{kx}{2} < x$ and since sin is strictly increasing on $(0, \pi/2)$, we have

$$\sin\big(\tfrac{kx}{2}\big) < \sin x,$$

hence

$$\varphi_k'(x) \geq 2k\sin(x)\Big(\frac{kx}{2} - \sin\big(\tfrac{kx}{2}\big)\Big).$$

which implies $\varphi_k'(x) > 0$. Therefore φ_k is strictly increasing on $(0, \pi/2)$, and since $\varphi_k(0) = 0$, we obtain $\varphi_k(x) > 0$.

Conversely, expanding at $x = 0$,

$$\varphi_k(x) = \frac{k(2-k)}{6}x^3 + o(x^3).$$

Thus, if $\varphi_k(x) > 0$ for $x > 0$ sufficiently small, then $k \in (0,2)$.

(2) Define

$$\psi_q(x) = \sinh(qx) + q^2\sinh(x) - q^2 x\cosh(x) - qx.$$

Then $\psi_q(0) = 0$, and the inequality is equivalent to $\psi_q(x) > 0$ for $x > 0$.

Differentiating, we obtain

$$\psi_q'(x) = q\big(\cosh(qx) + qx\sinh(x) - 1\big) = q\big(2\sinh^2(\tfrac{qx}{2}) - qx\sinh x\big).$$

For $q \geq 2$ and $x > 0$, we have $\frac{qx}{2} \geq x$, hence $\sinh(\frac{qx}{2}) \geq \sinh(x)$. Therefore

$$\psi_q'(x) \geq q\big(2\sinh(\tfrac{qx}{2}) - qx\big)\sinh(\tfrac{qx}{2}).$$

Since $\sinh u > u$ for all $u > 0$, it follows that $2\sinh(\frac{qx}{2}) - qx > 0$, and thus $\psi_q'(x) > 0$ for $x > 0$.

Hence ψ_q is strictly increasing on $(0, \infty)$ and $\psi_q(x) > 0$ for all $x > 0$. Moreover, ψ_q is odd, so $\psi_q(x) < 0$ for $x < 0$.

Conversely, expanding at $x = 0$,

$$\psi_q(x) = \frac{q^2(q-2)}{6}x^3 + o(x^3),$$

which implies that positivity near 0 requires $q \geq 2$.

(3) It is clear that it suffices to show the inequality for $x > 0$. If the inequality is valid for all $x > 0$, then the function

$$H_q(x) = q + \cosh(x)^q - \frac{\sinh(qx)}{qx} - q\frac{\sinh(x)}{x},$$

is positive for all $x > 0$. As x goes to infinity $H_q(x) = \frac{e^x}{2x}\big(\frac{e^{(q-1)x}}{2^q} - \frac{e^{(q-1)x}}{qx} - 1\big) + o(1)$. So, for $q < 1$, $\lim_{x\to\infty} H_q(x) = -\infty$. Therefore, $q \geq 1$. Expanding at $x = 0$, we get $H_q(x) = \frac{q}{6}(2-q)x^2 + o(x^2)$ and then $q \in [1, 2]$.

The converse. Assume $q \in [1, 2]$. For $x > 0$, let

$$g(q) = q\log(\cosh(x)) - \log(\cosh(qx)) + (1-q)\log 2.$$

Differentiating twice yields

$$g'(q) = \log(\cosh(x)) - x\tanh(qx) - \log 2,$$

and

$$g''(q) = -x^2(1 - \tanh(qx)^2) \leq 0.$$

Then, g' is strictly decreasing. Moreover, $g'(1) = \log(\cosh(x)) - x\tanh(x) - \log 2 := \theta(x)$. Differentiating yields

$$\theta'(x) = -x(1 - \tanh(x)^2) < 0,$$

and $\theta(0) = -\log 2$. So, $g'(q) \leq g'(1) < 0$ and g is strictly decreasing and $g(1) = 0$, exponentiate yields for $q \in (1, 2)$ $\cosh(x)^q \geq 2^{q-1}\cosh(qx)$. So, it suffices to prove

$$q + 2^{q-1}\cosh(qx) - \frac{\sinh(qx)}{qx} - q\frac{\sinh(x)}{x} > 0.$$

Let

$$h_q(x) = q + 2^{q-1}\cosh(qx) - \frac{\sinh(qx)}{qx} - q\frac{\sinh(x)}{x}.$$

Using the Taylor series we get

$$h_q(x) = 2^{q-1} - 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{2^{1-q}} - \frac{1}{2n+1} - \frac{1}{q^{2n-1}(2n+1)}\Big)q^{2n}\frac{x^{2n}}{(2n)!}.$$

For $q \in (1,2]$, the sequence

$$u_n(q) = \frac{1}{2^{1-q}} - \frac{1}{2n+1} - \frac{1}{q^{2n-1}(2n+1)},$$

is strictly increasing and $u_1(q) = \frac{1}{2^{1-q}} - \frac{q+1}{3q} = \frac{3q-(q+1)2^{1-q}}{3\cdot 2^{1-q}q}$. Since $q > 1$, then $3q - (q+1)2^{1-q} \geq 2q - 1 > 0$ and then $h_q(x) \geq h_q(0) = 2^{q-1} - 1 > 0$.

For $q = 1$,

$$h(x) := xh_1(x) = xH_1(x) = x + x\cosh(x) - 2\sinh(x),$$

then, $h'(x) = 1 - \cosh(x) + x\sinh(x)$ and $h''(x) = x\cosh(x) > 0$ for $x > 0$. Moreover, $h'(0) = 0$ and $h(0) = 0$. Then $H_1(x) > 0$ for $x > 0$. This completes the proof. □

Proposition 3.2 (Weighted extensions)**.**

1. *Let $x \in (0,\pi/2)$, $a \geq 1$, $b > 0$, and assume $2b \geq a^2$. Then*

$$\frac{\sin(ax)}{ax} + b\frac{\sin(x)}{x} > 1 + b\cos(x).$$

2. *Let $p \geq 0$, $q \geq \max(\sqrt{2}, \sqrt{2p})$, and $x \in \mathbb{R}$. Then*

$$\frac{\sinh(qx)}{qx} + p\frac{\sinh(x)}{x} > 1 + p\cosh(x).$$

Proof. **(1)** Define

$$\varphi_{a,b}(x) = \sin(ax) + ab\sin(x) - abx\cos(x) - ax.$$

Then $\varphi_{a,b}(0) = 0$, and the desired inequality is equivalent to $\varphi_{a,b}(x) > 0$ for $x \in (0,\pi/2)$.

Differentiating, we obtain

$$\varphi'_{a,b}(x) = a\big(\cos(ax) + bx\sin(x) - 1\big) = a\big(bx\sin x - 2\sin^2(\tfrac{ax}{2})\big).$$

Using the assumption $2b \geq a^2$, we get

$$\varphi'_{a,b}(x) \geq a\Big(\tfrac{a^2}{2} x \sin x - 2\sin^2(\tfrac{ax}{2})\Big).$$

Set

$$h(a) = \tfrac{a^2}{2} x \sin x - 2\sin^2\big(\tfrac{ax}{2}\big)\,.$$

Then

$$h'(a) = x\big(a \sin x - \sin(ax)\big).$$

For $a \geq 1$ and $x \in (0, \pi/2)$, it is well known that the function

$$t \mapsto \frac{\sin t}{t}$$

is strictly decreasing on $(0, \pi/2)$; see, for example, [5, 12]. Therefore, for $a \geq 1$ and $x \in (0, \pi/2)$, we have

$$\frac{\sin(ax)}{ax} \leq \frac{\sin x}{x},$$

which implies

$$a \sin x - \sin(ax) \geq 0.$$

Therefore $h'(a) \geq 0$, so $h(a)$ is increasing in a. Since $h(1) = 0$, we obtain $h(a) \geq 0$ for all $a \geq 1$.

Consequently, $\varphi'_{a,b}(x) \geq 0$ on $(0, \pi/2)$, so $\varphi_{a,b}$ is increasing. Since $\varphi_{a,b}(0) = 0$, we conclude that $\varphi_{a,b}(x) > 0$ for $x \in (0, \pi/2)$.

(2) Define

$$\psi_{p,q}(x) = \sinh(qx) + pq \sinh(x) - pqx \cosh(x) - qx.$$

For $x > 0$, the inequality is equivalent to

$$pq \leq \frac{\sinh(qx) - qx}{x \cosh(x) - \sinh(x)} =: \frac{u(x)}{v(x)}.$$

We study the monotonicity of u/v. We compute

$$\frac{u'(x)}{v'(x)} = q\, \frac{\cosh(qx) - 1}{x \sinh(x)} =: q\, H_q(x).$$

A direct computation gives

$$(\log H_q(x))' = q \coth\big(\tfrac{qx}{2}\big) - \coth(x) - \frac{1}{x}.$$

The function $q \mapsto q \coth(\frac{qx}{2})$ is increasing for $x > 0$, hence for $q \geq \sqrt{2}$,

$$(\log H_q(x))' \geq (\log H_{\sqrt{2}}(x))' = \sqrt{2} \coth\Big(\tfrac{x}{\sqrt{2}}\Big) - \coth(x) - \frac{1}{x}.$$

Using the expansion (see, e.g., [11])

$$\coth t = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 + n^2\pi^2},$$

we obtain

$$(\log H_{\sqrt{2}}(x))' = 2x^2 \sum_{n=1}^{\infty} \left(\frac{1}{\frac{x^2}{2} + n^2\pi^2} - \frac{1}{x^2 + n^2\pi^2} \right).$$

Then

$$(\log H_q(x))' > 0 \quad \text{for all } x > 0.$$

Thus H_q and u'/v' are increasing, and since $u(0) = v(0) = 0$, by L'Hôpital monotone rule it follows that u/v is increasing on $(0, \infty)$, see, for example, [2, 6].

Finally,

$$\lim_{x \to 0} \frac{u(x)}{v(x)} = \frac{q^3}{2}.$$

Hence the inequality holds for all $x > 0$ if and only if $pq \leq \frac{q^3}{2}$, i.e. $q \geq \sqrt{2p}$. The condition $q \geq \sqrt{2}$ ensures the required monotonicity.

The case $x < 0$ follows by oddness. □

4 Conclusion

The behavior of trigonometric functions along complex circles reflects a subtle interplay between oscillatory and exponential phenomena. Indeed, when $z = re^{i\theta}$, the real part $r\cos\theta$ governs oscillations, while the imaginary part $r\sin\theta$ induces hyperbolic growth. As a result, quantities such as $|\sin z|$ and $|\cos z|$ exhibit a mixed structure combining trigonometric and hyperbolic components.

The phase transition observed at the critical value r_0 corresponds to a balance point where these two competing effects have comparable influence. For $r < r_0$, oscillatory behavior dominates, while for $r > r_0$, exponential growth becomes predominant. This transition explains the qualitative change in monotonicity and extremal behavior of the functions studied.

5 Open Problem

The weighted inequalities established in Proposition 3.2 provide sufficient conditions ensuring the validity of double-sided trigonometric and hyperbolic inequalities. In particular, the conditions

$$2b \geq a^2, \qquad a \geq 1,$$

and

$$q \geq \max(\sqrt{2}, \sqrt{2p})$$

guarantee that the corresponding inequalities hold for all admissible x.

However, these conditions are not known to be optimal in full generality.

Problem. Determine the exact necessary and sufficient conditions on the parameters (a, b) and (p, q) such that the inequalities

$$\frac{\sin(ax)}{ax} + b\frac{\sin x}{x} > 1 + b\cos x \quad \text{for all } x \in (0, \pi/2),$$

and

$$\frac{\sinh(qx)}{qx} + p\frac{\sinh x}{x} > 1 + p\cosh x \quad \text{for all } x \in \mathbb{R},$$

hold.

In particular, it is unknown whether the sufficient conditions

$$2b \geq a^2, \quad a \geq 1, \qquad \text{and} \qquad q \geq \max(\sqrt{2}, \sqrt{2p})$$

are also necessary, or whether sharper threshold relations exist.

A deeper question is whether the optimal admissible region in the parameter space admits a geometric characterization, possibly in terms of convexity or monotonicity properties of associated auxiliary functions.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1979.

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Monotonicity rules in calculus*, Amer. Math. Monthly 113 (2006), 805–816.

[3] Y. J. Bagul and C. Chesneau, *Two double sided inequalities involving sinc and hyperbolic sinc functions*, Int. J. Open Problems Comput. Math. **12**(4) (2019).

[4] J. B. Conway, *Functions of One Complex Variable I*, Springer-Verlag, New York, 1978.

[5] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[6] R. G. E. Pinch, *Monotonicity properties of the ratio of functions*, Math. Gazette 77 (1993), 208–211.

[7] F. Qi, *On bounds for norms for sine and cosine along a circle on the complex plane*, Kragujevac J. Math. **48**(2) (2024), 255–266.

[8] F. Qi, *On bounds of the sine and cosine along straight lines on the complex plane*, Acta Univ. Sapientiae Math. **11**(2) (2019), 371–379.

[9] F. Qi and B.-N. Guo, *A criterion to justify a holomorphic function*, Glob. J. Math. Anal. **5**(1) (2017), 24–26.

[10] R. Redheffer, *Problem 5642*, Amer. Math. Monthly 76 (1969), 422.

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1927.

[12] L. Zhu, *Sharpening some inequalities related to the sinc function*, J. Math. Inequal. 3 (2009), 317–324.