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On Certain properties of Finite Rank Operators and Bishop-Phelps-Bollobas Criterion in Banach Spaces

Orina Moraa, Benard Okelo, Priscah Omoke

School of Biological, Physical, Mathematics and Actuarial Sciences,
Jaramogi Oginga Odinga University of Science and Technology,
Box 210-40601, Bondo-Kenya.
e-mail: benard@aims.ac.za

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Abstract

For bounded linear operators, it is observed that, Bishop-Phelps-Bollobas criterion has not been studied when operators are of finite rank exhaustively. Therefore, it is an area of interest in the concept of denseness of norm achieving mappings to determine whether every finite rank operator between Banach spaces can be estimated by those that achieve their norms. Hence, it has not been shown whether the set of norm achieving finite rank operators is dense in the whole space of mappings of finite rank. We analyzed this properties particulary norm-attainability. We showed that rank one mappings achieve their norms on certain Banach lattices. Since finite rank operators are obtained by summing rank one mappings, our results seeks to clarify the behavior of such operators especially in relation to norm-attainability and operator structure with regard to Bishop-Phelps-Bollobas property.

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1 Introduction

Finite rank operators have been studied by various researchers in terms of solvability, decomposability, compactness, continuity and Bishop-Phelps-Bollobas property (BPBp). In [1] the authors studied the relation between compactness, finite rank, the located kernel for a linear transformation into a normed space with finite dimension. It was shown that linear mappings in normed spaces with finite dimension are completely continuous given that their nullspace can be obtained. The authors gave examples showing the differences between a mapping into a space with finite dimension and a mapping with finite dimensional range. In [2], compactness of linear mappings was characterized on Banach spaces but in this note we characterize finite rank mappings on Banach spaces. Also [3] characterizes compactness of linear mappings on Hilbert spaces. It was shown that linear mapping are compact if they have a uniform limit and given that they posses a finite dimensional range but in this work we consider mapping with finite rank.

The study by [4] demonstrated that operators of finite rank are solvable indicating that they have solutions to certain equations on Hilbert spaces. Conditions were given for solvability for finite matrices and for self adjoint mappings. The study was further extended to normal mappings and mappings with finite rank on Banach spaces. Further study by [5] shows that finite rank operators were proved to be solvable but did not characterize denseness of finite rank operator. The solvability of finite rank operators on complex Hilbert spaces was characterized, but the study was limited to estimation of closeness of finite rank mappings. Also, the author characterized solvability of bounded linear operators but in [6] characterized denseness of operators that are of finite rank finite rank.

The study by [7] showed that there exist mappings of finite rank on the algebra of continuous linear operators, this enabled the Authors in defining spectral properties of Riez operators. Later, the Authors illustrated that operators of finite rank are decomposable. The author discussed when the subalgebra of operator of finite rank is non-zero, and provided an example showing that this subalgebra can be zero for finite structures that are characterized by ordered elements or points. The author in [8] then gave conditions for expressing a mapping of finite rank F into a sum of a mapping with a rank of one and a mapping with a smaller range. In [9] the work characterized the existence of mappings of finite rank on a normed linear space but this has not been done in sequence spaces of Banach spaces. In [10], they characterized finite rank mappings that attain their norm in Banach spaces. Also in [11] it was discussed that there exist a finite rank mapping on the algebra of A_L but the study was limited to sequence spaces of Banach spaces. Moreover, it was proved that finite rank mappings on nest algebras are decomposable but in our study we shall extend this result to sequence spaces.

The work of [12] characterized finite rank operators on Hilbert modules. Their main aim was to find out if $f(K)$ is of finite rank given that K is a completely continuous normal mapping. From [13], it was shown that completely continuous normal mappings can be mapped to normal finite rank operators but the study was limited to $BPBp$ for finite rank operators. In [14] compact mappings of finite rank were characterized on Hilbert modules but the study was limited to sequence spaces of Banach spaces. In [15] completely continuous mappings were characterized on abelian C^* algebra of Hilbert modules but in our study we extend this result to sequences spaces of Banach spaces.

The study by [16] characterized decomposability of finite rank mappings on Banach algebras. Some conditions were given for finite rank operators to be decomposable. Conditions were obtained such that an operator T has a rank one summand in a reflexive algebra X and conditions showing that X is strongly decomposable were given. In [17], rank one mappings have been characterized in Banach algebras but [18] they studied we shall extend this result to sequence spaces. Also in [19] finite rank mappings were characterize in Banach Algebras and not in sequence spaces. Lastly, in [20] decomposability of finite rank mappings have been characterized in subspace lattices on Hilbert spaces but the study was limited to sequence spaces of Banach spaces.

In [21] they characterized mappings with finite dimensional range in terms of decomposability in commutative lattice subspace algebra. In was shown that linear mappings with a rank of two are decomposable provided that they only have finitely many correlation coefficients. In [22] decomposability of rank two mappings was characterized on Banach algebras but [23] extended this result to sequence spaces of Banach spaces. From [24] it was shown that rank one mappings exist on Banach Algebras but [25] extended this result to sequence spaces. In [26] decomposability of finite rank mappings was characterized on a Banach Algebra but [27] extended this result to sequence spaces of Banach spaces.

In [28], they discussed mappings of finite rank in a triangular algebra. Decomposability of these operators was also discussed. In [29] they characterized the existence finite rank operators in irreducible triangular algebra, [30] considered those finite rank operators and determine whether they attain their norms in order to characterize their density. Finally, in [31] the study characterizes the existence of finite operators on a closed maximal triangular algebra but in our study we shall characterize the numerical radii of finite rank mappings.

The work of [32] characterized finite rank operators in μ-ideals. In [33] finite rank operators were characterized in μ-ideals but in our study we shall characterize finite rank operators on Banach spaces and we shall estimate the closeness between them on Banach spaces. Also in [34] finite rank operators were characterized in μ-ideals but [35] extended this result to sequence spaces of Banach spaces.

In [36] the author characterized commutativity of finite rank operators using Eigen Matrices. It was also examined that Toeplitz mappings do not commute with transformations whose range is of finite dimension. In [36] they described the commutativity of linear mappings but [37] considered mappings of finite rank. In Corollary [38] conditions under which a bounded linear mapping commutes with a rank one mapping but in our study we shall consider characterization of mappings of finite rank on Banach spaces. In [39] commutativity between linear mappings and mappings with rank one was characterized but [40] extended this result to sequence spaces of Banach spaces. It was shown in [41] that Toeplitz mappings do not commute with mappings with finite dimensional range but in this study we extended this result to sequence spaces of Banach spaces.

In [42] characterized finite rank operators in ideals of a nest algebra. This was achieved through decomposition method and it was emphasized in the following results. In [43] they showed that there exist finite rank transformations in a nest algebra. The author used the method of tensor products in the characterization but [44] employed other methods such as polar decomposition. In [45] the work was extended to characterization rank one operators in a nest algebra but the result was limited to other Banach spaces such as l_p spaces hence [46] extended this result to function spaces such as l_p spaces. Moreover, the study characterized operators of finite rank in lie ideas but our study considered l_p and determine whether there exist finite rank operators in those spaces.

The authors of [47] characterized the denseness of finite rank mappings of p-compact mappings on complete vector spaces. As a result the k_p-approximation property of complete vector spaces was characterized through the density of finite rank mapping and through this, the relationship the k_p-approximation property and Saphar's approximation property was obtained. In [48], approximation property was characterized for finite rank mappings in the space of p-completely continuous mappings but [49] extended this result to sequence spaces of Banach spaces. In [50], approximation property was characterized on Lebesque spaces but [51] investigated whether finite rank mappings can be estimated by those that attain their norms on Banach spaces.

The work of [52] proved that p-compact mappings can be estimated by norm achieving finite rank mappings. Hence the author proved that p-compact operators have the approximation property. In [53] Banach spaces were characterized, it was discussed that complete normed vector spaces posses approximation property provided that it has (p, p)-approximation property but the study was limited to finite rank mappings. In [54], approximation property was characterized in Lebesgue spaces but [55] estimated finite rank operators.

In [56] the authors provided conditions under which an adjoint mapping on a complete normed vector space is of finite rank. Hence, Riesz operators were characterized between Banach spaces and it was shown that they have a finite dimensional range implying that they are of finite rank. Again [57] showed that a Riesz mappings can be mapped to a codimension with finite dimension implying that they have a finite rank but the study was limited to characterization of properties of mappings of finite rank on Banach spaces. Also [58] in assumptions were given under which Riesz mappings can be considered to be of finite rank but our study shall determine the closeness of finite rank mappings whether mappings of finite rank are dense in the dual space and by use of tensors products the author proved that a Riesz mappings on a C^*-algebra are said to be of finite rank if they preserve distinctness between Banach spaces. In our study we further characterize mappings with finite dimensional rank by use of other methods such as known inequalities.

Lausten and Raubenheimer [44] discussed conditions under which a Riezs operator is a finite rank operator. In [8] the author proved that their exist Reizs operators with a finite ascent but in our study we shall characterize finite rank mappings on Banach spaces. Also [12] characterizes Riezs mappings but [9] considered mappings with finite dimensional range. In [27] it was shown that a Riezs operator is of finite rank if it is normal but [2] characterized finite rank mappings on Banach spaces.

Bachir *et al* [11] showed that linear mappings are differentiable implying that their derivative is a constant function nd that the mappings have a constant slope, the author later proved a similar result for mappings of finite rank and in this this way Banach-Stone-representation theorem which relates to the representation of isometric isomorphism was obtained. From [34]in Theorem 3.14 showed the relation between compactness and boundedness of an operator. In our study the notion of compactness and boundedness shall help us in determining continuity for finite rank operators. Also linear transformations were characterizes in terms of compactness but in our study we shall prove whether this linear mappings have a finite rank and we shall investigate their density in Banach spaces.

Kadet *et al* [40] showed that there exist mappings of finite rank that achieve their norms on Banach spaces, conditions were given for their existence on a Banach space Q especially those mappings whose range space is of dimension two. One of the conditions they came up with was the existence of a cone that is not trivial comprising of continuous functionals that achieve their norms on Q. The authors further discussed denseness of rank-two mappings that achieve their norms, which is true for a dense linear subspace comprising of functionals that achieve their norms on Q. The authors considered mappings on Hilbert spaces where these properties were completely characterized and obtained. From Proposition 2.5 in [10] it was shown that those mappings that are characterize by compactness are norm-attainable but in our study we shall determine if mappings of finite rank can be approximate by those that attain their norms. This is further emphasized by the Corollary below:

Corollary 1.1 ([40], Corollary 2.6) *Consider Banach spaces W and Z and consider $J \in U(W, W)$ as a compact operator. If $[ker J]^{\perp} \subset NA(X)$, then J has norm attainment.*

Corollary 1.1 characterizes norm attainability of compact operators but [5] characterized finite rank operators on Banach spaces. Finally, [4] showed that every operator which is compact can be estimated by those that achieve their norms but the study was limited to denseness of finite rank operator.
Otieno, Ambogo and Nyamwala [49] characterized continuity of finite rank operator on Hilbert spaces. It was lastly shown that finite rank mappings are Cauchy. The following are some of their results:

Proposition 1.2 ([49], Proposition 3.1) *Suppose $P \in N(H)$ is a mapping with finite dimension. Then P is continuous if its mapping norm is finite.*

Proposition 1.2 characterizes continuity of finite rank mappings but the result was limited to density of finite rank operators. The continuity of finite rank operators was characterized in relation to its adjoint but our study shall consider the use of known inequalities in the characterization of finite rank mappings. Also in [33] finite rank mappings are characterized in terms of continuity but the result was limited to sequence spaces. Finally in [21] the relation between continuity and boundedness of mappings with finite rank was investigated but the result was limited to density of finite rank mappings.

2 Preliminaries

This section provides basic concepts and definitions that will be useful throughout the study.

Definition 2.1 ([36]) *Let Q and W be Banach spaces and $P : Q \to W$ be a linear transformation. Then P is a* fro *if its range is of finite dimension.*

Definition 2.2 ([10]) *Let Q be a BS, then if $\epsilon > 0$ and $r^* \in Q^*$ there is $r_o^* \in Q^*$ such that $\|r_o^*(r_o)\| = \|r_o^*\|$ and $\|r_o^* - r^*\| < \epsilon$.*

Definition 2.3 *A real normed Riesz space Ω has the Hereditary norm attaining property if for $(q, q^\star) \in S_\Omega \times S_{\Omega^\star}$, $q^\star(q) = \|q^\star\|\|q\| \Rightarrow q^\star(q^+) = \|q^{\star+}\|\|q^+\|$ and $q^{\star-}(q^-) = \|q^{\star-}\|\|q^-\|$.*

Definition 2.4 *Let Ω be a Banach space, $w \in S_\Omega$ is said to be an extreme point of $L(\Omega)$ if $w = \gamma u + (1-\gamma)v$, for $u, v \in B(\Omega)$ and $\gamma \in (0,1)$ implying $u = v$. This is denoted as $w \in extL(\Omega)$.*

3 Research methodology

In this section, we discuss the methods that we will use to successfully obtain our objectives. The methodology involves the use of some inequalities and technical approaches of tensor products and direct sum decomposition. We will also use Property β of Lindenstrauss and Approximate Hyperplane Series Property in characterizing properties for finite rank operator. Certain known inequalities and equalities that are instrumental in obtaining norm estimates of finite rank operator are presented as follows.

3.1 Cauchy-Schwarz inequality

Suppose $(X, \langle ., . \rangle)$ is an inner product space over a field $\mathcal{F}$ then [36], for all $e, f \in X$, $|\langle e, f \rangle| \leq \|e\|\|f\|$.

3.2 Triangle inequality

For all $e, h \in \mathcal{R}$, $\|e + h\| \leq \|e\| + \|h\|$.

3.3 Hölder's inequality for sequences

Let $(g_n) \in l_p$ and $(h_n) \in l_q$, where $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then [36],

$$\sum_{k=1}^{\infty} |g_k h_k| \leq (\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |h_k|^q)^{\frac{1}{q}}.$$

3.4 Minkowski's inequality for sequences

Let $p > 1$ and $(g_n), (h_n)$ be sequences in l_p. Then [36],

$$(\sum_{k=1}^{\infty} |g_k + h_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |h_k|^p)^{\frac{1}{p}}.$$

3.5 Parallelogram law

Suppose X is an inner product with induced norm $\|.\|$ then [52], $\|j + q\|^2 + \|j - q\|^2 = 2(\|j\|^2 + \|q\|^2)$ for all $j, q \in X$.

3.6 Polarization identity

Suppose X is an inner product with induced norm $\|.\|$ then [52], if X is real
$4\langle j, p \rangle = \|j + p\|^2 + \|j - p\|^2$ while if X is complex
$4\langle j, p \rangle = \|j + p\|^2 + \|j - p\|^2 + i\|j + p\|^2 + i\|j - ip\|^2$.

The above identities played a key role in analyzing properties of finite rank operator especially in norm or nr calculation. Next, we discuss technical approaches and they include Direct sum decomposition and tensor products.

3.7 Direct sum decomposition

Suppose P and Q are subspaces of W. Then [36], W is said to be the direct sum of P and Q if $W = P+Q$ and $P \cap Q = 0$ and we denote this by $W = P \oplus Q$. This was used to reduce $BPBp$ and $BPBp - nr$ for fro from the entire space to individual component spaces.

3.8 Tensor products

Given $Q : W \to W$ and $R : Z \to Z$ [52], there exist an operator $Q \otimes R : W \otimes Z \to W \otimes Z$ such that $(Q \otimes R)(w \otimes z) = Q(w)R(z)$.
Tensor product approach enabled us to study the properties fro. It was instrumental in providing a representation of fro through the relationship between vectors and functionals. It also played a key role in nr analysis of fro.

3.9 Property β (of Lindenstrauss)

A BS Z satisfies property β if there exist two sets $\{z_k : k \in K\} \subset S_z$, $\{z_{k^*} : k \in K\} \subset S_{z^*}$ and $\rho \in (0, 1)$ satisfying the conditions listed below. [54]

(i). $z_{k^*}(z_k) = 1$.

(ii). $|z_{k^*}(z_r)| \leq \rho < 1$ if $k \neq r$.

(iii). $\|z\| = sup_{k \in K}|z_{k^*}(z)|$, for all $z \in Z$.

3.10 Approximate Hyperplane Series Property (AHSP)

A BS Y satisfies $AHSP$ if for each $\epsilon > 0$ there exists $0 < \eta < \epsilon$ such that for each $(x_k) \subset S_Y$ and every convex series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\|\Sigma_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta(\varepsilon)$$

there exist a subset $G \subset \mathcal{N}$ and a subset $\{z_k : k \in G \subset S_Y\}$ satisfying [55]

(i). $\sum_{k=1}^{\infty} \alpha_k > 1 - \eta(\varepsilon)$,

(ii). $\|z_k - x_k\| < \epsilon$ for $k \in G$,

(iii). $y^*(z_k) = 1$, $y^* \in S_{Y^*}$ for $k \in G$.

The above geometrical properties served as a basis for studying the $BPBp$ by providing functionals that almost achieve their norms which is essential in proving $BPBp$ results.

4 Main results

We provide new orthogonality relations in this section. We begin with the following proposition.

Proposition 4.1 *Let Ω be a normed Riesz space. Let $(r_o, p^*) \in S_\Omega \times S_{\Omega^*}$ and $p^*(r_o) = 1$, then $|p^*||r_o|$=1. Therefore, $p^{*+}(r_o^-) = p^{*-}(r_o^+) = 0$ and $\|p^{*+}\|\|r_o^+\| + \|p^{*-}\|\|r_o^-\| \leq 1$. Hence, Ω has the hereditary norm attaining property.*

Proof. Since $(r_o, p^*) \in S_\Omega \times S_{\Omega^*}$ we have,

$$\begin{aligned}
1 &= p^*(r_o) \\
&= (p^{*+} - p^{*-})(r_o^+ - r_o^-) \\
&\leq p^{*+}(r_o^+) + p^{*-}(r_o^-) \\
&\leq (p^{*+} + p^{*-})(r_o^+ + r_o^-) \\
&= |p^*||r_o|
\end{aligned}$$

Therefore, $|p^*||r_o| = 1$ and $p^{*+}(r_o^-) = p^{*-}(r_o^+) = 0$.
We now prove that $\|p^{*+}\|\|r_o^+\| + \|p^{*-}\|\|r_o^-\| \leq 1$. We first check if the condition in Definition 2.3 holds. Take $(r_o, p^*) \in S_\Omega \times S_{\Omega^*}$ such that $p^\star(r_o) = 1$. Then from this assumption we have,

$$\begin{aligned}
1 &= p^*(r_o) \\
&= (p^{*+} - p^{*-})(r_o^+ - r_o^-) \\
&\leq p^{*+}(r_o^+) + p^{*-}(r_o^-) \\
&\leq \|p^{*+}\|\|r_o^+\| + \|p^{*-}\|\|r_o^-\|
\end{aligned}$$

Therefore, $p^{*+}(r_o^+) = \|p^{*+}\|\|r_o^+\|$ and $p^{*-}(r_o^-) = \|p^{*-}\|\|r_o^-\|$ implying that Ω has Hereditary norm attaining property.
In the following result, we give equivalent conditions on norm attainability.

Lemma 4.2 *The following are equivalent for every Banach lattice Ω and and for every $g^\star \in L(\Omega^\star)$.*

(i). $g^\star$ is norm-attainable

(ii). Both $g^{\star+}$ and $g^{\star-}$ are norm-attainable

(iii). $g^{\star+}$ or $g^{\star-}$ is norm one

Proof. $(i) \Rightarrow (ii)$. Consider $g^* \in \Omega^\star$ for which $g^\star(r_o) = \|g^\star\| = 1$. Let $r_o = e_o - f_o$. We show that

$$\|g^\star\| = g^\star(r_o) = g^{\star+}(e_o) + g^{\star-}(f_o) - g^{\star+}(f_o) - g^{\star-}(e_o)$$

$$\begin{aligned} &\leq \|g^{\star+}\|\|e_o\| + \|g^{\star-}\|\|f_o\| \\ &\leq \|g^{\star}\|(\|e_o\| + \|f_o\|) \\ &= \|g^{\star}\|\|r_o\| \\ &= \|g^{\star}\| = 1. \end{aligned}$$

Since $e_o = r_o^+$ and $f_o = r_o^-$ then, from Proposition 4.1 we have that $g^{\star+}(f_o) = g^{\star-}(e_o) = 0$. Hence, both g^{*+} and g^{*-} attain their norms.
(ii) $\Rightarrow$ *(iii)*. Let both g^{*+} and g^{*-} be norm-attainable. Then, for every $e_o, f_o \in S_\Omega$ we have, $\|g^{*+}e_o\| = \|g^{*+}\|$ and $\|g^{*-}f_o\| = \|g^{*-}\|$. Assume that neither g^{*+} nor g^{*-} has a norm one. Then $\|g^*\| \neq 1$. If $\|g^*\| \neq 1$, then g^* cannot be norm-attainable which is a contradiction since g^* is norm-attainable. Hence, atleast either g^{*+} or g^{*-} is norm one.
(iii) $\Rightarrow$ *(i)*. Let $g^{\star+}$ be norm one and norm-attainable. Then $g^{\star+}(r_o) = \|g^{\star+}\| = 1$. We show that $g^{\star-}(r_o) = 0$. Indeed

$$\begin{aligned} g^{\star+}(r_o) + g^{\star-}(r_o) &\leq |g^{\star}||r_o| \leq \|g^{\star}\| = 1 \\ 1 + g^{*-}(r_o) &\leq 1 \end{aligned}$$

which implies that $g^{\star-}(e_o) = g^{\star+}(f_o) = 0$. Hence, $g^{\star}(r_o) = g^{\star+}(r_o) = \|g^{\star}\| = 1$. Next we present a theorem which gives an estimate on the distance between two operators.

Theorem 4.3 *Let $\Omega \in AL$ and $\varepsilon > 0$. Let $\varphi \in S_\Omega$ and $\varphi^* \in S_{\Omega^*}$ be positive such that $\varphi^*(\varphi) > 1 - \frac{\varepsilon^2}{2}$, then there are elements $\alpha \in \Omega \cap S_\Omega$ and $\alpha^* \in \Omega^* \cap S_{\Omega^*}$ satisfying the following:*

(i). $\alpha^*(\alpha) = 1$

(ii). $\|\alpha - \varphi\| < \varepsilon$

(iii). $\|\alpha^* - \varphi^*\| < \varepsilon$.

Proof. From BPB, there exist elements $v \in S_\Omega$ and $v^* \in S_{\Omega^*}$ for which $v^*(v) = 1$, $\|v - \varphi\| < \varepsilon$ and $\|v^* - \varphi^*\| < \varepsilon$. For (i), let $\alpha \in \Omega \cap S_\Omega$ and $\alpha^* \in \Omega^* \cap S_{\Omega^*}$. Suppose that $\alpha = \lambda\varphi$ and $\alpha^* = \lambda\varphi^*$ where $\lambda = \frac{1}{\varphi^*(\varphi)}$. Then, $\alpha(\alpha^*) = \lambda\varphi(\lambda\varphi^*) = \lambda^2\varphi(\varphi^*) = \lambda^2\varphi^*(\varphi) = 1$. Hence, from Proposition 4.1 we have $\alpha(\alpha^*) = 1$.
For (ii), Let $\alpha \in \Omega \cap S_\Omega$ and $\alpha = \lambda\varphi$ where $\lambda = \frac{1}{\varphi^*(\varphi)}$. Let $\varphi \in S_\Omega$, then $\|\varphi\| = 1$. Therefore, $\|\alpha\| = |\lambda|\|\varphi\| = \frac{1}{\varphi^*(\varphi)} < \frac{2}{2-\varepsilon^2}$. Indeed $\|\alpha - \varphi\| = \|\frac{\varphi}{\varphi^*(\varphi)} - \varphi\| = \|\frac{\varphi - \varphi(\varphi^*)\varphi}{\varphi^*(\varphi)}\|$. Since $\varphi^*(\varphi) > 1 - \frac{\varepsilon^2}{2}$ we have $\|\varphi - \varphi\varphi^*(\varphi)\| < \varepsilon$. Thus $\|\alpha - \varphi\| < \varepsilon$.
We now prove (iii), suppose $\alpha^* \in \Omega^* \cap S_{\Omega^*}$ and let $\alpha^* = \lambda\varphi^*$ where $\lambda = \frac{1}{\varphi^*(\varphi)}$. Then $\|\alpha^* - \varphi^*\| = \|\lambda\varphi^* - \varphi^*\| = |\lambda - 1|\|\varphi^*\|$. Since $\|\varphi^*(\varphi)\| > 1 - \frac{\varepsilon^2}{2}$. We obtain

$|\frac{1-\varphi^*(\varphi)}{\varphi^*(\varphi)}| < \varepsilon$. Thus $\|\alpha^* - \varphi^*\| < \varepsilon$. Hence, the elements α and α^* satisfies the required conditions.
It the following result, we show that the negative and positive parts of r are strictly less than ε if r itself is less than ε.

Corollary 4.4 *If Ω is a Banach Lattice and $r \in \Omega^*$, then for every $\varepsilon > 0$ a subspace S of Ω exists for which $\Omega = S + S^\perp$ and $\|r^+|_S\| < \varepsilon$ and $\|r^-|_{S^\perp}\| < \varepsilon$.*

Proof. Suppose $m \in S_\Omega$ for which $r(m) + \varepsilon > \|r\|$. Let $\Omega = S + S^\perp$. Since $r = r^+ - r^-$ and $m \in S_\Omega$. From proposition 4.1 we have

$$\begin{aligned} r(m) + \varepsilon &= r^+|_S(m) + r^+|_{S^\perp}(m) - r^-|_S(m) - r^-|_{S^\perp}(m) + \varepsilon \\ &> \|r^+|_S\| + \|r^+|_{S^\perp}\| + \|r^-|_S\| + \|r^-|_{S^\perp}\| \\ &= \|r^+\| + \|r^-\| = \|r\| \end{aligned}$$

Since $\|r^+\| = \|r^+|_S\| + \|r^+|_{S^\perp}\|$ and $\|r^-\| = \|r^-|_S\| + \|r^-|_{S^\perp}\|$. We have, $\|r^+|_S\| = \|r^+\| - \|r^+|_{S^\perp}\|$ and $\|r^-|_{S^\perp}\| = \|r^-\| - \|r^-|_S\|$. Adding the two we obtain,

$$\begin{aligned} \|r^+|_S\| + \|r^-|_{S^\perp}\| &= \|r^+\| - \|r^+|_{S^\perp}\| + \|r^-\| - \|r^-|_S\| \\ &< r^+|_S(m) - r^-|_{S^\perp}(m) + \varepsilon < \varepsilon \end{aligned}$$

The following result provides a necessary condition for the norm attainment of functionals.

Proposition 4.5 *Suppose $\Omega \in AM$ is order complete, then $\alpha \in \Omega^\star$ attains is norm if and only if a subspace S of Ω exists for which $\alpha^+ = \alpha|_S$, $\alpha^- = -\alpha|_{S^\perp}$.*

Proof. $\Longrightarrow$ Let α be norm-attainable, we show that a subspace S of Ω exists for which $\alpha^+ = \alpha|_S$, $\alpha^- = -\alpha|_{S^\perp}$. Since α is norm-attainable, there exists $m \in L(\Omega)$ for which $\|\alpha(m)\| = \|\alpha\|$. Let $\Omega = S + S^\perp$ and $m^+ \in S$, $m^- \in S^\perp$. From Corollary 4.4, it follows that $\|\alpha\| = \|\alpha|_S\| + \|\alpha|_{S^\perp}\|$. Now, to prove that $\alpha^+ = \alpha|_S$, and $\alpha^- = -\alpha|_{S^\perp}$, we show that $\alpha|_S \geq 0$ and $-\alpha|_{S^\perp} \geq 0$. Clearly, if there is $k \geq 0 \in L(\Omega)$ for which $\alpha|_S(k) < 0$ then we consider $k \in S$. So, $v = -m^- - k$ and $\|v\| = max\{\|m^-\|\|k\|\} = 1$. Thus, $\|\alpha^-\| \geq \alpha^-(-v) = \alpha(v) - \alpha^+(v) \geq \alpha(v) = \alpha|_{S^\perp}(-m^-) - \alpha|_S(k) > \alpha|_{S^\perp}(-m^-) = -\alpha|_{S^\perp}(m)$ since $\|\alpha^+\| \geq \alpha(m|_S) = \alpha|_S(m)$ which is a contradiction to $\|\alpha\| = \|\alpha^+\| \geq \alpha|_S(m) - \alpha|_{S^\perp}(m) = \alpha(m) = \|\alpha\|$. Hence, also $-\alpha|_{S^\perp} \geq 0$.
$\Longleftarrow$ Let S be a subspace of Ω such that $\alpha^+ = \alpha|_S$, $\alpha^- = -\alpha|_{S^\perp}$, then we show that α is norm-attainable. Suppose that a subspace S of Ω exists for which $\alpha^+ = \alpha|_S$, $\alpha^- = -\alpha|_{S^\perp}$. Let $m, k(\geq 0) \in L(\Omega)$ such that $\|\alpha(m)\| = \|\alpha^+\|$ and $\|\alpha^-(k)\| = \|\alpha^-\|$. Since $\alpha^+ = \alpha|_S$ and $\alpha^- = -\alpha|_{S^\perp}$, we take $m \in S$ and $k \in S^\perp$ and $v = m - k$. Then we have, $\|v\| = \|m - k\| = max\{\|m\|, \|k\|\} = 1$. Hence,

we obtain $\|\alpha\| = \|\alpha^+\| + \|\alpha^-\| = \alpha^+(m) - \alpha^-(k) = \alpha|_S(m) - \alpha|_{S^\perp}(k) = \alpha(v)$

In the next result, we give the necessary condition for a point to lie in the extreme point of a given set.

Lemma 4.6 *Suppose $\Omega \in AM$ is $\sigma-$complete and $\alpha \in D(\Omega^*)$. Then $\alpha \in extL(\Omega^*)$ if and only if $\alpha(m) \cdot \alpha(l) = 0$ for each $m, l \in \Omega$ satisfying $m \wedge l = 0$.*

Proof. $\Longrightarrow$ If $\alpha \in$ext$L(\Omega^*)$, then we show that $\alpha(m) \cdot \alpha(l) = 0$ for each $m, l \in \Omega$. Let $m, l \in \Omega$ for which $m \wedge l = 0$. Then we define $\Omega = S + S^\perp$. Let $p = \alpha|_S$ and $q = \alpha|_{S^\perp}$. Then p and q are both nonnegative since $m \in S$ and $l \in S^\perp$. Consequently, $\alpha = \{\|p\| \frac{p}{\|p\|} + \|q\| \frac{q}{\|q\|}\}$ and $\|p\| + \|q\| = \|\alpha\| = 1$. Hence $\alpha \in$ext$L(\Omega^*)$. Assume that $m, l \in \Omega$ with $m \wedge l = 0$ and $\alpha(m) \cdot \alpha(l) \neq 0$. Then $\alpha(m)$ and $\alpha(l)$ are nonzero since $\alpha(m) \cdot \alpha(l) \neq 0$. This implies that $\alpha \notin L(\Omega^*)$. This is a contradiction since $\alpha \in extL(\Omega^*)$ hence $\alpha(m) \cdot \alpha(l) = 0$ whenever $m \wedge l = 0$.
$\Longleftarrow$ If $\alpha(m) \cdot \alpha(l) = 0$ for each $m, l \in \Omega$, then $\alpha \in extL(\Omega^*)$. First we show that $\alpha(m) \cdot \alpha(l) = 0$ ie $\|\alpha^+\| \|\alpha^-\| = 0$. From Corollary 4.4, there exist two orthogonal subspaces S and $S^\perp$ of Ω such that $\Omega = S + S^\perp$ and $\|\alpha|_S\| < \varepsilon$, $\|\alpha|_{S^\perp}\| < \varepsilon$. Take $m \in L(\Omega)$ such that $\alpha(m) > \|\alpha\| - \varepsilon$ and let $m = j - k$ in which $j \in S$ and $k \in S^\perp$. Then $\alpha(j) \cdot \alpha(k) = 0$ since $j \wedge k = 0$. If $\alpha(k) = 0$ then

$$\begin{aligned} \|\alpha\| - \varepsilon &< \alpha^+(j) - \alpha^-(j) \\ &\leq \|\alpha^+\| + \|\alpha^-\| < \|\alpha^+\| + \varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$, we obtain, $\|\alpha^-\| = \|\alpha\| - \|\alpha^+\| = 0$. Also if $\alpha(j) = 0$ then $\|\alpha^+\| = 0$. Hence, we may assume that $\alpha = \alpha^+$. Let $p, q \in \Omega^*$ for which $2\alpha = p + q$. Then $2\alpha = p^+ + q^+ - p^- - q^-$. Hence,

$$\begin{aligned} \|2\alpha\| &\leq \|p^+\| + \|q^+\| + \|p^-\| + \|q^-\| \\ &= \|p\| + \|q\| = 2 = \|2\alpha\|. \end{aligned}$$

Clearly, $p^+ + q^+ = 2\alpha$ and $p^- = q^- = 0$. We now prove that $p = q = \alpha$ i.e $\alpha \in extL(\Omega)$. Hence we show that $p(l) = q(l) = 0$ whenever $\alpha(l) = 0$. This implies that $\alpha = ap = bq$ but $\alpha, p, q \in D(\Omega^*)$ and $2\alpha = p + q$ and hence $a = b = 1$. Assuming that $l \geq 0$, we have $p(l) \geq 0$, $q(l) \geq 0$ and $p(l) + q(l) = 2\alpha(l) = 0$. We obtain $p(l) = q(l) = 0$. For the general case we have $\alpha(l^+) = \alpha(l^-) = 0$, since $\alpha(l) = 0$ and from the condition that $\alpha(l^+) \cdot \alpha(l^-) = 0$ and hence the condition $p(l) = q(l) = 0$ holds from the first case.

Theorem 4.7 *Let $\Omega \in AM$. The following conditions on $\alpha \in L(\Omega^*)$ are equivalent.*

(i). α is norm-attainable

(ii). There exist $j \neq 0$ for which $\alpha(m) = \|m\|$ for each $m \in \overline{S_\Omega}$, the norm closure of S_Ω where $0 \leq m \leq tj$ for some $t \geq 0$

(iii). There exist $j(\neq 0) \in \Omega^+$ such that among $L(\Omega^\star) = \varphi \in \Omega^*; \|\varphi\| = 1$, α is maximal on $\overline{S_\Omega}$

Proof. $(i) \Rightarrow (ii)$ Let α be norm-attainable. Then for each $j \in S_\Omega$, $\alpha(j) = \|\alpha\| = \|j\| = 1$. From Proposition 4.1 it follows that, $1 = \alpha(j) = \alpha(j^+) - \alpha(j^-) \leq \|\alpha(j^+)\| + \|\alpha(j^-)\| = \|\alpha\|(\|j^+\| + \|j^-\|) = \|\alpha\|(\|j\|) \leq 1$. Assume that $m \in S_\Omega$, then we need to show that $\alpha(m) = \|\alpha\|$. Since α is continuous, let $m \in \overline{S_\Omega}$ that is $0 \leq m \leq tj$ for some $t \geq 0$. But since $t = \alpha(tm) = \alpha(tm - j) + \alpha(j) < \|tm - j\| + \|j\| = \|tm\| = \|t\|$ and hence we have $\alpha(m) = \|m\|$. $(ii) \Rightarrow (iii)$ Let $j \neq 0$ such that $\alpha(m) = \|m\|$ for all $m \in \overline{S_\Omega}$. Since $\alpha(m) = \|m\|$ for all $m \in S_\Omega$, α attains its maximum on S_Ω. Let $\varphi \in L(\Omega^*)$ with $\|\varphi\| = 1$. Then $\alpha(m) < \|t\|$ for all j and by assumption $\alpha(m) = \|\alpha\|$. Hence α is maximal on S_Ω. $(iii) \Rightarrow (i)$ Let $j(\neq 0) \in \Omega^+$ such that α is maximal on $\overline{S_\Omega}$ among elements $\varphi \in L(\Omega^*)$ with $\|\varphi\| = 1$. Since α is maximal, $\alpha(r) = \|r\|$ for some $r \in S_\Omega$. Hence by $\alpha(r) \geq \varphi(r) = \|r\|$, α is norm-attainable at $\frac{j}{\|j\|}$.

In the above result, we have given equivalent conditions on norm attainability. As an extension of the above result, we present a result on norm attainability of functionals

Corollary 4.8 *If Ω is a bounded σ-lattice, then $\alpha \in \Omega^{*+}$ attains its norm if $L(\Omega)$ is order bounded.*

Proof. Let $t_n(\geq 0) \in L(\Omega)$ such that $\alpha(t_n) \to \|\alpha\|$. Given that Ω^* is a bounded $\sigma-$ lattice and $L(\Omega)$ is order bounded we have $y = V_n(t_n)$ and $\|y\| = 1$. Thus $y \geq t_n \geq 0$ and $\alpha \geq 0$ implying that $\|\alpha\| \geq \alpha(y) \geq \alpha(t_n) \to \|\alpha\|$.

We now extend the above results to norm attainment properties of operators and their adjoints.

Proposition 4.9 *An operator $Q \in F(H)$ is norm-attainable if and only if $Q^* \in F(H)$ is norm-attainable.*

Proof. If Q is norm-attainable, then Q^* is norm-attainable. Let $Q \in F(H)$ be norm-attainable, then for each $f_o \in S_H$ we have $\|Qf_o\| = \|Q\|$. But for an adjoint operator Q^* satisfies $\langle Qf, g\rangle = \langle f, Q^*g\rangle$ for all $f, g \in H$. Thus, $QQ^*f = \|Q\|^2 f$. Let $g_o = \frac{Qf_o}{\|Q\|}$ be a unit vector. Then, $\|Q^*(g_o)\| = \|Q(f_o)\| = \|Q\| = \|Q^*\|$. Hence Q^* is norm-attainable. Conversely, let Q^* be norm-attainable, then for any $g_o \in S_H$ we have $\|Q^*g_o\| = \|Q^*\|$. Now, by the property of adjoint $\langle Qf, g\rangle = \langle f, Q^*g\rangle$ for each $f, g \in H$. Let $f_o = \frac{Q^*g_o}{\|Q^*\|}$ then, we have $\|Q(f_o)\| = \|Q^*(g_o)\| = \|Q^*\| = \|Q\|$. Hence, Q is norm-attainable.

Next, we present a result that shows the connection between normality and norm attainability.

Lemma 4.10 *Let $J \in F(H)$ be normal. Then it is norm-attainable.*

Proof. If $J \in F(H)$ is normal, we show that it is norm-attainable. Let J be normal, then $JJ^* = J^*J$. But from Proposition 4.9 and from the property of adjoint, for each $m \in S_H$ we have $J(Jm) = J(J^*m)$. Hence, $\langle Jm, Jm\rangle = \langle JJ^*m, m\rangle = \|J(J^*m)\|\|m\| = \|Jm\|^2 = \|J\|^2$. Applying square roots on both sides we have, $\|Jm\| = \|J\|$. Hence, J is norm-attainable.

With Lemma 4.10 proved , we proceed to the following result.

Theorem 4.11 *Suppose that $\varphi, \varphi^* \in NA(C(K), Y)$ for any finite dimensional Banach space Y. Then $\varphi \in (C(K), Y)$ is normal.*

Proof. Let φ and φ^* be norm-attainable. Then from Proposition 4.9, there exists $w_o \in S_{(C(K),Y)}$ and $z_o \in S_{(C(K),Y)}$ for which $\|\varphi\| = \|\varphi w_o\| = \|\varphi^* z_o\| = \|\varphi^*\|$. Let $\varphi \in (C(K), Y)$ be normal, then $\varphi^*\varphi = \varphi\varphi^*$. Assume that Y is of finite dimension and that $\varphi, \varphi^* \in NA(C(K), Y)$ then, $\varphi \in (C(K), Y)$ is normal. If $\varphi \in NA(C(K), Y)$. Then also $\varphi^* \in NA(C(K), Y)$ indicating that $\varphi^*\varphi = \varphi\varphi^* \in NA(C(K), Y)$. Therefore φ is normal.

In what follows, we present a result on boundedness and linearity of operators.

Corollary 4.12 *If G and H are Asplund spaces. Then $\tau \in NA(G, H)$ is bounded and linear.*

Proof. Let τ be norm-attainable. Then, for each $w_o \in S_G$ we have $\|\tau w_o\| = \|\tau\|$. Indeed, $\|\tau w_o\| = \|\tau\|\|w_o\| = \|\tau\|.1 = 1.\|\tau\|$. Hence, τ is bounded. Now, we show that τ is linear. Since τ is bounded and norm attaining. Then, it is linear i.e for every $s, r \in G$ and $\alpha, \beta \in \mathcal{K}$ we have, $\tau(\alpha s + \beta r) = \alpha\tau s + \beta\tau r$. Conversely, Let τ be bounded and linear. Then, for a complex number $\lambda \geq 0$. $\|\tau w_o\| \leq \lambda\|w_o\|$ for all $w_o \in G$. But $\|w_o\| = 1 \Rightarrow \|\tau w_o\| \leq \lambda$. Thus, $\lambda = sup\|\tau(w_o)\|$. We now show that τ is norm-attainable. From Theorem 4.11 there exist a unit vector $w_o \in G$ for which $\|\tau w_o\| \leq \lambda = \|\tau\| = sup\|\tau\|$ thus τ is norm-attainable.

In Proposition 4.13, Lemma 4.14 and Theorem 4.15, we further examine norm attainability of operators and conditions that ensure norm attainment.

Proposition 4.13 *Let $Q \in K(X, Y)$ for Asplund spaces X and Y. Let K be a closed subspace of X. Then Q attains its norm if $K^\perp \subseteq NA(X)$.*

Proof. Let $Q \in L(X,Y)$, then from James theorem $\frac{X}{K}$ is reflexive. The operator Q maps via $\frac{X}{K}$. Thus, there is an operator $P : \frac{X}{K} \to Y$ and $Q = P.\sigma$. Then $\|Q\| = \|P\|$ and P is compact in the case that Q is compact. From Proposition 4.9 we have that P achieves its norm and as a consequence its adjoint P^* also achieves its norm. Hence, there is a $g^* \in S_{Y^*}$ for which $\|P^*g^*\| = \|P\| = \|Q\| = \|Q^*g^*\|$. The functional $h^* = Q^*g^* = (\sigma.P^*)g^* \in X$ disappears on X implying that it is in $K^\perp \subseteq NA(X)$ i.e $K^\perp = \{h^*; Qh^* = 0\}$. Therefore, there exist $h \in S_X$ for which $\|h^*(h)\| = \|h^*\| = \|\sigma^*(P^*g^*)\| = \|P^*g^*\| = \|P\|$ where σ^* is a norm preserving map. Consequently, $\|Q\| = \|(Q^*g^*)h\| = \|g^*(Qh)\|$ and hence $\|Qh\| = \|h\|$.

Lemma 4.14 *Suppose that $\varphi \in L(c_o(X), Y)$ attains its norm at $y_o \in B_Y$ for a complex Banach space Y. Then, $\|\varphi\| = \|\varphi(y_o + y)\|$ for every $y \in Y$ with $\|y_o + \sigma y\| \leq 1$ for all $\sigma \in \mathcal{C}, |\sigma| \leq 1$.*

Proof. Let $y \in Y$ such that $\|y_o + \sigma y\| \leq 1$. Suppose that D is the open disk in $\mathcal{C}$ for which $D = \{\sigma \in \mathcal{C} : |\sigma| \leq 1\}$. Let the function $K : \overline{D} \to \mathcal{C}$ be such that $k(\sigma) = \varphi(y_o + \sigma y)$ for all $\sigma \in \mathcal{C}$ and $|\sigma| \leq 1$. Since $\varphi \in (c_o(X), Y)$, then k is continuous on D and holomorphic on D and hence $k_n : \overline{D} \to \mathcal{C}$ can be defined as $k_n(\sigma) = \varphi(t_n(y_o + \sigma y))$ where t_n is a sequence in $(0, 1\}$ that converges to 1. Since φ is uniformly continuous on D we have that k_n is holomorphic on D and continuous on D. Indeed by [58], φ attains its norm at y_o and hence from Proposition 4.13 we obtain $\|\varphi\| = \|\varphi y_o\| = \|k(o)\| \leq max\{|k\sigma| : \sigma \in \overline{\mathcal{C}}\} = \|\varphi\|$. From the Principle of maximum modulus we have that k is constant on $\overline{D}$. Thus $\|\varphi\| = \|\varphi y_o\| = \|k(o)\| = \|k(1)\| = \|\varphi(y_o + y)\|$.

Theorem 4.15 *Suppose Z is a complex BS and $Q \in L(c_o(Y), Z)$ such that Q is norm attainable at $w_o \in B_{(c_o(Y),Z)}$ with $w_o(t) \neq 0$ for each $t \in c_o(Y)$. Then Q is also norm attainable at z if z is a function given by $z(t) = \frac{w_o(t)}{\|w_o(t)\|}$.*

Proof. Suppose that Q is normalized. Since $\|w_o\| \leq 1$ we have by [57]that $\|Qw_o\| = \|Q\| = 1$. From Lemma 4.14 we have have $\|y_o + \sigma y\| \leq 1$. We now consider w_o and $z - w_o$ to play the role of y_o and y in Lemma 5.4. Thus for the element $z \in \overline{D}$. It follows that

$$\begin{aligned} \|w_o(t) + (z - w_o)t\| &\leq \|w_o(t)\| + \|(z - w_o)t\| \\ &= \|w_o(t)\| + \|1 - w_o(t)\| \\ &= \|w_o(t)\| + 1 - \|w_o(t)\| = 1. \end{aligned}$$

Therefore by [56] we have $(w_o + (z - w_o)) \in B_{(c_o(Y),Z)}$ implying that Q is norm attainable at z i.e $\|Qw_o\| = \|Qz\| = 1$.

5 Open Problems

For bounded linear operators, it is observed that, Bishop-Phelps-Bollobas criterion has not been studied when operators are of finite rank exhaustively. Therefore, it is an area of interest in the concept of denseness of norm achieving mappings to determine whether every finite rank operator between Banach spaces can be estimated by those that achieve their norms. Hence, it has not been shown whether the set of norm achieving finite rank operators is dense in the whole space of mappings of finite rank. We analyzed this properties particulary norm-attainability. We showed that rank one mappings achieve their norms on certain Banach lattices. Since finite rank operators are obtained by summing rank one mappings, our results seeks to clarify the behavior of such operators especially in relation to norm-attainability and operator structure with regard to Bishop-Phelps-Bollobas property. **Problem:** Firstly, do these results hold in Lorentz algebras?

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