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On geometrical Aspects of Various Operators and their Orthogonality in Complex Normed Spaces

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Abstract

Studies involving orthogonality of operators is an area with various applications with regard to the ever dynamic and emerging technological research outputs. In normed spaces (NS) there are different types of orthogonality. Useful results have come up where operators possessing given conditions are chosen for Range-Kernel orthogonality to be established. However, most of the results have been focussing on one type of orthogonality called the Birkhoff-James which we have given more results on. In this paper we give results on various notions of orthogonality by considering certain geometrical aspects in NS.

Keywords: *Derivation, FP-property, Orthogonality, Normality.*
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1 Introduction

Studies involving geometrical aspects of NS have been given considerations over decades by mathematicians. A lot of different operators have also been involved and their properties unveiled. The authors of [6] employed approximate proper vectors in the analysis of normal operators. When the spectrum of a normal operator is non-empty, its study can be reduced to the elementary case of self-adjoint operators. In [7], it was shown that every quasi-normal operator is subnormal and that every normal operator is paranormal. The work in [15] further explored properties of both paranormal and normal operators and provided conditions for extending related concepts. In [4], the backward extension problem for subnormal weighted shifts was generalized to subnormal operators, while [9] later developed a structure theorem for p-paranormal operators. Another line of investigation by [13] introduced the concept of square normal operators and examined their spectral characteristics. For distinct eigenvalues z and w of T with respective eigenvectors $a, b \in H$, it was established that $\langle a, b\rangle = 0$. In addition, [11] demonstrated that T is normal whenever there exists a continuous function f on the relevant set.

The work of [66] on range kernel orthogonality for derivations implemented by normal operators using Birkhoff-James (BJ)- orthogonality formed a cornerstone for our study. They recommended extending this research to other types of orthogonality and exploring adjoints of derivations induced by normal operators [67]. Accordingly, our work focuses on various orthogonality concepts in connection with range and kernel properties for derivations and the Fuglede Putnam (FP)-property. To broaden the scope, we also investigate how normal operators influence properties of the derivations they induce. The foundational concepts summarized below provide the basis for our results.

With the advancement of operator theory and quantum theory, derivations have become an essential analytical tool on spaces, particularly Hilbert spaces. It is well known that derivations may be inner or generalized [17], and many of their properties are discussed in [19]. Their norms and derivation ranges were studied comprehensively in [18]. The kernels of derivations were further examined in [20], which focused on structural properties of elementary operators, while [21] later analyzed generalized derivations and their numerical ranges. Moreover, [24] provided a detailed study on kernels of generalized derivations.

The study in [26] revealed that the norm-closure of the range of inner derivations always contains lower triangular compact operators. Furthermore, derivations on commutative von Neumann algebras are induced by bounded operators, and similar results hold for non-self-adjoint commutative algebras via similarity. In [27], spatial derivations on $*$-algebras were characterized using positive linear functionals. The authors of [46] investigated properties of inner derivations strictly implemented by norm-attainable operators and determined corresponding norms. In [28], formulas were given for approximating the norms of $\delta_{T,S}$, showing that this derivation is bounded and compact whenever T and S are compact.

The Fuglede Putnam theorem relates properties of arbitrary operators to those of normal operators. Fuglede showed that for bounded S and normal N, if S commutes with operators T_i, i.e., $ST_i = T_iS$, then S also commutes with any function of N [29]. For the case of unbounded operators, commutativity may fail, and several authors have studied asymptotic forms of the Fuglede theorem. Further extensions were made by researchers in [2], who analyzed

Fuglede Putnam results for various classes of operators.

In [33], it was proved that for a pair (T, S^*) satisfying the FP-property, any $C \in \ker \delta_{T,S} \cap B(H)$ satisfies $|\delta_{T,S} + C| \geq |C|$, and moreover $T^2X = XS^2$ and $T^3X = XS^3$. In [34], it was shown that a quasi-normal and normal operator is normal and unitarily equivalent. In [36], the FP-theorem was extended to (p, w)-normal operators, while [3] later examined the FP-property for N-class $A(k)$ operators, including range kernel results for the generalized derivations they induce. Furthermore, [37] provided a complete characterization of left-symmetric points under strong Birkhoff orthogonality. In [38], relationships between two functionals and the existence of b-Birkhoff orthogonal elements in two-normed linear spaces were established. The work of [39] generalized properties of j-orthogonality, examined their relationship with metric projection in smooth uniform complex normed spaces, and characterized the j-orthogonal complement.

Several authors introduced orthogonality concepts in Banach spaces that generalize classical Hilbert-space orthogonality. Such notions are vital, as they allow eigenvectors associated with eigenvalues to serve as basis elements of Hilbert spaces. In [40], the authors characterized inner derivations with respect to orthogonality for normal operators and established range kernel orthogonality for inner derivations in the sense of [43]. In normed spaces, [42] showed that Birkhoff orthogonality is equivalent to best approximation, highlighting its significance. Range properties of derivations were discussed in [41]. Our investigation extends these ideas by interpreting derivations as projections, forming a basis for orthogonality relations. We also incorporate other forms of orthogonality beyond those previously mentioned.

Operators in $R(\delta_T)$, where δ_T is an inner derivation induced by T, were used in [44] to show that elements of $R(\delta_T) \cap T'$ (the commutant of T) are nilpotent whenever $P(T)$ is normal, isometric, or co-isometric for some polynomial P. Normal operators form a broad class encompassing many subclasses. Range kernel orthogonality conditions for such classes have been derived using minimization techniques, the power norm inequality, and compactness arguments. In particular, [45] obtained approximation results for paranormal operators, which were then applied to orthogonality problems.

The present work focuses on derivations induced by normal operators. Normal operators possess properties related to spectrum, boundedness, invertibility, and density, as discussed in [47]. Similar properties hold for derivations induced by such operators. These derivations may be inner or generalized, as examined in [48], while their ranges were studied in [50]. Orthogonality for these derivations has been considered in [51] and extended to linear functionals in [30]. In [52], range kernel orthogonality for these derivations was analyzed.

Certain commutants that intersect with closed derivation ranges include normal operators, as shown in [23] in the context of normal derivations, isometries, cyclic subnormal operators, and Jordan operators. The results of [22] established that when the range of a derivation is closed, its intersection with its kernel is trivial whenever the associated operators are normal. In [25], it was shown that the adjoint of the kernel of a derivation is zero provided that the involved polynomial satisfies the Fuglede Putnam property, a result later extended to $\delta_{T,S}$. This motivated our interest in the behavior of derivations under normality assumptions.

A comprehensive summary of relationships between different orthogonality concepts is given in [53], including symmetry [54], homogeneity [31], and additivity ([56] [55]). These studies established range kernel orthogonality for elementary operators under unitary invariant norms [60]. In [62], the notion of orthogonal mappings in isosceles orthogonal spaces was introduced. The stability of orthogonally constant mappings was established, and BJ-orthogonality in weak C^*-algebras was characterized. Further investigations examined domain sets, dual target spaces, and orthogonality with respect to operator norms and numerical radius norms.

Berberian obtained a FP-theorem for operators whose adjoints are normal, under the additional condition that one of the operators belongs to the Hilbert Schmidt class. Later, [1] proved that if X is Hilbert Schmidt and $TX = XS$, then $T^*X = XS^*$. A similar conclusion holds when $T, S^* \in B(H)$ are injective and X is arbitrary, as shown in [32]. Some authors showed that Berberian s result can be achieved without restrictions on X. The work of [10] on derivation ranges highlighted gaps and facilitated our analysis of connections between FP-properties and range kernel orthogonality for derivations induced by normal operators. These generalizations have been instrumental in linking normal and normal operators [35], thereby simplifying their study. Consequently, we extended these ideas to the derivations they induce.

The authors of [61] studied isosceles and BJ-orthogonality for positive linear operators, analyzing their interrelationships. They noted that BJ-orthogonality encapsulates essential features of smooth norms in reflexive Banach spaces and is relevant to determining dimensions of NS. The study also characterized right-symmetric and left-symmetric operators under this notion of orthogonality.

In [63], it was shown that certain linear operators strongly preserve reverse orthogonality, and conditions for orthogonality were characterized. The unitary Carlsson orthogonality was introduced to characterize real inner product spaces, along with the concept of Birkhoff orthogonal sets.

Several authors defined a new orthogonal geometric constant $\Omega(X)$, based on the parallelogram law and isosceles orthogonality. In normed spaces, this constant equals one when the norm arises from an inner product. It has also been shown that orthogonality is preserved in Krein spaces [64], where four types of orthogonality were studied. In particular, [65] examined approximate symmetry and its relation to geometric properties of the space X. The authors of [12] introduced a new class of operators termed D-normal and D-quasi-normal, and outlined their fundamental properties. In [14], the notion of normality was examined, and its equivalence to an operator S under isometric conditions was established, together with the concept of unitary quasi-equivalence. The study in [15] also defined quasi-normal operators on Hilbert spaces as a unifying extension of paranormal and k-quasi-paranormal classes, presenting matrix representations for such operators. In [16], it was shown that normal operators satisfy various forms of Weyl s and Browder s theorems. It is clear from these studies that the geometrical aspects including orthogonality in Ns have not been exhausted. In this work we continue our work in the same spirit.

2 Preliminaries

At this point, we discuss some useful definitions to this work.

Definition 2.1 *([7]) If a vector space V is given over any field of real scalars $\mathcal{R}$, then the the function which is positive and real valued that takes vectors to real scalars represented by $\|.\|$ is referred to as a norm if it obeys the following conditions:*

(i). Positive definite: $\|a\| \geq 0, \ \forall \ a \in V$,

(ii). Zero property : $\|a\| = 0$ *iff* $a = 0, \ \forall \ a \in V$,

(iii). Homogeneity: $\|\beta a\| = |\beta|\|a\|, \ \forall \ a \in V$, *and* $\beta \in \mathcal{R}$,

(iv). Triangle inequality: $\|a + b\| \leq \|a\| + \|b\|, \ \forall \ a, b \in V$.

We provide preliminary concepts that are key to the study.

Definition 2.2 *([58], Definition 2.2) Two maps on a vector space are considered orthogonal with regard to a specific inner product if their action on any pair of vectors a and b in that space results in the inner product of their images being zero, expressed as $\langle Ta, Sb\rangle = 0$.*

Definition 2.3 *([59], Definition 3.1) A mapping $S \in B(H)$ is hyponormal if $(SS^{\star})^p - (S^{\star}S)^p \geq 0$ and $p = 1$.*

Definition 2.4 *(([9], Definition 1.4) A mapping D is an inner derivation if it satisfies the Leibnitz rule for all vectors in its domain, that is, $D(\phi\phi) = (D\phi)\phi + \phi(D\phi)$ where ϕ are vectors in the domain of D and all vector ϕ in the Hilbert space H.*

Definition 2.5 *([6], Definition 1.2.1) T has Fuglede-Putnam property if $SC = CS$, then $TSC = SCT$ for some self-adjoint maps C and S. It suggests that if two self-adjoint operators commute (product and order do not matter), then when one of this operators is multiplied by a third self-adjoint operator, the order of multiplication still does not matter.*

3 Research methodology

In this study, we used some fundamental principles and known results that are deemed useful to execute our tasks. Several technical approaches have been employed and the known results which are useful to our study utilized.

3.1 Known fundamental theorems

Lemma 3.1 *([50]). Let any two non empty sets W and S be bounded in the plane and $\alpha \in U$, $\beta \in V$ then $\exists\, t_0$ (positive number) then $t_0 \in\ bndry(W)$, $\beta | t_0 \in S$ or, $t_0 \in W$, $\beta | t_0 \in\ bndry(S)$.*

Lemma 3.2 *([16]). For some operators $X\ A_n$ and B_n if bounded, it holds that $\Pi_{i=1}^{k} s_i^p \left(\Sigma_{i=1}^{n} AXB\right) \leq \Pi_{i=1}^{k} s_i^p \left(A*\right) s_i(B^*) s^p_{i+[\frac{i-1}{N}]}(X)$.*

Theorem 3.3 *([64]). For every $n \in \mathcal{N}$ and $\alpha > 0$, the following inequality holds: $\sum_{k=1}^{n} s_k^{\alpha} \left(\int_{\Omega} CXD d\mu\right) \leq \sum^{\frac{\alpha}{2}} \left(\int_{\Omega} C|X^*|^{2-1\theta} C^* d\mu\right) s_k^{\frac{\alpha}{2}} \left(\int_{\Omega} D^*|X|^{2\theta} D^* d\mu\right)$.*

Theorem 3.4 ***Fuglede-Putnam Theorem*** *([4]). Let $A \in \mathcal{C}^{mn}$, $Q \in \mathcal{C}^{nn}$ and $T \in \mathcal{C}^{nm}$ where $\mathcal{C}^{mn}$ is a set comprising of all complex matrices. If R and Q are normally represented and $RT = TQ$, then $R^*T = TQ^*$.*

Theorem 3.5 ***Fuglede-Putnam Theorem*** *([57]). $A \in \mathcal{C}^{mn}$ and $B \in \mathcal{C}^{nn}$, where $\mathcal{C}^{mn}$ is a set comprising complex matrices, then AB and BA are normal if and only if $A^*AB = BBA^*$ and $ABB^* = B^*BA$.*

Proposition 3.6 *([14]). If A, B, C, D are normal operators, then*

(i). The sum of normal operators is normal. I.e, $A + B$ is normal

(ii). The product of any normal operators is normal. I.e, AB is normal. Similarly $C \times D$ are normal operators.

Theorem 3.7 *([20]). Any two definite sequences with m-tuples power have the property that $\sum_{k=1}^{m} a_k, b_k = 0$.*

In the next part we give the technical approaches which are very instrumental to this work.

3.2 Technical approaches

Tensor products: Any space containing all linear maps taking elements from the cross product of X and Y to another vector space Z is naturally isomorphic in relation to a space containing linear maps from the tensor product to another space. This is a construction; $X \otimes Y$ to Z which will be linear [21].

Theorem 3.8 *([66]). For any two vector spaces X and Y over a field $\mathcal{K}$ there exist a tensor product $X \otimes Y$ with a canonical bilinear homeomorphism distinguished up to an isomorphism by the following universal properties: Every bilinear homeomorphism $\phi : X \times Y \to Z$ lifts to a unique homeomorphism, and $\phi : X \otimes Y \to Z$.*

Orthogonal direct sum: Let$\{M_i\}_{i\in I}$ be a collection of closed subspaces of H such that $M_i \perp M_j$ whenever $i \neq j$ [62]. Then the orthogonal direct sum of the M_i is the smallest closed subspace which contains every M_i. This space is $\oplus_{i\in I} = \overline{span}(\cup_{i\in I} M_i)$

Splitting lemma: Let $A = A_1+A_2+...+A_g$ and $B = B_1+B_2+...+B_g$. If all A_i and B_i are symmetric positive semi-definite and if for each i, A_i is in the range of B_i then $\sigma(A, B) \leq max_i\sigma(A_i, B_i)$. The splitting lemma is normally used for decomposition of a matrix A into the sum of rank-one matrices with each corresponding to one off-diagonal and by decomposing B into path matrices [34].

Example 3.9 *Let* $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$ *to have U and V complete to a triplet which is symmetric we shall have that* $W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ *then the 2-norm of* $\|W\|_2 = \sqrt{3}$. *by choosing* $\epsilon = \frac{1}{\sqrt{2}}$ *it yields,* $\|W\|_2 = 2$. *We use splitting to achieve* $\|\tilde{W}\|_2 = \|W\|_2 = \sqrt{3}$.

Let $S = \begin{pmatrix} \sqrt{\frac{1}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} \end{pmatrix}$ *So*

$$W = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & & \\ 0 & & \\ & 0 & \\ & \sqrt{3} & \\ & & \frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} \end{pmatrix}$$

The W is shown without zeros to emphasize the structure of its columns [8].

Frobenius Heuristic : Let W be a matrix and M a diagonal matrix s.t (i, j) the value in the j^{th}-matrix will be given by:

$$(D_j)_{i,j} = \frac{\sqrt{|W_{i,j}|}}{\sqrt{\sum_{c=1}^{m} |W_{i,c}|}}.$$

Example 3.10 *Frobenius Heuristic minimizes* $\|\tilde{W}\|_F$ *over all fractional splittings of* W*. To illustrate this, we already know that each row is minimized to* $\|\tilde{W}\|_F\ \forall i, j \in \tilde{W}$*, are non zero elements of the vectors* $D_+^j W_{i,j}$ *for* $j = 1, ..., m$*. The* i^{th} *element of this vector* $(D_+^j)_{i,i} W_{i,j}$ *is* $(D_+^j)_{i,i} W_{i,j}$ *given that* $W_{i,j} = 0$*, so we have that* $(D_+^j)_{i,i} W_{i,j} = 0$*, otherwise* $(D_+^j) W_{i,i} W_{i,j} = \frac{W_{i,j}}{(D)_{i,j}}$*. On the second part, it follows that by breaking the minimization problem into k-independent sub-problems. It can be seen that* $\sum_{j=1}^{m} x^2 = 1$*. Let* $f(x_1, ..., x_m, \lambda) = \sum_{j=1}^{m} (\frac{c_j}{x_j})^2 + \lambda(\sum_{j=1}^{m} x^2 - 1)$*. The minimizer satisfies*

$$0 = \frac{\delta f}{\delta x_i} = -2.\frac{c^2}{x_j^3} + 2\lambda i,$$

$$0 = \frac{\delta f}{\delta \lambda} = \sum_{j=1}^{m} x^2 j - 1$$

then it follows that $C_i^2 = \lambda x^4 i$*, therefore* $x_i^2 = \frac{|c_i|}{\sqrt{\lambda}}$*. Since* $\sum_{j=1}^{m} x^2 j = 1$*, it follows in [58] that*

$$\sum_{j=1}^{m} \frac{|c_j|}{\sqrt{\lambda}} = 1.$$

and hence,

$$\frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m} |c_j| = 1$$

and so'

$$\lambda = \sum_{j=1}^{m} |c_j|.$$

Now, we have that $x_i^2 = \frac{|c_i|}{\sqrt{\lambda}}$ *so it follows that,*

$$x_i^2 = \frac{|c_i|}{\sum_{j=1}^{m} |c_i|}$$

and thus

$$x_i = \frac{\sqrt{|c_i|}}{\sqrt{\sum_{j=1}^{m} = 1|c_j|}}.$$

4 Main results

In this section we give results of our study. We begin with geometrical aspect and FP-property. In particular we consider derivations as our operators.

Proposition 4.1 *Suppose that $C, D \in G_H(H)$, then the derivation $\delta_{C,D}$ is bounded from above for all $Y \in G_H(H)$.*

Proof. Let C, D and Y be induced by c_n, d_n respectively and f_n be arbitrary elements of $B(H)$. By definition of δ, we have for $[\|\epsilon_n f_n\|^2] = 1$ and $\|\epsilon_n f_n\| \leq 1$ that $\|\delta_{C,D}\|^2 = \|\epsilon_n(c_n f_n - f_n d_n)\|^2 \leq \|\epsilon_n c_n f_n\|^2 + \|\epsilon_n f_n d_n\|^2 \leq [\epsilon_n|c_n|^2+\epsilon_n|d_n|^2][\|\epsilon_n f_n\|^2] = [\epsilon_n|c_n|^2+\epsilon_n|d_n|^2]\ [\epsilon_n|c_n|+\epsilon_n|d_n|]$. Taking the supremum of both sides of the inequality gives us $\|\delta_{C,D}(Y)\| \leq [\epsilon_n|c_n|^2]^{\frac{1}{2}} + [\epsilon_n|d_n|^2]^{\frac{1}{2}}$.

By considering normal operators we give some geometrical aspects involving FP-criterion.

Lemma 4.2 *Every derivation is bounded from below*

Proof. By definition of $\delta_{C,D}$, we see that $\delta_{C,D}(Y) = c_n - d_n$ for the bases c_n and d_n of C and D respectively with $\epsilon_n\|f_n\|^2 = 1$. Since c_n and d_n are bounded, from the definition of $\delta_{C,D}(Y)$, we have $\|\delta_{C,D}(f_n)\|^2 = \|\epsilon_n(c_n f_n - f_n d_n)\|^2 \geq \epsilon_n\|c_n f_n\|^2 - \epsilon_n\|f_n d_n\|^2 = [\epsilon_n|c_n|^2 - \epsilon_n|d_n|^2]\|f_n\|^2$ Since the difference of finite summation of c_n and d_n is also bounded, and clearly, $\|\delta_n\| \geq [\epsilon_n|c_n|^2] - [\epsilon_n|d_n|^2]^{\frac{1}{2}}$.

Lemma 4.3 *The FP-criterion suffices for all derivations implemented by normal operators.*

Proof. It is known from [9] that

$$\delta_C(XY) = CXY - XYC. \tag{1}$$

Now,

$$\delta_D(XY) = DXY - XYD \tag{2}$$

subtracting Equation 2 from Equation 1 we have:

$$\delta_C(XY) - \delta_D(XY) = CXY - DXY - XYC + XYD$$

$$\Rightarrow (\delta_C - \delta_D)(XY) = (C - D)XY - XY(C - D)$$

$$\Rightarrow (\delta_C - \delta_D)(XY) = \delta_{C-D}(XY)$$

$$\Rightarrow \delta_C - \delta_D = \delta_{C-D}$$

which is a derivation. The converse is true, that is if δ is a derivation in $G_H(H)$ then their exists $C \in G_H(H)$ such that $\delta = \delta_C$.
Conversely, suppose for $C, D \in G_H(H)$, we have $\delta_C = \delta_D$, then this implies

$$\delta_C - \delta_D = \delta_{C-D} = 0$$

Hence for all $Y \in G_H(H)$, we have

$$\delta_{C-D}(Y) = (C - D)Y - Y(C - D) = 0$$

$$\Rightarrow (C - D)Y = Y(C - D)$$

Setting $C - D = E$ We have $EY = YE$ implying $E = \lambda I$ thus

$$C - D = \lambda I \Rightarrow D = C - \lambda I$$

On the other hand, if $D = C - \lambda I$, then by applying derivation on both sides, we have:

$$\delta_D(Y) = \delta_{C-\lambda I}(Y)$$

$$\Rightarrow DY - YD = (C - \lambda I)Y - Y(C - \lambda I)$$

$$\Rightarrow DY - YD = CY - \lambda Y - YC + Y\lambda$$

$$\Rightarrow DY - YD = CY - YC$$

$$\Rightarrow \delta_D = \delta_C. \tag{3}$$

It has been shown in [30] that the identities $\delta_C + \delta_D = \delta_{C+D}$, $\delta_C\delta_D - \delta_D\delta_C = \delta_{CD-DC}$ indicate that the sum and Lie product of two inner derivations is a derivation. However, the product $\delta_C\delta_D$ is a derivation only in trivial cases.

Theorem 4.4 *Every finite rank normal derivation is linear and bounded.*

Proof. For linearity, let $X, Y \in G_H(H)$, then for scalars $\alpha, \beta \in C$. We have

$$\delta_C(\alpha X + \beta Y) = C(\alpha X + \beta Y) - (\alpha X + \beta Y)C \tag{4}$$

$$= \alpha CX - \alpha XC + \beta CY - \beta YC$$

$$= \alpha(CX - XC) + \beta(CY - YC)$$

$$= \alpha\delta_C + \beta\delta_C(Y)$$

Hence δ_C is linear. But a derivation is a linear map $\delta : G_H(H)$ satisfying the leibniz rule:

$$\delta(XY) = \delta(X)Y + X\delta(Y) \tag{5}$$

and is $\delta : G_H(H) \Rightarrow G_H(H)$ is a derivation, then their exists $C \in G_H$ such that $\delta = \delta_C$. Thus

$$\delta_C(XY) = \delta_C(X)Y + X\delta_C(Y)$$

$$\Rightarrow \delta_C(XY) = (CX - XC)Y + X(CY - YC)$$

But C is finite implying existence of $I \in G_H(H)$ such that

$$\|CX - XC - I\| \geq I$$

and

$$\|CY - YC - I\| \geq I$$

and hence from line 1, we have

$$\|\delta(XY)\| \leq \|CX - XC - I\|\|Y\| + \|X\|\|CY - YC - I\| \Rightarrow \|\delta_C(XY)\| \leq \|Y\| + \|X\|$$

Thus their exists a positive integer $n \in N$ such that

$$\|\delta(XY)\| \leq n. \tag{6}$$

Corollary 4.5 *Let C, D be a normal operators. Then $\delta(C) = \|D\|$ if $\delta(C) = \|D\|$.*

Proof. Suppose that $\delta(C) = \|D\|$. We have $[y_n]_n$ in H with $\|y_n\| = 1$ for each n and such that $< Cy_n, y_n > \to 0$ and $\|Cy_n\| \to \|C\|$ as $n \to \alpha$. So $< DRy_n, Ry_n > \to 0$ as $n \to \alpha$. Moreover, since $\|Ry_n\| \to 1$ and $\|DRy_n\| \to \|D\|$ as $n \to \alpha$. Also, $\delta(D) = \|D\|$. Conversely, suppose that $\delta(D) = \|D\|$. We have $R_D = \|D\|$. Since $\sigma(D) \subseteq \sigma(C)$, then $N_D \leq R(C)$. We obtain $\|C\| \leq R_C$. Hence $\delta(C) = \|C\|$ which completes the proof.

Next we consider universality as geometrical property of operators.

Proposition 4.6 *Every normal operator is S-universality.*

Proof. Since S-universality and normality are preserved under translations, we may assume that $\delta(C) = \|C\|$ and hence $\delta(D) = \|D\|$. Suppose that T is S-universal. By [47], we have; $\|\delta_{2,C}\| = \|\delta_C\| = 2\|C\| = 2\|D\| = \|\delta_D\|$. Consider $[y_n]_n$ in $C_2(H)$ with $\|y_n\|_2 = 1$ for which $\|Cy_n - y_nC\| \to 2\|C\|$ as $n \to \alpha$. Since, $\|Cy_n - y_nC\|_2 \leq \|Cy_n\|_2 + \|y_nC\|_2 \leq \|C\| + \|y_nC\|_2 \leq 2\|C\|$. We deduce that, $\|Cy_n\|_2 \to \|C\|$. Similarly, we get $\|Y_nC\|_2 \to \|C\|$. Now, from the identity $\|Cy_n - y_nC\|_2^2 = \|Cy_n\|_2^2 + \|y_nC\|_2^2 - 2R(< Cy_n, y_nC >)$, we conclude that $-R(< Cy_n, y_nC >) \to \|C\|^2$ as $n \to \alpha$ where R denotes the real part. Consider the operator $Ry_nR^\star \in L(H)$. Since $y_n \in C_2(H)$ and $\|y_n\| = 1$, then $Ry_nR^\star \in C_2(K)$ and $\|Ry_nR^\star\|_2 \leq 1$. Furthermore, $< NRy_nR^\star, Ry_nR^\star D >= tr(DRy_nR^\star(Ry_nR^\star D)^\star) =< Cy_n, y_nC >$. Hence $R(< DRy_nR^\star, Ry_nR^\star D >) \to -\|D\|^2$ as $n \to \alpha$. Since $|R(< DRy_nR^\star, Ry_nR^\star D >)| \leq \|DRy_nR^\star\|^2\|Ry_nR^\star D\|_2 \leq \|D\|^2$, and so $\|DRy_nR^\star\|_2 \to \|D\|$, $\|\|_2 \to \|Ry_nR^\star D\|_2 \to \|D\|$ as $S \to \alpha$. Whence we infer $\|\delta_{2,D}(Ry_nR^\star)\|_2 \to 2\|D\|$ as $n \to \alpha$. That is $\|\delta_{2,D}\| = 2\|D\|$. Since D is normal, it is guaranteed that $diam(\sigma(D)) = \|\delta_2(D)\|$. On the other hand, we see that $diam(\sigma(D)) \leq diam(\sigma(C)) \leq \|\delta_{2,D}\| \leq 2\|D\|$. Therefore, $diam(\sigma(C)) = 2\|D\| = 2\|C\| = 2R_C$. The sufficient condition follows trivially.

Next we consider isoloidity as an instrumental property.

Lemma 4.7 *Every normal derivation satisfies isoloidity criterion.*

Proof. If $\lambda \in iso\sigma(\delta_{CD})$, then $0 \in iso\sigma(\delta_{CD} - \lambda)$. Now $\lim_{n\to\infty} \|(\delta_{CD} - \lambda)^n x_{11}\|^{\frac{1}{n}} = 0$. The operator C_1 and $D_1 + \lambda$ in $\delta_{C_1(D_1+\lambda)} = \delta_{C_1D_1} - \lambda$ being normal,

$$Lim_{n\to\infty}\|(\delta_{C_1}(D_1+\lambda)^n x_{11}\|^{\frac{1}{n}} \leftrightarrow (\delta_{C_1}(D_1+\lambda)^n x_{11} = 0).$$

That is if and only if 0 is an eigenvalue of $\delta_{C_1}(D_1+\lambda)$. Now $(\delta_{CD}-\lambda)x_{11}\oplus 0 = 0$. In the case in $\lambda \neq -1$, then

$$P(B(H)) = \begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & 0 \end{pmatrix} \in B(H) : (\triangle_{CD} - \lambda)x = \phi_{ID}(x)$$

or

$$\phi_{CD}(x) = 0$$

Thus, if $0 \in iso\sigma(\delta_{CD})$ then $P(B(H)) = \delta_{CD}^{-1}(0)$. So, $\delta_{CD} = \delta_{C_0D_0} \oplus \delta_{C_1D_1}$ where $\delta_{C_0D_0}$ is nilpotent and $\delta_{C_1D_1}$ is invertible.

Remark 4.8 *Operators satisfying isoloid property fall in several subclasses of other operators like normal, hyponormal, subnormal and compact which are key properties.*

Now we consider FP-property. We start by giving a key relationship. $Normal \subset M - normal \subset dorminant$ and $normal \subset p - normal \subset w - normal$. We consider a result on this relationship.

Proposition 4.9 *Let C be M-normal and let $D^\star$ be w-normal operators in $B(H)$. Then $\delta_{C,D}(X) = 0$ entails $\delta_{C^\star,D^\star} = 0$. Moreover, it satisfies the unitary property.*

Proof. Invariant subspaces for C and D are $\bar{r}an(X)$ and $(\ker(X))^\perp$ respectively because $\delta_{C,D}(X) = 0$. We can write $C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 \\ D_2 & D_3 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : H_2 \to H_1$ under the decompositions $H = H_1 = \bar{r}an(X) \oplus ran(X)^\perp$, $H = H_2 = (\ker X)^\perp \oplus \ker X$. From $\delta_{C,D}(X) = 0$, we get $C_1X_1 = X_1D_1$ where C_1 is M-hyponormal and D_1 is w-hyponormal. Let $C_1X_1 = X_1|D_1^\star|U$. Multiplying the two sides of this equation at right by $|D_1^\star|^{\frac{1}{2}}$, we obtain $C_1(X_1|D_1^\star|^{\frac{1}{2}}) = X_1|D_1^\star|U|D_1^\star|^{\frac{1}{2}} = (X_1|D_1^\star|^{\frac{1}{2}}\bar{D}_1^\star)$. The Alugthe transform $\bar{D}_1^\star$ of $D_1^\star$ is semi-hyponormal. Hence the pair $(C, \bar{D}_1^\star)$ satisfies the FPP. Thus, the restriction $C_1|_{\bar{r}an(X_1|D_1^\star|^{\frac{1}{2}})}$ and $\bar{D}_1^\star|_{\ker(X_1|D_1^\star|^{\frac{1}{2}})^\perp}$ are equivalent normal operators. Since X_1 is a quasi-affinity and $|D_1^\star|^{\frac{1}{2}}$ is injective, $\bar{r}an(X_1|D_1^\star|^{\frac{1}{2}}) = \bar{r}anX_1 = \bar{r}anX$ and $\ker(X_1|D_1^\star|^{\frac{1}{2}}) = \ker X_1 = \ker X$. The operator $D^\star$ and its restriction $D_1^\star$ on $(\ker X)^\perp$ is normal. Consequently, $\ker X$ reduces $D^\star$. Hence $D_2 = 0$. Similarly, C is M-normal and its restriction C_1 on $\bar{r}anX$ is normal. Then $\bar{r}anX$ reduces T. Thus $T_2 = 0$. Since the pair (C_1, D_1) satisfies FP-property, $C_1^\star X_1 = X_1D_1^\star$. Finally $C^\star X = XD^\star$.

Lemma 4.10 *Every (p, w)-normal operator and a p-normal operator satisfies FP-Property.*

Proof. FP-Property holds for Consider $H = H_2 = (\ker C)^\perp \oplus (\ker C)$, $H = H_2 = (\ker D^\star)^\perp \oplus (\ker D^\star)$. From equation $CX = XD$, we get $C_1X_1 = X_1D_1$ and $C_1X_2 = X_3D_1 = 0$. Since C_1 and D_1 are one to one, $X_2 = X_3 = 0$, C_1 is a one to one p-normal operator. Let $C_1 = U|C_1|$ be the polar decomposition of C_1. Equation above can be written as $U|C_1|X_1 = X_1D_1$. Multiplying the two sides of this equation on the left by $|C_1|^{\frac{1}{2}}$, we get $|C_1|^{\frac{1}{2}}U|C_1|^{\frac{1}{2}}|C_1|^{\frac{1}{2}}X_1 = |C_1|^{\frac{1}{2}}X_1D_1$ So $\bar{C}_1(|C_1|^{\frac{1}{2}}X_1) = (|C_1|^{\frac{1}{2}}X_1)D_1$. The Alugthe transform $\bar{C}_1$ of C_1 is $\frac{p}{2} - hyponormal$ and $D_1^\star$ is p-normal. The pair $(\bar{C}_1, D_1)$ satisfies the FP-propery. Thus, $\bar{C}_1^\star(|C_1|^{\frac{1}{2}}X_1) = (|C_1|^{\frac{1}{2}}X_1)D_1^\star$. Consequently, restrictions $\bar{C}_1|_{\bar{r}an(|C_1|^{\frac{1}{2}}X_1)}$ and $D_1|_{\ker(|C_1|^{\frac{1}{2}})^\perp}$ are unitarily equivalent normal operators. Since the operator $|C_1|^{\frac{1}{2}}$ and X_1 are one to one, the operator $|C_1|^{\frac{1}{2}}X_1$ so is. Thus, $(\ker(|C_1|^{\frac{1}{2}}X_1))^\perp = [0]^\perp = (\ker X_1)^\perp = (\ker X)^\perp$ and $\bar{r}an(|\bar{C}_1|^{\frac{1}{2}}X_1) = (\ker(|C_1|^{\frac{1}{2}}X_1))^\perp = [0]^\perp = \bar{r}an(X_1) = \bar{r}an(X)$. Thus, $\bar{C}_1$ is a normal operator. The operator C_1 so is. Therefore, $\bar{r}anX$ reduces C_1 and $(\ker X_1)^\perp$ reduces $D_1^\star$. Since C_1 is normal and $D_1^\star$ is p-normal, the FP-property holds for the pair (C_1, D_1). Thus, $C_1^\star X_1 = X_1D_1^\star$ and then $C^\star X = XD^\star$. The converse is true since pair (C, D) satisfies FP-Property.

Theorem 4.11 *Every (p, w)-normal operator is a log-normal operator.*

Proof. Let the restriction $C|_M$ be log-normal. This property helps us to prove this theorem in the sequel. Let $D_1 = U|D_1|$ be of D_1. $C_1X_1 = X_1|D_1^\star|U$. Multiplying the two sides of this equation on the right by $|D_1^\star|^{\frac{1}{2}}$, we get, $C_1(X_1|D_1^\star|^{\frac{1}{2}}) = (X_1|D_1^\star|^{\frac{1}{2}})|D_1^\star|^{\frac{1}{2}}U|D_1^\star|^{\frac{1}{2}} = (X_1|D_1^\star|^{\frac{1}{2}})\bar{D}_1^\star$ C_1 is p-w-normal and the Alugthe transform $\bar{D}_1^\star$ of $D_1^\star$ is $\frac{1}{2}$-hyponormal.By theorem above, the FP-property holds for the pair $(C_1, \bar{D}_1^\star)$. Hence $C_1^\star(X_1|D_1^\star|^{\frac{1}{2}}) = (X_1|D_1^\star|^{\frac{1}{2}})\bar{D}_1^\star$.

Further more, $C_1|_{\bar{r}an((X_1|D_1^\star|^{\frac{1}{2}}))}$ and $\bar{D}_1^\star|_{((X_1|D_1^\star|^{\frac{1}{2}}))^{\frac{1}{2}}}$ are unitarily equivalent normal operators. Since $|D_1^\star|^{\frac{1}{2}}$ and X_1 are one to one, the operator $(X_1|D_1^\star|^{\frac{1}{2}})$ so is.

Theorem 4.12 *Every injective derivation is normal for some positive operator X.*

Proof. Let decompositions $H = (\ker X)^\perp \oplus \ker X$ and $K = \bar{r}anX \oplus (\bar{r}anX)^\perp$ be considered. Then we have the following matrix representations: $C = \begin{pmatrix} C_1 & 0 \\ C_2 & C_3 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $C_1^\star$ is p-normal, D_1 is injective (p,k)-quasi-normal and X_1 is injective with dense range. Therefore, we have $X_1C_1x = XCx = DXx = D_1X_1x$ for $x \in (\ker X)^\perp$. That is, $X_1C_1 = D_1X_1$ and hence C_1 and D_1 are normal and $X_1C_1^\star = D_1^\star X_1$ by the FPP. $(\ker X)^\perp$ and $\bar{r}anX$ reduces $C^\star$ and D respectively. Hence, we obtain $XC^\star = D^\star X$. Therefore we recapture a generalized FP-property for p-normal operators.

The theorem can be proved for generalized scalar operators.

Corollary 4.13 *Normal derivations and their adjoints are orthogonal via FP-property.*

Proof. Let $\bar{C} \in B(\bar{H})$ and $\bar{D} \in Q(\bar{K})$ be positive and normal and $\bar{H} \subset H$, $\bar{K} \subset K$,

$$\bar{C} = \begin{pmatrix} C & C_1 \\ 0 & C \end{pmatrix}$$

and

$$\bar{D} = \begin{pmatrix} D & 0 \\ D_1 & D_2 \end{pmatrix}$$

Define $\bar{Y} : \bar{K} \to \bar{H}$ by

$$\bar{Y} = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$$

It can be seen that

$$\delta^n_{\bar{C},\bar{D}}(\bar{Y}) = \begin{pmatrix} \delta^n_{\bar{C},\bar{D}}(\bar{Y}) & 0 \\ 0 & 0 \end{pmatrix}$$

for all n, hence

$$lim_n \|\delta^n_{\bar{C},\bar{D}}(\bar{Y})\|^{\frac{1}{n}} = 0.$$

The rest follow trivially.

Now we consider the range-kernel orthogonality of normal operators in complex NS.

Proposition 4.14 *Let C, D, $N \in B(H)$ where N has the polar decomposition $N = U|N|$. Then the pair $(C, D) \in FPP(\delta(N))$ and*

(i). $[C, |N^\star|] = 0$.

(ii). $[D, |N|] = 0$.

(iii). $\delta_{C,D}(U) = 0$.

Proof. If $N \in \ker(\delta_{C,D})$ and $(C, D) \in FPP(\delta(N))$, then

$$\delta_{CD}(N) = 0 = \delta_{C^\star D^\star}(N) \tag{7}$$

and so let

$$D : \ker N (= \ker U) \to \ker N$$

Hence

$$\delta_{CD}(U) = 0$$

Since $\overline{r}anN$ reduces C (by (i)) and $\ker^\perp N$ reduces D by (ii), it follows from $\delta_{C,D}(N) = 0$ that $\delta_{C_1,D_1}(N) = 0$ where $C_1 = C|_{\overline{r}anN}$

$$D_1 = D|_{\ker^\perp N}$$

and the quasi-affinity

$$N_1 : \ker^\perp N \to \overline{r}anN$$

is

$$N_1 x = Nx$$

Let N_1 have the polar decomposition

$$N_1 = U_1|N_1|$$

then U_1 is a unitary and $|N_1|$ is a quasi-affinity. Clearly

$$[D_1, |D_1|] = 0 \tag{8}$$

Hence, $S\delta_{C_1D_2}(N_1) = 0$ implies that $\delta_{C_1D_2}(U_1) = 0$, that is

$$D_1 = U_1^\star$$

Thus,

$$D_1^\star|N_1| = |N_1|D_1^\star$$

implies

$$U_1^\star C_1^\star U_1|N_1| = |N_1|D_1^\star$$

or

$$\delta_{C_1^\star D_2^\star}(N_1) = 0$$

This implies that

$$\delta_{C^{\star}D^{\star}}(N) = 0. \tag{9}$$

Theorem 4.15 *Every normal derivation is BJ-orthogonal via FP-property.*

Proof. Since FP-property gives orthogonality in a general set-up, then by Proposition 4.14, the proof follows trivially.

5 Open Problems

Characterizations involving orthogonality of operators is an area with various applications with regard to the ever dynamic technological advances. In NS we are different types of orthogonality. Useful results have come up where operators possessing given conditions are chosen for Range-Kernel orthogonality to be established. However, most of the results have been focussing on one type of orthogonality called the Birkhoff-James which we have given more results on. The following question arise always naturally however we ask them in our context: **Problem 1:** Could there be a possibility for studying other types of orthogonality with respect to the range and the kernel of norm-attainable operators apart from the BJ-orthogonality? This problem has been partially solved by other authors. **Problem 2:** Can the orthogonality be obtained via CBS-inequality be deduce in the space of norm-attainable operators?

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