Int. J. Open Problems Compt. Math., Vol. 19, No. 1, March 2026
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Tracing the Origins and Generalizations of a Class of Moving Point Inequalities

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Received 10 October 2025; Accepted 3 November 2025

Abstract

This research begins with the Guggenheimer inequality and systematically reviews its developmental progress, which has generally undergone stages of degree generalization, form strengthening, and weighted generalization. Following this, inequalities structurally similar to the Guggenheimer inequality have continuously emerged. The Hungarian Mathematical Competition of 2015–2016 presented an inequality with a similar structure. A higher-degree generalization of this inequality leads to a more general form of moving point inequality. By combining the structure of the strengthened generalization of the Guggenheimer inequality, five conjectures with similar structures are proposed.

Keywords: *Guggenheimer Inequality, Moving Point Inequality, Generalization of Inequalities*

2010 Mathematics Subject Classification: 51M16.

1 Introduction

Let P be an arbitrary point inside triangle $\triangle ABC$. Inequalities related to the moving point P are generally referred to as moving point inequalities.

These inequalities represent a hot and challenging topic in the field of geometric inequality research. Well-known examples such as the Erdős-Mordell inequality and the Guggenheimer inequality belong to this category, having inspired a series of research questions and results. Such inequalities also frequently appear in mathematical competitions, attracting widespread attention from mathematics researchers and participants. The Hungarian Mathematical Competition of 2015–2016 featured the following problem: Let P be an arbitrary point inside triangle $\triangle ABC$. Lines PA, PB, PC are extended to meet BC, AC, AB at A_1, B_1, C_1 respectively. If a, b, c are the side lengths of the triangle, then

$$PA \cdot PA_1 + PB \cdot PB_1 + PC \cdot PC_1 < \frac{1}{3}(a^2 + b^2 + c^2).$$

This is a moving point inequality where the upper bound is expressed as a combination of the squares of the sides, structurally similar to the Guggenheimer inequality. The problem's author was likely inspired by it. So, what is the Guggenheimer inequality? What are the latest research developments regarding it? This paper addresses these questions by systematically reviewing the developmental context of the Guggenheimer inequality.

1.1 Research Context of the Guggenheimer Inequality

The research lineage of the Guggenheimer inequality has generally progressed through stages of degree generalization, form strengthening, and weighted generalization. In the following, we consistently adopt the convention: P is an arbitrary point inside triangle $\triangle ABC$, a, b, c are the side lengths, and n is a positive integer.

1.1.1 Degree Generalization Stage: Upper bounds using homogeneous sums of sides

In 1967, H. W. Guggenheimer first used the sides of the triangle to formulate an upper bound, obtaining Inequality A, which we call the Guggenheimer inequality. In 1971, M. S. Klamkin generalized the Guggenheimer inequality to a quadratic form, obtaining Inequality B. In 1989, Ji Chen provided a unified generalization of the Guggenheimer and Klamkin inequalities to higher degrees, obtaining Inequality C.

- **Inequality A** [1]: $PA + PB + PC < a + b + c$
- **Inequality B** [2]: $PA^2 + PB^2 + PC^2 < a^2 + b^2 + c^2$
- **Inequality C** [3]: $PA^n + PB^n + PC^n < a^n + b^n + c^n$

1.1.2 Form Strengthening Stage: Upper bounds using homogeneous sums of two sides

While generalizing the inequalities, researchers have also explored strengthened forms. As early as 1980, Zhun Shan provided a strengthened form of Inequality A, obtaining Inequality D. In 2001, Bo Deng provided an alternative proof for Inequality D and simultaneously strengthened Inequality B, obtaining Inequality E. Actually, Gangsong Leng had already strengthened the higher-degree form earlier, obtaining Inequality F.

- **Inequality D** [4, 5]: If $a \geq b \geq c$, then $PA + PB + PC < a + b$.
- **Inequality E** [5]: If $a \geq b \geq c$, then $PA^2 + PB^2 + PC^2 < a^2 + b^2$.
- **Inequality F** : If $a \geq b \geq c$, then $PA^n + PB^n + PC^n < a^n + b^n$.

1.1.3 Weighted Generalization Stage: Upper bounds using weighted homogeneous sums of two sides

The Guggenheimer inequality was generalized from linear to higher-degree forms, yielding a series of inequalities. Naturally, one considers whether this series can be further generalized. Many scholars began considering further generalizations using weighted forms. In 2018, Shanpeng Zeng and Hongliang Fei generalized Inequalities D and E using weighted forms, obtaining Inequalities G and H, achieving weighted generalizations for the linear and quadratic cases of the Guggenheimer inequality. However, their proof method could not solve the higher-degree weighted generalization problem. In 2019, Hongliang Fei, Shanpeng Zeng, Xuezhi Yang using a method of constructing local inequalities, obtained a weighted generalization of Inequality F, yielding Inequality I, thus achieving a higher-degree weighted generalization of the Guggenheimer inequality. Can the weighted generalization form of the Guggenheimer inequality be further strengthened? While a global strengthening is difficult, a local strengthening can be considered. In 2025, Hongliang Fei obtained a local strengthening of Inequality G by restricting the range of the moving point P, resulting in Inequality J.

- **Inequality G** [6]: If the side lengths of $\triangle ABC$ satisfy $c \leq b \leq a$, and x, y, z are arbitrary positive real numbers, then for any point P inside the triangle:

$$x \cdot PA + y \cdot PB + z \cdot PC < \max\{y, z\} \cdot a + \max\{x, z\} \cdot b.$$

- **Inequality H** [6]: Under the same conditions as Inequality G:

$$x \cdot PA^2 + y \cdot PB^2 + z \cdot PC^2 < \max\{y, z\} \cdot a^2 + \max\{x, z\} \cdot b^2.$$

- **Inequality I** [7]: A generalization to the n-th power. Under the conditions of Inequality G and for any positive integer n:

$$x \cdot PA^n + y \cdot PB^n + z \cdot PC^n < \max\{y, z\} \cdot a^n + \max\{x, z\} \cdot b^n.$$

- **Inequality J** [8]: Let A', B', C' be the midpoints of BC, CA, AB respectively, and let P be any point inside $\triangle A'B'C'$. Under the side length and positive real number conditions of Inequality G:

$$x \cdot PA + y \cdot PB + z \cdot PC < \max\{x, y\} \cdot \left(\frac{1}{2}c + \frac{1}{2}b\right) + \max\{y, z\} \cdot a.$$

2 Preliminaries

Lemma 2.1 *If a, b, c, d are all positive numbers with $a \neq c$ and $b \neq d$, and $\frac{a}{b} = \frac{c}{d}$, then*

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$$

proof. Let $\frac{a}{b} = \frac{c}{d} = k$. Then $a = kb$ and $c = kd$ $(k \neq 0)$. Since $a \neq c$ and $b \neq d$, we have

$$\frac{a+c}{b+d} = \frac{kb+kd}{b+d} = k, \quad \frac{a-c}{b-d} = \frac{kb-kd}{b-d} = k.$$

Thus, $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$.

Lemma 2.2 *If a, b, c are the side lengths of a triangle, and u, v, w are positive numbers such that $u + v + w = 1$, then*

$$\sum \left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^n \leq \sum \frac{(uv)^n c^2 + (uw)^n b^2}{a^2 + b^2 + c^2}. \tag{1}$$

proof. We first prove the local inequality

$$\left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^n < \frac{(uv)^n c^2 + (uw)^n b^2}{a^2 + b^2 + c^2}. \tag{2}$$

1. For $n = 1$, inequality (2) holds trivially. 2. For $n = 2$, computing the difference between the left-hand side and right-hand side of (2) yields:

$$\begin{aligned} &\left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^2 - \frac{(uv)^2 c^2 + (uw)^2 b^2}{a^2 + b^2 + c^2} \\ =& \frac{-(uv - uw)^2 b^2 c^2 - (uv)^2 a^2 c^2 - (uw)^2 a^2 b^2}{(a^2 + b^2 + c^2)^2} < 0. \end{aligned}$$

Thus, (2) holds for $n = 2$.

3. Assume (2) holds for $n = k$, i.e.,

$$\left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^k \leq \frac{(uv)^k c^2 + (uw)^k b^2}{a^2 + b^2 + c^2}. \tag{3}$$

For $n = k + 1$, we have:

$$\begin{aligned}
&\left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^{k+1} - \frac{(uv)^{k+1}c^2 + (uw)^{k+1}b^2}{a^2 + b^2 + c^2} \\
&= \left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^k \left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right) - \frac{(uv)^{k+1}c^2 + (uw)^{k+1}b^2}{a^2 + b^2 + c^2} \\
&\leq \frac{(uv)^k c^2 + (uw)^k b^2}{a^2 + b^2 + c^2} \cdot \frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2} - \frac{(uv)^{k+1}c^2 + (uw)^{k+1}b^2}{a^2 + b^2 + c^2} \quad \text{(by inductive hypothesis)} \\
&= \frac{(uv)^k uwb^2c^2 + (uw)^k uvb^2c^2 - (uv)^{k+1}(a^2 + b^2)c^2 - (uw)^{k+1}(a^2 + c^2)b^2}{(a^2 + b^2 + c^2)^2} \\
&\leq \frac{(uv)^k uwb^2c^2 + (uw)^k uvb^2c^2 - (uv)^{k+1}b^2c^2 - (uw)^{k+1}b^2c^2}{(a^2 + b^2 + c^2)^2} \\
&= \frac{u^{k+1}(w - v)(v^k - w^k)b^2c^2}{(a^2 + b^2 + c^2)^2}.
\end{aligned}$$

Since $(w - v)$ and $(v^k - w^k)$ have opposite signs, the expression is negative. Therefore,

$$\left(\frac{uvc^2 + uwb^2}{a^2 + b^2 + c^2}\right)^{k+1} < \frac{(uv)^{k+1}c^2 + (uw)^{k+1}b^2}{a^2 + b^2 + c^2},$$

which completes the induction. Thus, (2) holds, and consequently, (1) holds.

Lemma 2.3 *If a, b, c are the side lengths of a triangle, and u, v, w are positive numbers such that $u + v + w = 1$, then*

$$wva^2 + uwb^2 + uvc^2 < \frac{1}{6}(a^2 + b^2 + c^2). \tag{4}$$

proof. To prove (4), it suffices to prove the following inequality:

$$\left(wv - \frac{1}{6}\right)a^2 + \left(uw - \frac{1}{6}\right)b^2 + \left(uv - \frac{1}{6}\right)c^2 < 0. \tag{5}$$

Note that $wv + uw + uv \leq \frac{1}{3}(u + v + w)^2 = \frac{1}{3}$. We consider two cases:

1. If wv, uw, uv are all less than or equal to $\frac{1}{6}$, then at least one of them is strictly less than $\frac{1}{6}$, and thus (5) holds.

2. If one of wv, uw, uv is greater than $\frac{1}{6}$, by the above inequality, at most one term can exceed $\frac{1}{6}$. Without loss of generality, assume $uv > \frac{1}{6}$. Then $uw < \frac{1}{6}$ and $vw < \frac{1}{6}$. Note that

$$2uv + uw = u(2v + w) < u(2v + 2w) = 2u(1 - u) \leq \frac{1}{2},$$

Hence,$2uv + uw < \frac{1}{2}$. Similarly, $2uv + vw < \frac{1}{2}$. Therefore,

$$2\left(uv - \frac{1}{6}\right) < \frac{1}{2} - uw - \frac{1}{3} = \frac{1}{6} - uw, \tag{6}$$

$$2\left(uv - \frac{1}{6}\right) < \frac{1}{2} - vw - \frac{1}{3} = \frac{1}{6} - vw. \tag{7}$$

Using the triangle inequality $c < a+b$ and the fact that $(a+b)^2 \leq 2(a^2+b^2)$, we have $c^2 < 2(a^2 + b^2)$. Combining (6), (7), and this observation:

$$\left(uv - \frac{1}{6}\right) c^2 < 2\left(uv - \frac{1}{6}\right) a^2 + 2\left(uv - \frac{1}{6}\right) b^2 < \left(\frac{1}{6} - vw\right) a^2 + \left(\frac{1}{6} - uw\right) b^2.$$

This implies (5), completing the proof.

Lemma 2.4 (Kooi's Inequality) *If a, b, c are the side lengths of a triangle, R is its circumradius, and x, y, z are arbitrary real numbers, then*

$$yza^2 + zxb^2 + xyc^2 \leq (x + y + z)^2 R^2.$$

3 Main results

Regarding the inequality from the Hungarian Mathematical Competition, it is natural to consider whether it can be generalized to the case of arbitrary positive integer powers. Thus, we generalize Inequality J to the n-th power, obtaining Theorem 3.1. Based on Theorem 3.1, we further consider some special cases. Corollary 3.1.1 follows directly from Theorem 3.1 and the power mean inequality. Let the perpendiculars from point P to lines BC, AC, and AB intersect at points D, E, and F, respectively. Define $R_1 = PA$, $R_2 = PB$, $R_3 = PC$, $r_1 = PD$, $r_2 = PE$, $r_3 = PF$. Since r_1, r_2, r_3 are the altitudes of triangles $\triangle PBC$, $\triangle PAC$, $\triangle PAB$ respectively, we have $r_1 \leq PA_1$, $r_2 \leq PB_1$, $r_3 \leq PC_1$. Combining this with Theorem 3.1 yields Corollary 3.1.2.

Theorem 3.1 *Let P be an arbitrary point in triangle $\triangle ABC$. Lines PA, PB, PC are extended to meet BC, AC, AB at points A_1, B_1, C_1, respectively. Let a, b, c be the side lengths of the triangle, and let n be an arbitrary positive integer. Then*

$$(PA \cdot PA_1)^n + (PB \cdot PB_1)^n + (PC \cdot PC_1)^n < \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} (a^2 + b^2 + c^2)^n. \tag{8}$$

Corollary 3.1.1. Under the conditions of Theorem 3.1, we have

$$(PA \cdot PA_1)^n + (PB \cdot PB_1)^n + (PC \cdot PC_1)^n < \frac{1}{3}\left(\frac{3}{4}\right)^{n-1}(a^{2n} + b^{2n} + c^{2n}). \quad (9)$$

Corollary 3.1.1. Under the conditions of Theorem 3.1, we have

$$(R_1 r_1)^n + (R_2 r_2)^n + (R_3 r_3)^n < \frac{1}{3}\left(\frac{3}{4}\right)^{n-1}(a^{2n} + b^{2n} + c^{2n}). \quad (10)$$

The upper bound in the Hungarian Mathematical Competition inequality is not sharp, as equality cannot be attained. This leads us to consider: does there exist a sharp upper bound with a concise form? Consequently, we attempt to incorporate the circumradius R of the triangle into the upper bound, resulting in Theorem 3.2.

Theorem 3.2 *Let P be an arbitrary point in triangle $\triangle ABC$. Lines PA, PB, PC are extended to meet BC, AC, AB at points A_1, B_1, C_1, respectively. Let a, b, c be the side lengths, and R be the circumradius of the triangle. Then*

$$PA \cdot PA_1 + PB \cdot PB_1 + PC \cdot PC_1 \leq 2R^2. \quad (11)$$

3.1 Proof of Theorem 3.1

Place the triangle in a suitable Cartesian coordinate system. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $A_1(x_1', y_1')$, $B_1(x_2', y_2')$, $C_1(x_3', y_3')$, $P(x, y)$. Denote the areas of $\triangle PBC$, $\triangle PAC$, $\triangle PAB$ by S_1, S_2, S_3, respectively.

By Lemma 2.1, we have

$$\frac{|\overrightarrow{BA_1}|}{|\overrightarrow{A_1C}|} = \frac{S_{\triangle ABA_1}}{S_{\triangle ACA_1}} = \frac{S_{\triangle PBA_1}}{S_{\triangle PCA_1}} = \frac{S_{\triangle ABA_1} - S_{\triangle PBA_1}}{S_{\triangle ACA_1} - S_{\triangle PCA_1}} = \frac{S_3}{S_2}. \quad (12)$$

From (12), it follows that

$$\frac{|\overrightarrow{BA_1}|}{|\overrightarrow{BC}|} = \frac{|\overrightarrow{BA_1}|}{|\overrightarrow{BA_1}| + |\overrightarrow{A_1C}|} = \frac{S_3}{S_2 + S_3}. \quad (13)$$

Using the vector relation $\frac{\overrightarrow{BA_1}}{|\overrightarrow{BA_1}|} = \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|}$ and equation (13), we get

$$\overrightarrow{BA_1} = \frac{|\overrightarrow{BA_1}|}{|\overrightarrow{BC}|}\overrightarrow{BC} = \frac{S_3}{S_2 + S_3}\overrightarrow{BC}.$$

Thus,

$$x_1' - x_2 = \frac{S_3}{S_2 + S_3}(x_3 - x_2),$$

which simplifies to

$$x_1' = \frac{S_2}{S_2+S_3}x_2 + \frac{S_3}{S_2+S_3}x_3. \tag{14}$$

Similarly,

$$y_1' = \frac{S_2}{S_2+S_3}y_2 + \frac{S_3}{S_2+S_3}y_3. \tag{15}$$

By Lemma **??**, we also have

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{AA_1}|} = \frac{S_2}{S_{\triangle ACA_1}} = \frac{S_3}{S_{\triangle ABA_1}} = \frac{S_2+S_3}{S_{\triangle ACA_1}+S_{\triangle ABA_1}} = \frac{S_2+S_3}{S_1+S_2+S_3}. \tag{16}$$

Using the vector relation $\frac{\overrightarrow{AP}}{|\overrightarrow{AP}|} = \frac{\overrightarrow{AA_1}}{|\overrightarrow{AA_1}|}$ and equation (16), we obtain

$$\overrightarrow{AP} = \frac{|\overrightarrow{AP}|}{|\overrightarrow{AA_1}|}\overrightarrow{AA_1} = \frac{S_2+S_3}{S_1+S_2+S_3}\overrightarrow{AA_1}.$$

Therefore,

$$x - x_1 = \frac{S_2+S_3}{S_1+S_2+S_3}\left(\frac{S_2}{S_2+S_3}x_2 + \frac{S_3}{S_2+S_3}x_3 - x_1\right),$$

which simplifies to

$$x = \frac{S_1x_1+S_2x_2+S_3x_3}{S_1+S_2+S_3}. \tag{17}$$

Similarly,

$$y = \frac{S_1y_1+S_2y_2+S_3y_3}{S_1+S_2+S_3}. \tag{18}$$

From (14), (15), (17), (18), we derive:

$$\overrightarrow{PA} = \left(\frac{S_2(x_1-x_2)+S_3(x_1-x_3)}{S_1+S_2+S_3}, \frac{S_2(y_1-y_2)+S_3(y_1-y_3)}{S_1+S_2+S_3}\right), \tag{19}$$

$$\overrightarrow{PA_1} = \left(\frac{S_1S_2(x_2-x_1)+S_1S_3(x_3-x_1)}{(S_2+S_3)(S_1+S_2+S_3)}, \frac{S_1S_2(y_2-y_1)+S_1S_3(y_3-y_1)}{(S_2+S_3)(S_1+S_2+S_3)}\right). \tag{20}$$

Using the Cauchy-Schwarz inequality and equations (19) and (20), we compute:

$$\begin{aligned}
&|\overrightarrow{PA}|\cdot|\overrightarrow{PA'}| = -\overrightarrow{PA}\cdot\overrightarrow{PA'} \\
&= \frac{S_1S_2^2c^2+S_1S_3^2b^2+2S_1S_2S_3[(x_1-x_2)(x_1-x_3)+(y_1-y_2)(y_1-y_3)]}{(S_2+S_3)(S_1+S_2+S_3)^2} \\
&\le \frac{S_1S_2^2c^2+S_1S_3^2b^2+2S_1S_2S_3\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\sqrt{(x_1-x_3)^2+(y_1-y_3)^2}}{(S_2+S_3)(S_1+S_2+S_3)^2} \\
&= \frac{S_1S_2^2c^2+S_1S_3^2b^2+2S_1S_2S_3bc}{(S_2+S_3)(S_1+S_2+S_3)^2} = \frac{S_1(S_2c+S_3b)^2}{(S_2+S_3)(S_1+S_2+S_3)^2}.
\end{aligned} \tag{21}$$

By Titu's lemma (the Engel form of Cauchy-Schwarz), we have:

$$\frac{S_1(S_2c+S_3b)^2}{(S_2+S_3)(S_1+S_2+S_3)^2} \le \frac{S_1}{(S_1+S_2+S_3)^2}\left(\frac{S_2^2c^2}{S_2}+\frac{S_3^2b^2}{S_3}\right) = \frac{S_1S_2c^2+S_1S_3b^2}{(S_1+S_2+S_3)^2}. \tag{22}$$

From (21) and (22), we obtain:

$$\sum(PA\cdot PA_1)^n \le \sum\left(\frac{S_1S_2c^2+S_1S_3b^2}{(S_1+S_2+S_3)^2}\right)^n. \tag{23}$$

Let $u=\frac{S_1}{S_1+S_2+S_3}$, $v=\frac{S_2}{S_1+S_2+S_3}$, $w=\frac{S_3}{S_1+S_2+S_3}$, so that $u+v+w=1$. Then

$$\sum\left(\frac{S_1S_2c^2+S_1S_3b^2}{(S_1+S_2+S_3)^2}\right)^n = \sum\left(uvc^2+uwb^2\right)^n. \tag{24}$$

By Lemma 2.2 and the mean inequalities, we have:

$$\begin{aligned}
&\sum(uvc+uwb)^n = \sum\left(\frac{uvc+uwb}{a^2+b^2+c^2}\right)^n(a^2+b^2+c^2)^n\\
&\le \sum\frac{(uv)^nc^2+(uw)^nb^2}{a^2+b^2+c^2}(a^2+b^2+c^2)^n\\
&= 2\left(v^nw^na^2+u^nw^nb^2+u^nv^nc^2\right)(a^2+b^2+c^2)^{n-1}\\
&\le 2\left[\left(\frac{v+w}{2}\right)^{2n-2}vwa^2+\left(\frac{u+w}{2}\right)^{2n-2}uwb^2+\left(\frac{u+v}{2}\right)^{2n-2}uvc^2\right](a^2+b^2+c^2)^{n-1}\\
&\le 2\left(\frac{u+v+w}{2}\right)^{2n-2}(vwa^2+uwb^2+uvc^2)(a^2+b^2+c^2)^{n-1}\\
&= \left(\frac{1}{2}\right)^{2n-3}(vwa^2+uwb^2+uvc^2)(a^2+b^2+c^2)^{n-1}.
\end{aligned} \tag{25}$$

Combining inequalities (23), (24),(25), and applying Lemma 2.3, we finally obtain inequality (8). This completes the proof of Theorem **??**.

3.2 Proof of Theorem 3.2

From inequalities (21), (22), and Lemma 2.4 (Kooi's Inequality), we have:

$$\begin{aligned}
\sum PA\cdot PA_1 &\le \sum\frac{S_1S_2c^2+S_1S_3b^2}{(S_1+S_2+S_3)^2}\\
&= \frac{2(S_2S_3a^2+S_1S_3b^2+S_1S_2c^2)}{(S_1+S_2+S_3)^2} \le 2R^2.
\end{aligned}$$

This proves Theorem 3.2.

4 Open Problem

Inequalities E and F strengthen Guggenheimer's inequality by reducing the number of triangle side elements in the upper bound and scaling it to be smaller than the original. Inspired by Inequalities E and F, we naturally consider whether Theorem 3.1 can be further strengthened, leading to Conjectures 1 and 2.

Conjecture 1. If $a \geq b \geq c$, then

$$\sum PA \cdot PA_1 < \frac{1}{3}(a^2 + b^2).$$

Conjecture 2. If $a \geq b \geq c$ and n is a positive integer, then

$$\sum (PA \cdot PA_1)^n < \frac{1}{3}\left(\frac{3}{4}\right)^{n-1}(a^{2n} + b^{2n}).$$

Inequalities H and I are weighted generalizations of the original inequality. If Conjectures 5.1 and 5.2 hold, inspired by Inequalities H and I, we naturally consider whether Conjectures 1 and 2 can be further generalized to weighted forms, leading to Conjectures 5.3 and 5.4.

Conjecture 3. If $a \geq b \geq c$, x, y, z are arbitrary positive real numbers, and n is a positive integer, then

$$\sum x(PA \cdot PA_1) < \frac{1}{3}\left(\max\{y, z\}a^2 + \max\{x, z\}b^2\right).$$

Conjecture 4. If $a \geq b \geq c$, x, y, z are arbitrary positive real numbers, and n is a positive integer, then

$$\sum x(PA \cdot PA_1)^n < \frac{1}{3}\left(\frac{3}{4}\right)^{n-1}\left(\max\{y, z\}a^{2n} + \max\{x, z\}b^{2n}\right).$$

Inequality J provides a local strengthening of the first weighted generalization of Guggenheimer's inequality by restricting the range of the moving point P and reducing the upper bound. Inspired by Inequality J, we naturally consider whether its higher-power form also holds, leading to Conjecture 5.5.

Conjecture 5. If $a \geq b \geq c$, x, y, z are arbitrary positive real numbers, A', B', C' are the midpoints of BC, AC, AB respectively, P is any point inside $\triangle A'B'C'$, and n is a positive integer, then

$$\sum xPA^n < \max\{x, y\}\left(\frac{1}{2}c + \frac{1}{2}b\right)^n + \max\{y, z\}a^n.$$

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