Int. J. Open Problems Compt. Math., Vol. 19, No. 1, March 2026
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Application of Kharrat-Toma Iterative Method for Solving Fractional Differential Equations

Ahmed ANBER

Laboratory of LAMDA-RO, University of Blida 1

Department of Mathematics University of USTO, Oran, 31000, Algeria

e-mail: ah.anber@gmail.com

Received 10 September 2025; Accepted 8 November 2025

Abstract

In this paper, we apply the Kharrat-Toma Iterative Method (KTIM) for solving some fractional differential equations with caputo derivative. This method is combined from the Iterative method and Kharrat-Toma Transform. The obtained results are compared with the exact solutions and some examples are given to show the accuracy of the method.
Keywords: *Caputo derivative, Fractional differential equation, Kharrat-Toma Iterative Method, Kharrat-Toma Transform.*
2010 Mathematics Subject Classification: 26A33, 35R11.

1 Introduction

In recent years, there is great interest in fractional differential equations due to their effective applications in many fields of science and engineering [8, 9, 10, 17, 18, 21]. Several numerical methods [1, 2, 3, 4, 5, 6, 7] exist for finding approximate solutions to these equations. Among these methods is the Kharrat-Thomas Iterative Method (KTIM) [16], which provides an efficient approach for finding explicit and numerical solutions for a wide class of fractional differential equations.

In this paper, we applied the Kharrat-Toma Iterative Method (KTIM) to obtain approximate solutions of fractional differential equations and system of fractional differential equations.

The paper is organized as follows: Section 2 we give some definitions and properties about fractional calculus. Section 3 we present the Kharrat-Toma Iterative Method. Section 4 we present some numerical examples of FDEs to show the effectiveness of the proposed method by means of some comparison with exact solution.

2 Preliminaries

In this section, we give some definitions and properties which will be used in this paper. For more details, we refer the interested reader to [12, 14, 15, 19].

Definition 1.1 *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$*, for a function* $f \in C([a,b])$ *is defined as*

$$J_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-s)^{\alpha-1} f(s)\, ds, & \alpha > 0, \\ \\ f(x), & \alpha = 0, \end{cases} \tag{2.1}$$

where $\frac{1}{\Gamma(\alpha)} = \int\limits_0^\infty e^{-t} t^{\alpha-1} dt.$

Definition 1.2 *The fractional derivative of* $f \in C^n([a,b])$ *in the sense of Caputo is defined as:*

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \qquad n-1 < \alpha < n, n \in \mathbb{N}^*, x \in [a,b] \tag{2.2}$$

Definition 1.3 [14] *The function* $f(x)$ *is said to have exponential order on every finite interval in* $[0, +\infty[$ *, If there exist a positive number* M *that satisfying:*

$$|f(x)| \leq M e^{\lambda x}, \quad \lambda > 0, x \geq 0.$$

Definition 1.4 [14]*The Kharrat-Toma transform of a function* $f(x)$ *is expressed as follows:*

$$B[f(x)] = G(s) = s^3 \int\limits_0^\infty f(x)\, e^{-\frac{x}{s^2}} dx, \quad x \geq 0, \tag{2.3}$$

The inverse Kharrat-Toma integral transform is defined as:

$$B^{-1}[G(s)] = f(x) = B^{-1}\left[s^3 \int\limits_0^\infty f(x)\, e^{-\frac{x}{s^2}} dx \right]. \tag{2.4}$$

The B integral transform states that, if $f(x)$ is piecewise continuous on $[0, +\infty)$ and has exponential order. The B^{-1} will be the inverse of the B integral transform.
The Kharrat-Toma transform of some functions is as follows:

$f(x)$	$B[f(x)] = G(s)$
1	s^5
x^n	$n!s^{2n+5} = \Gamma(n+1)\, s^{2n+5}, n \geq 0,$
$\sin(\lambda x)$	$\frac{\lambda s^7}{1+\lambda^2 s^4}$
$\cos(\lambda x)$	$\frac{s^5}{1+\lambda^2 s^4}$
$\sinh(\lambda x)$	$\frac{\lambda s^7}{1-\lambda^2 s^4}$
$\cosh(\lambda x)$	$\frac{s^5}{1-\lambda^2 s^4}$

TABLE 1: Kharrat-Toma transform of some functions

The Kharrat-Toma Transform is a linear operator, we have

$$B\left[\sum_{k=0}^{n} \lambda_k f_k(x)\right] = \sum_{k=0}^{n} \lambda_k B[f_k(x)], \tag{2.5}$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are non-zero constants.
Theorem 1.1 *The Kharrat-Toma Transform of Caputo fractional derivative of $f(t)$ of order α is given by*

$$B\left[{}^C D^\alpha(f(x))\right] = s^{-2\alpha} G(s) - \sum_{k=0}^{n-1} s^{2k-2\alpha+5} f^{(k)}(0), n-1 < \alpha < n, n \in \mathbb{N}^*. \tag{2.6}$$

Definition 1.5 *A one-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_\alpha(t) = \sum_{n \geq 0} \frac{(t^\alpha)^n}{\Gamma(n\alpha+1)!}. \tag{2.7}$$

3 Basic Idea of Kharrat-Toma Iterative Method

This section discusses a Kharrat-Toma Iterative Method to solve fractional differential equation numerically. For more details, we refer the interested reader to [16].
To clarify the basic ideas of KTIM, we consider the following nonlinear fractional differential equation

$${}^C D_t^\alpha(u(t,x)) + R(u(t,x)) + N(u(t,x)) = f(t,x), \quad n-1 < \alpha \leq n, n \in \mathbb{N}, \tag{3.1}$$

with the initial condition

$$u^{(k)}(0,x) = b_k(x), \tag{3.2}$$

where ${}^{C}D^{\alpha}$ denotes the fractional order derivative in Caputo sense. R is a linear operator. N is a nonlinear operator and $f(t,x)$ is a known function. Applying the Kharrat-TomaTransform to both sides of $Eq.\,(3.1)$ and by using the linearity of KharratToma Transform, the result is

$$B\left[{}^{C}D_t^{\alpha}(u(t,x))\right] + B\left[R(u(t,x))\right] + B\left[N(u(t,x))\right] = B\left[f(t,x)\right]. \tag{3.3}$$

Using (2.6), we get

$$B\left[u(t,x)\right] = \frac{1}{s^{-2\alpha}}\left(\sum_{k=0}^{n-1} s^{2k-2\alpha+5}u^{(k)}(0,x) + B\left[f(t,x)\right] - B\left[R(u(t,x))\right] - B\left[N(u(t,x))\right]\right) \tag{3.4}$$

The KTIM represents the solution as an infinite series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x). \tag{3.5}$$

Substituting Eq. (3.5) in Eq. (3.4), we have

$$B\left[\sum_{n=0}^{\infty} u_n(t,x)\right] = \frac{1}{s^{-2\alpha}}\left(\begin{array}{c} \sum_{k=0}^{n-1} s^{2k-2\alpha+5}u^{(k)}(0,x) + B\left[f(t,x)\right] - B\left[\sum_{n=0}^{\infty} R(u_n(t,x))\right] \\ -B\left[N(u_0(t,x)) + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} N(u_n(t,x)) - \sum_{k=0}^{n-1} N(u_n(t,x))\right)\right] \end{array}\right) \tag{3.6}$$

Hence the iterations are defined by the following recursive algorithm

$$B\left[u_0(t,x)\right] = \frac{1}{s^{-2\alpha}}\left(\sum_{k=0}^{n-1} s^{2k-2\alpha+5}u^{(k)}(0,x) + B\left[f(t,x)\right]\right)$$

$$B\left[u_1(t,x)\right] = -\frac{1}{s^{-2\alpha}}B\left[R(u_0(t,x)) + N(u_0(t,x))\right]$$

.

.

.

$$B\left[u_n(t,x)\right] = -\frac{1}{s^{-2\alpha}}B\left[R(u_{n-1}(t,x)) + \sum_{k=0}^{n} N(u_k(t,x)) - \sum_{k=0}^{n-1} N(u_k(t,x))\right], n \geq 1 \tag{3.7}$$

Using the initial conditions (3.2) and applying the inverse Kharrat-TomaTransform to equations (3.7) we obtain the values $u_i\left(t,x\right), i \in \{0,1,...,n\}$.
Therefore the n-term approximate solution is given by

$$u\left(t,x\right) = u_0\left(t,x\right) + u_1\left(t,x\right) + u_2\left(t,x\right) + ... \tag{3.8}$$

Theorem 3.1 *Let B be a Banach space. If there exists k, $0 \leq k < 1$ such that, $\|u_n\| \leq k\|u_{n-1}\|$ for $\forall n \in \mathbb{N}$, then the approximate solution $u\left(t,x\right)$ converges to S.*
Preuve: Define the sequence S_i, $i = 0,1,...,n$

$$\begin{aligned} S_1 &= u_0 \\ S_2 &= u_0 + u_1 \\ S_3 &= u_0 + u_1 + u_2 \\ &. \\ &. \\ &. \\ S_n &= u_0 + u_1 + u_2 + ... + u_{n-1}, \end{aligned}$$

and prove that $\left(S_i\right)_{i\geq 0}$ is a Cauchy sequence, and we consider

$$\|S_n - S_{n-1}\| \leq \|u_n\| \leq k^n u_0,$$

for $p > q > 0$, we have

$$\begin{aligned} \|S_p - S_q\| &= \|S_p - S_{p-1} + S_{p-1} - S_{p-2} + ... + S_{q+1} - S_q\| \\ &\leq \|S_p - S_{p-1}\| + \|S_{p-1} - S_{p-2}\| + ... + \|S_{q+1} - S_q\| \\ &\leq \left(k^p + k^{p-1} + ... + k^{q+1}\right) u_0 \\ &\leq \left\| \tfrac{k^{q+1}\left(1-k^{p-q}\right)}{k-1} \right\| u_0, \end{aligned}$$

where u_0 is bounded, and we have

$$\lim_{p,q\to\infty} \|S_p - S_q\| = 0.$$

Therefore, the sequence $\left(S_i\right)_{i\geq 0}$ is a Cauchy sequence in B, so the solution of Eq. (3.1) is convergent.
Remark Similar proofs can be found in [11, 22] .

4 Numerical examples

In this section, we apply the KTIM method to get the solutions of fractional partial differential equations.

4.1 Example 1

Consider the fractional Klein-Gordon equation [20]

$$ {}^{C}D_t^{\alpha}\left(u\left(t,x\right)\right) = u\left(t,x\right) + u_{xx}\left(t,x\right), \quad 0 < \alpha \leq 1, \tag{4.1}$$

with the initial condition:

$$ u\left(0,x\right) = 1 + \sin x. \tag{4.2}$$

The exact solution of (4.1) for the special case $\alpha = 1$ is

$$ u\left(t,x\right) = \sin x + e^{t}. \tag{4.3}$$

Applying the Kharrat-Toma Transform in the Eq. (4.1), then

$$ B\left[{}^{C}D_t^{\alpha}\left(u\left(t,x\right)\right)\right] = B\left[u\left(t,x\right) + u_{xx}\left(t,x\right)\right], \tag{4.4}$$

Using (2.7) and the initial conditions (4.2), then we have

$$ B\left[\left(u\left(t,x\right)\right)\right] = \tfrac{1}{s^{-2\alpha}}B\left[u_{xx}\left(t,x\right)\right] + \tfrac{1}{s^{-2\alpha}}B\left[u\left(t,x\right)\right] + \tfrac{1}{s^{-2\alpha}}s^{-2\alpha+5}\left(1+\sin x\right). \tag{4.5}$$

Applying the inverse Kharrat-Toma Transform in Eq. (4.5) we obtain

$$ u\left(t,x\right) = B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{xx}\left(t,x\right)\right] + \tfrac{1}{s^{-2\alpha}}B\left[u\left(t,x\right)\right] + \tfrac{1}{s^{-2\alpha}}s^{-2\alpha+5}\left(1+\sin x\right)\right]. \tag{4.6}$$

In the view of the recurrence relations (3.7) we get

$$\begin{aligned}
&u_0\left(t,x\right) = \left(1+\sin x\right)B^{-1}\left[\tfrac{1}{s^{-2\alpha}}s^{-2\alpha+5}\right] = 1+\sin x \\
&u_1\left(t,x\right) = B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_0\left(t,x\right)\right]\right] + B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{0xx}\left(t,x\right)\right]\right] = \tfrac{1}{\Gamma(\alpha+1)}t^{\alpha} \\
&u_2\left(t,x\right) = B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_1\left(t,x\right)\right]\right] + B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{1xx}\left(t,x\right)\right]\right] = \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
&u_3\left(t,x\right) = B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_2\left(t,x\right)\right]\right] + B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{2xx}\left(t,x\right)\right]\right] = \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\
&. \\
&. \\
&. \\
&u_n\left(t,x\right) = B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{n-1}\left(t,x\right)\right]\right] + B^{-1}\left[\tfrac{1}{s^{-2\alpha}}B\left[u_{(n-1)xx}\left(t,x\right)\right]\right] = \tfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}
\end{aligned} \tag{4.7}$$

Thus the approximate solution of (4.1) is

$$\begin{aligned} u(x,t) &= 1+\sin x+\tfrac{1}{\Gamma(\alpha+1)}t^{\alpha}+\tfrac{1}{\Gamma(2\alpha+1)}t^{2\alpha}+\tfrac{1}{\Gamma(3\alpha+1)}t^{3\alpha}+\ldots \\ &= \sin x+\sum_{n\geq 0}\tfrac{1}{\Gamma(n\alpha+1)}t^{n\alpha}=\sin x+E_{\alpha}(t). \end{aligned} \tag{4.8}$$

When $\alpha = 1$, the exact solution of the linear fractional Klein–Gordon equation is as follows:

$$u(x,t) = \sin x + E_1(t) = \sin x + e^t.$$

(x,t)	**Numerical solution with KTIM for $n = 10$**	**Exact solution** $u(x,t)$	**Error**
$(0,0)$	1	1	0
$(0,0.25)$	1.284025417	1.284025417	0
$(0,0.5)$	1.648721271	1.648721271	0
$(0,0.75)$	2.117000017	2.117000017	0
$(0,1)$	2.718281828	2.718281828	0
$(0,1.5)$	4.481689067	4.481689070	0.3×10^{-8}
$(0,75)$	5.754602643	5.754602676	0.33×10^{-7}
$(0,2)$	7.389055882	7.389056099	0.217×10^{-6}
$(0,2.5)$	12.18248885	12.18249396	0.511×10^{-5}
$(0,3)$	20.08546859	20.08553692	0.6833×10^{-4}

TABLE 2: Describe a comparison between the exact solution and the numerical solution using the KTIM of Eq.(4.1) for $\alpha = 1$

Figure 4.1 is a graph of the exact solution (4.3) and the numerical solution (4.8) using KTIM method for $\alpha = 0.5, 0.75$ and 1.

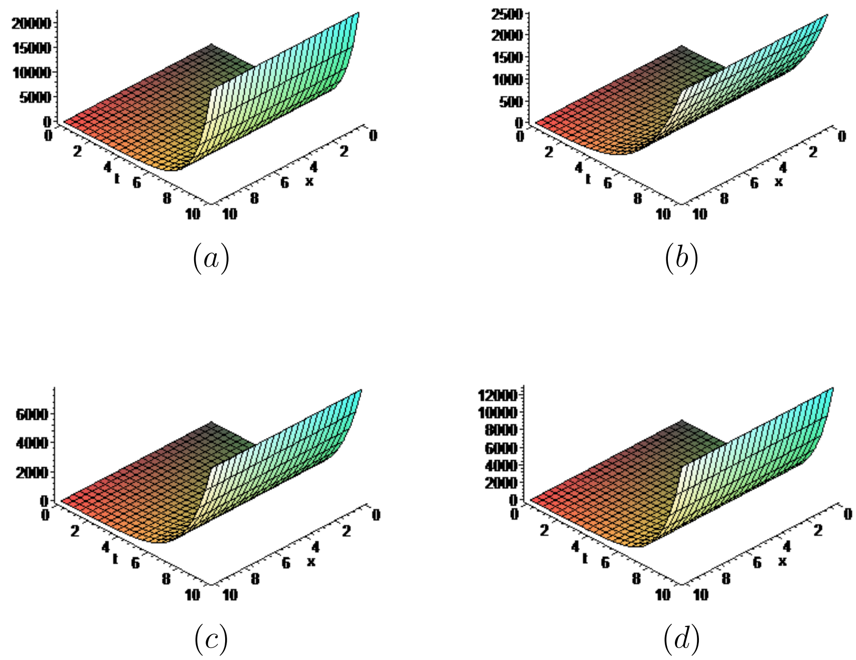


Fig 4.1: Graph of (4.11) and (4.15)
(a) Exact solution (4.3), (b) Numerical solution (4.8) for $\alpha = 0.5$,
(c) Numerical solution (4.8) for $\alpha = 0.75$, (d) Numerical solution (4.8) for $\alpha = 1$.

4.2 Example 2

Consider the following system of linear fractional partial differential equations [13]

$$\begin{cases} {}^{C}D_{t}^{\alpha}\left(u\left(t,x\right)\right) = v_{x}\left(t,x\right) - u\left(t,x\right) - v\left(t,x\right) \\ {}^{C}D_{t}^{\beta}\left(v\left(t,x\right)\right) = u_{x}\left(t,x\right) - u\left(t,x\right) - v\left(t,x\right) \end{cases}, \ 0 < \alpha, \beta \leq 1, \tag{4.9}$$

with the initial condition:

$$u\left(0,x\right) = \sinh x, \ \ v\left(0,x\right) = \cosh x. \tag{4.10}$$

The exact solution of (4.9) for the special case $\alpha = \beta = 1$ is

$$\begin{cases} u\left(t,x\right) = \sinh x \cosh t - \cosh x \sinh t \\ v\left(t,x\right) = \cosh x \cosh t - \sinh x \sinh t \end{cases} \tag{4.11}$$

Applying the Kharrat-Toma Transform in the Eq. (4.9), then

$$\begin{cases} B\left[{}^{C}D_t^{\alpha}\left(u\left(t,x\right)\right)\right] = \frac{1}{s^{-2\alpha}}B\left[v_x\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[v\left(t,x\right)\right] \\ B\left[{}^{C}D_t^{\beta}\left(v\left(t,x\right)\right)\right] = \frac{1}{s^{-2\beta}}B\left[u_x\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[v\left(t,x\right)\right] \end{cases} \tag{4.12}$$

Using (2.6) and the initial conditions (4.10), then we have

$$\begin{cases} B\left[\left(u\left(t,x\right)\right)\right] = \frac{1}{s^{-2\alpha}}B\left[v_x\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[v\left(t,x\right)\right] + \frac{1}{s^{-2\alpha}}s^{-2\alpha+5}\sinh x \\ B\left[\left(v\left(t,x\right)\right)\right] = \frac{1}{s^{-2\beta}}B\left[u_x\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[v\left(t,x\right)\right] + \frac{1}{s^{-2\beta}}s^{-2\beta+5}\cosh x \end{cases} \tag{4.13}$$

Applying the inverse Kharrat-Toma Transform in Eq. (4.13) we obtain

$$\begin{cases} u\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[v_x\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[v\left(t,x\right)\right] + \frac{1}{s^{-2\alpha}}s^{-2\alpha+5}\sinh x\right] \\ v\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[u_x\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[u\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[v\left(t,x\right)\right] + \frac{1}{s^{-2\beta}}s^{-2\beta+5}\cosh x\right] \end{cases} \tag{4.14}$$

In the view of the recurrence relations (3.7) we get

$$\begin{cases} u_0\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}s^{-2\alpha+5}\sinh x\right] = \sinh x, \\ v_0\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\beta}}s^{-2\beta+5}\cosh x\right] = \cosh x. \end{cases}$$

For $n = 1$, we have

$$\begin{cases} u_1\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[v_{0x}\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[u_0\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[v_0\left(t,x\right)\right]\right] \\ = B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[\sinh x\right] - \frac{1}{s^{-2\alpha}}B\left[\sinh x\right] - \frac{1}{s^{-2\alpha}}B\left[\cosh x\right]\right] = -\frac{\cosh x}{\Gamma(\alpha+1)}t^{\alpha} \\ v_1\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\beta}}B\left[u_{0x}\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[u_0\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[v_0\left(t,x\right)\right]\right] \\ = B^{-1}\left[\frac{1}{s^{-2\beta}}B\left[\cosh x\right] - \frac{1}{s^{-2\beta}}B\left[\sinh x\right] - \frac{1}{s^{-2\beta}}B\left[\cosh x\right]\right] = -\frac{\sinh x}{\Gamma(\beta+1)}t^{\beta}. \end{cases}$$

For $n = 2$, we have

$$\begin{cases} u_2\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[v_{1x}\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[u_1\left(t,x\right)\right] - \frac{1}{s^{-2\alpha}}B\left[v_1\left(t,x\right)\right]\right] \\ = \frac{\sinh x - \cosh x}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta} + \frac{\cosh x}{\Gamma(2\alpha+1)}t^{2\alpha} \\ v_2\left(t,x\right) = B^{-1}\left[\frac{1}{s^{-2\beta}}B\left[u_{1x}\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[u_1\left(t,x\right)\right] - \frac{1}{s^{-2\beta}}B\left[v_1\left(t,x\right)\right]\right] \\ = \frac{\cosh x - \sinh x}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta} + \frac{\sinh x}{\Gamma(2\beta+1)}t^{2\beta} \end{cases}$$

And for $n=3$, we get

$$\begin{cases} u_3(t,x)=B^{-1}\left[\frac{1}{s^{-2\alpha}}B\left[v_{2x}(t,x)\right]-\frac{1}{s^{-2\alpha}}B\left[u_2(t,x)\right]-\frac{1}{s^{-2\alpha}}B\left[v_2(t,x)\right]\right] \\ =\frac{\sinh x-\cosh x}{\Gamma(2\alpha+\beta+1)}t^{2\alpha+\beta}+\frac{\cosh x-\sinh x}{\Gamma(\alpha+2\beta+1)}t^{\alpha+2\beta}-\frac{\cosh x}{\Gamma(3\alpha+1)}t^{3\alpha} \\ \\ v_3(t,x)=B^{-1}\left[\frac{1}{s^{-2\beta}}B\left[u_{2x}(t,x)\right]-\frac{1}{s^{-2\beta}}B\left[u_2(t,x)\right]-\frac{1}{s^{-2\beta}}B\left[v_2(t,x)\right]\right] \\ =\frac{\sinh x-\cosh x}{\Gamma(2\alpha+\beta+1)}t^{2\alpha+\beta}+\frac{\cosh x-\sinh x}{\Gamma(\alpha+2\beta+1)}t^{\alpha+2\beta}-\frac{\sinh x}{\Gamma(3\beta+1)}t^{3\beta} \end{cases}$$

Thus the approximate solution of (4.9) is

$$\begin{cases} u(t,x)=(\sinh(x))\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}-\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+...\right) \\ -(\cosh(x))\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}-\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...\right) \\ \\ v(t,x)=(\cosh(x))\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+...\right) \\ -(\sinh(x))\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}+\frac{t^{3\beta}}{\Gamma(3\beta+1)}+...\right) \end{cases} \tag{4.15}$$

When $\alpha=\beta=1$, the exact solution of (4.9) is as follows:

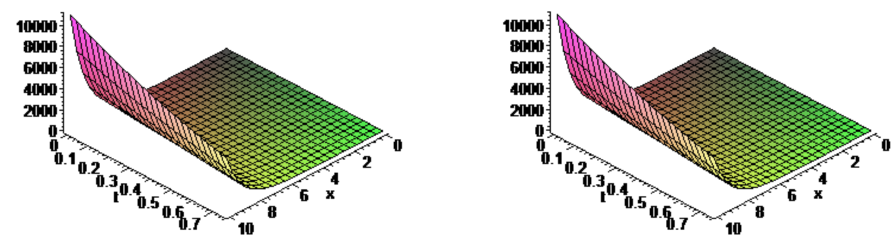
$$\begin{cases} u(t,x)=(\sinh x)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+...\right)-(\cosh x)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+...\right) \\ =(\sinh x)\sum\limits_{n\geq 0}\frac{x^{2n}}{(2n)!}-(\cosh x)\sum\limits_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}=\sinh x\cosh t-\cosh x\sinh t \\ \\ v(t,x)=(\cosh x)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+...\right)-(\sinh x)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+...\right) \\ =(\cosh x)\sum\limits_{n\geq 0}\frac{x^{2n}}{(2n)!}-(\sinh x)\sum\limits_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}=\cosh x\cosh t-\sinh x\sinh t \end{cases} \tag{4.16}$$

(x,t)	**Exact Solution** $u\,(x,t)$	**Numerical solution with KTIM for** $n=2$	**Error**
$(0,0)$	0	0	0
$(0,0.01)$	-0.01000016667	-0.1	$0.16667{\times}10^{-6}$
$(0,0.02)$	-0.02000133336	-0.02	$0.133336{\times}10^{-5}$
$(0,0.03)$	-0.03000450020	-0.03	$0.450020{\times}10^{-5}$
$(0,0.04)$	-0.04001066752	-0.04	$0.1066752{\times}10^{-4}$
$(0,0.05)$	-0.05002083594	-0.05	$0.2083594{\times}10^{-4}$
$(0,0.06)$	-0.06003600648	-0.06	$0.3600648{\times}10^{-4}$
$(0,0.07)$	-0.07005718067	-0.07	$0.5718067{\times}10^{-4}$
$(0,0.08)$	-0.08008536064	-0.08	$0.8536064{\times}10^{-4}$
$(0,0.09)$	-0.09012154922	-0.09	$0.12154922{\times}10^{-3}$
$(0,0.1)$	-0.10016675	-0.1	$0.1667500{\times}10^{-3}$

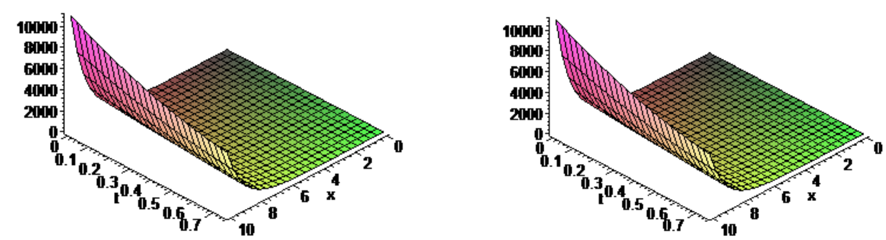
(x,t)	**Exact Solution** $v\,(x,t)$	**Numerical solution with KTIM for** $n=2$	**Error**
$(0,0)$	1	1	0
$(0,0.01)$	1.00005	1.00005	0
$(0,0.02)$	1.000200007	1.0002	$0.7{\times}10^{-8}$
$(0,0.03)$	1.000450034	1.00045	$0.34{\times}10^{-7}$
$(0,0.04)$	1.000800107	1.0008	$0.107{\times}10^{-6}$
$(0,0.05)$	1.00125026	1.00125	$0.26{\times}10^{-6}$
$(0,0.06)$	1.00180054	1.0018	$0.54{\times}10^{-6}$
$(0,0.07)$	1.002451001	1.00245	$0.1001{\times}10^{-5}$
$(0,0.08)$	1.003201707	1.0032	$0.1707{\times}10^{-5}$
$(0,0.09)$	1.004052734	1.00405	$0.2734{\times}10^{-5}$
$(0,0.1)$	1.005004168	1.005	$0.4168{\times}10^{-5}$

TABLE 3: Describe a comparison between the exact solution and the numerical solution using the KTIM of Eq.(4.9) for $\alpha=\beta=1$

Figure 4.2 is a graph of the exact solution (4.11) of Eq.(4.9) for $\alpha=\beta=1$ and the numerical solution (4.15) using KTIM method for $\alpha=\beta=0.5$ and $\alpha=\beta=1$.



(a)



(b)

Fig 4.2: Graph of (4.11) and (4.15)
(a) Exact solution (4.11), (b) Numerical solution (4.15) for $\alpha = \beta = 1$

4.3 Example 3

Consider the following nonlinear system of fractional differential equations [13]

$$\begin{cases} {}^{C}D_t^{\alpha}(u) = v_y w_x - v_x w_y - u \\ {}^{C}D_t^{\beta}(v) = -u_y w_x - u_x w_y + v \quad , 0 < \alpha, \beta, \gamma \leq 1, \\ {}^{C}D_t^{\gamma}(w) = -u_y v_x - u_x v_y + w \end{cases} \tag{4.17}$$

with the initial conditions

$$u(0, x, y) = e^{x+y},\ v(0, x, y) = e^{x-y}, w(0, x, y) = e^{-x+y}. \tag{4.18}$$

The exact solution of (4.17) for the special case $\alpha = \beta = 1$ is

$$\begin{cases} u(t, x, y) = e^{x+y-t} \\ v(t, x, y) = e^{x-y+t} \\ w(t, x, y) = e^{-x+y+t} \end{cases} \tag{4.19}$$

Applying the Kharrat-Toma Transform in the Eq. (4.17), then

$$\begin{cases} B\left[{}^{C}D_{t}^{\alpha}\left(u\left(t,x,y\right)\right)\right] = \frac{1}{s^{-2\alpha}}B\left[v_{y}w_{x}\right] - \frac{1}{s^{-2\alpha}}B\left[v_{x}w_{y}\right] - \frac{1}{s^{-2\alpha}}B\left[u\right] \\ B\left[{}^{C}D_{t}^{\beta}\left(v\left(t,x,y\right)\right)\right] = -\frac{1}{s^{-2\beta}}B\left[u_{y}w_{x}\right] - \frac{1}{s^{-2\beta}}B\left[u_{x}w_{y}\right] + \frac{1}{s^{-2\beta}}B\left[v\right] \\ B\left[{}^{C}D_{t}^{\gamma}\left(w\left(t,x,y\right)\right)\right] = -\frac{1}{s^{-2\gamma}}B\left[u_{y}v_{x}\right] - \frac{1}{s^{-2\gamma}}B\left[u_{x}v_{y}\right] + \frac{1}{s^{-2\gamma}}B\left[w\right] \end{cases} \tag{4.20}$$

Using (2.6) and the initial conditions (4.18), then we have

$$\begin{cases} B\left[\left(u\left(t,x,y\right)\right)\right] = \frac{1}{s^{-2\alpha}}B\left[v_{y}w_{x}\right] - \frac{1}{s^{-2\alpha}}B\left[v_{x}w_{y}\right] - \frac{1}{s^{-2\alpha}}B\left[u\right] + \frac{1}{s^{-2\alpha}}s^{-2\alpha+5}e^{x+y} \\ B\left[\left(v\left(t,x,y\right)\right)\right] = -\frac{1}{s^{-2\beta}}B\left[u_{y}w_{x}\right] - \frac{1}{s^{-2\beta}}B\left[u_{x}w_{y}\right] + \frac{1}{s^{-2\beta}}B\left[v\right] + \frac{1}{s^{-2\beta}}s^{-2\beta+5}e^{x-y} \\ B\left[\left(w\left(t,x,y\right)\right)\right] = -\frac{1}{s^{-2\gamma}}B\left[u_{y}v_{x}\right] - \frac{1}{s^{-2\gamma}}B\left[u_{x}v_{y}\right] + \frac{1}{s^{-2\gamma}}B\left[w\right] + \frac{1}{s^{-2\gamma}}s^{-2\gamma+5}e^{-x+y} \end{cases} \tag{4.21}$$

Applying the inverse Kharrat-Toma Transform in Eq. (4.21) we obtain

$$\begin{cases} u\left(t,x,y\right) = B^{-1}\left(\frac{1}{s^{-2\alpha}}B\left[v_{y}w_{x}\right] - \frac{1}{s^{-2\alpha}}B\left[v_{x}w_{y}\right] - \frac{1}{s^{-2\alpha}}B\left[u\right] + \frac{1}{s^{-2\alpha}}s^{-2\alpha+5}e^{x+y}\right) \\ v\left(t,x,y\right) = B^{-1}\left(-\frac{1}{s^{-2\beta}}B\left[u_{y}w_{x}\right] - \frac{1}{s^{-2\beta}}B\left[u_{x}w_{y}\right] + \frac{1}{s^{-2\beta}}B\left[v\right] + \frac{1}{s^{-2\beta}}s^{-2\beta+5}e^{x-y}\right) \\ w\left(t,x,y\right) = B^{-1}\left(-\frac{1}{s^{-2\gamma}}B\left[u_{y}v_{x}\right] - \frac{1}{s^{-2\gamma}}B\left[u_{x}v_{y}\right] + \frac{1}{s^{-2\gamma}}B\left[w\right] + \frac{1}{s^{-2\gamma}}s^{-2\gamma+5}e^{-x+y}\right) \end{cases} \tag{4.22}$$

In the view of the recurrence relations (3.7) we get

$$\begin{cases} u_{0}\left(t,x,y\right) = B^{-1}\left[\frac{1}{s^{-2\alpha}}s^{-2\alpha+5}e^{x+y}\right] = e^{x+y}, \\ v_{0}\left(t,x,y\right) = B^{-1}\left[\frac{1}{s^{-2\beta}}s^{-2\beta+5}e^{x-y}\right] = e^{x-y}, \\ w_{0}\left(t,x,y\right) = B^{-1}\left[\frac{1}{s^{-2\gamma}}s^{-2\gamma+5}e^{-x+y}\right] = e^{-x+y}, \end{cases}$$

for $n = 1$, we have

$$\begin{cases} u_{1}\left(t,x,y\right) = B^{-1}\left(\frac{1}{s^{-2\alpha}}B\left[v_{0y}w_{0x}\right] - \frac{1}{s^{-2\alpha}}B\left[v_{0x}w_{0y}\right] - \frac{1}{s^{-2\alpha}}B\left[u_{0}\right]\right) \\ = -\frac{e^{x+y}}{\Gamma(\alpha+1)}t^{\alpha} \\ v_{1}\left(t,x,y\right) = B^{-1}\left(-\frac{1}{s^{-2\beta}}B\left[u_{0y}w_{0x}\right] - \frac{1}{s^{-2\beta}}B\left[u_{0x}w_{0y}\right] + \frac{1}{s^{-2\beta}}B\left[v_{0}\right]\right) \\ = \frac{e^{x-y}}{\Gamma(\beta+1)}t^{\beta} \\ w_{1}\left(t,x,y\right) = B^{-1}\left(-\frac{1}{s^{-2\gamma}}B\left[u_{0y}v_{0x}\right] - \frac{1}{s^{-2\gamma}}B\left[u_{0x}v_{0y}\right] + \frac{1}{s^{-2\gamma}}B\left[w_{0}\right]\right) \\ = \frac{e^{-x+y}}{\Gamma(\gamma+1)}t^{\gamma}. \end{cases}$$

For $n=2$, we obtain

$$\begin{cases} u_2(t,x,y) = B^{-1}\begin{pmatrix} \frac{1}{s^{-2\alpha}}B\left[(v_0+v_1)_y(w_0+w_1)_x\right] - \frac{1}{s^{-2\alpha}}B\left[(v_0)_y(w_0)_x\right] \\ -\frac{1}{s^{-2\alpha}}B\left[(v_0+v_1)_x(w_0+w_1)_y\right] + \frac{1}{s^{-2\alpha}}B\left[(v_0)_x(w_0)_y\right] - \frac{1}{s^{-2\alpha}}B[u_1] \end{pmatrix} \\ = \frac{e^{x+y}}{\Gamma(2\alpha+1)}t^{2\alpha}, \\ v_2(t,x,y) = B^{-1}\begin{pmatrix} -\frac{1}{s^{-2\beta}}B\left[(u_0+u_1)_y(w_0+w_1)_x\right] + \frac{1}{s^{-2\beta}}B\left[(u_0)_y(w_0)_x\right] \\ -\frac{1}{s^{-2\alpha}}B\left[(u_0+u_1)_x(w_0+w_1)_y\right] + \frac{1}{s^{-2\alpha}}B\left[(u_0)_x(w_0)_y\right] + \frac{1}{s^{-2\beta}}B[v_1] \end{pmatrix} \\ = \frac{e^{x-y}}{\Gamma(2\beta+1)}t^{2\beta}, \\ w_2(t,x,y) = B^{-1}\begin{pmatrix} -\frac{1}{s^{-2\beta}}B\left[(u_0+u_1)_y(v_0+v_1)_x\right] + \frac{1}{s^{-2\beta}}B\left[(u_0)_y(v_0)_x\right] \\ -\frac{1}{s^{-2\alpha}}B\left[(u_0+u_1)_x(v_0+v_1)_y\right] + \frac{1}{s^{-2\alpha}}B\left[(u_0)_x(v_0)_y\right] + \frac{1}{s^{-2\gamma}}B[w_1] \end{pmatrix} \\ = \frac{e^{-x+y}}{\Gamma(2\gamma+1)}t^{2\gamma}. \end{cases}$$

Thus the approximate solution of (4.17) is

$$\begin{cases} u(t,x,y) = e^{x+y} - \frac{e^{x+y}}{\Gamma(\alpha+1)}t^{\alpha} + \frac{e^{x+y}}{\Gamma(2\alpha+1)}t^{2\alpha} + ... = e^{x+y}\sum\limits_{n\geq 0}\frac{(-t^{\alpha})^n}{\Gamma(n\alpha+1)!} = e^{x+y}E_{\alpha}(-t) \\ v(t,x,y) = e^{x-y} + \frac{e^{x-y}}{\Gamma(\beta+1)}t^{\beta} + \frac{e^{x-y}}{\Gamma(2\beta+1)}t^{2\beta} + ... = e^{x-y}\sum\limits_{n\geq 0}\frac{\left(t^{\beta}\right)^n}{\Gamma(n\beta+1)!} = e^{x-y}E_{\beta}(t) \\ w(t,x,y) = e^{-x+y} + \frac{e^{-x+y}}{\Gamma(\gamma+1)}t^{\gamma} + \frac{e^{-x+y}}{\Gamma(2\gamma+1)}t^{2\gamma} + ... = e^{-x+y}\sum\limits_{n\geq 0}\frac{(t^{\gamma})^n}{\Gamma(n\gamma+1)!} = e^{-x+y}E_{\gamma}(t) \end{cases} \tag{4.23}$$

When $\alpha=\beta=\gamma=1$, the exact solution of (4.17) is as follows:

$$\begin{cases} u(t,x,y) = e^{x+y}E_1(-t) = e^{x+y-t} \\ v(t,x,y) = e^{x-y}E_1(t) = e^{x-y+t} \\ w(t,x,y) = e^{-x+y}E_1(t) = e^{-x+y+t}. \end{cases}$$

5 Conclusion and Open Problem

In this paper, we have presented Kharrat-Toma Iterative Method for solving fractional differential equations. The illustrative examples confirm the validity of this method.
At the end of this paper, we shall propose the following open question: We think it is important to address a comparative study with other numerical methods.

References

[1] A. Anber, I. Jebril, Z. Dahmani, N. Bedjaoui, A. Lamamri: *The Tanh Method and the (G'/G)-Expantion Method for Solving the Space -Time Conformable FZK and FZZ Evolution Equations*. International Journal of Innovative Computing, Information and Control, Vol. 20, Iss. 2, pp.557 - 573 , 2024.

[2] A. Anber, Z. Dahmani, M.Z. Sarikaya: *Two numerical methods for solving conformable fractional KGE equation*. Journal of Interdisciplinary Mathematics, Vol. 27, Iss. 6, pp. 1361–1370, 2024.

[3] A. Anber, Z. Dahmani: *The LDM and the CVIM Methods for Solving Time and Space Fractional Wu–Zhang Differential System*, Int. J. Open Problems Compt. Math., Vol. 17, No. 3, pp. 1-18, 2024.

[4] A. Anber, Z. Dahmani: *The SGEM Method For Solving Some Time and Space-Conformable Fractional Evolution Problems*, Int. J. Open Problems Compt. Math., Vol. 16, No. 1, pp. 33-43, 2023.

[5] A. Anber, Z. Dahmani: *Two Numerical Methods for Solving the Fractional Thomas-Fermi Equation.* Journal of Interdisciplinary Mathematics, Vol. 18, Iss. 1-2, pp.35-41, 2015.

[6] A. Anber, Z. Dahmani: *The Homotopy Analysis Method for Solving Some Fractional Differential Equations.* Journal of Interdisciplinary Mathematics, Vol. 17, Iss. 3, pp.255-269, 2014.

[7] Z. Dahmani, A. Anber, I. Jebril: *Solving Conformable Evolution Equations by an Extended Numerical Method,* Jordan Journal of Mathematics and Statistics (JJMS), Vol.15, Iss.2, pp.363-280, 2021.

[8] K.R. Cheneke, K.P. Rao, G.K. Edessa: *Application of a new generalized fractional derivative and rank of control measures on Cholera transmission dynamics.* Int. J. Math. Math. Sci. 2021, 2021, 2104051.

[9] K. Hattaf: *On the Stability and Numerical Scheme of Fractional Differential Equations with Application to Biology.* Computation 2022, 10, 97.

[10] K. Hattaf, N. Yousfi: *Global stability for fractional diffusion equations in biological systems.* Complexity 2020, 2020, 5476842.

[11] Y. He and W. Zhang: *Application of the Elzaki iterative method to fractional partial differential equations.* Boundary Value Problems, 6, 2023.

[12] R. Gorenflo, F. Mainardi, Fractional calculus: *integral and differential equations of fractional order*. Springer Verlag, Wien, 223-276, 1997.

[13] H. Jafari and S. Seifi: *Solving a system of nonlinear fractional partial differential equations using homotopy analysis method*, Communication in Nonlinear Science and Numerical Simulation, 14, 1962-1969, 2009.

[14] B.N. Kharrat and G.A. Toma: *A new integral transform: Kharrat-Toma transform and its properties*, World Applied Sciences Journal, 38(5), 436-443, 2020.

[15] B.N. Kharrat: *Combine Kharrat-Toma Transform and Variational Iteration Method to Solve Nonlinear Boundary Value Problems*, World Applied Sciences Journal 40 (1): 41-46, 2022.

[16] S.K. Lydia, M.M. Jancirani and A.A. Anitha: *Numerical Solution Of Nonlinear Fractional Differential Equations Using Kharrat-Toma Iterative Method,* Nat. Volatiles & Essent. Oils, 8(4), 9878-9890, 2021.

[17] R.L. Magin: *Fractional calculus models of complex dynamics in biological tissues.* Comput. Math. Appl. 2010, 59, 1586-1593.

[18] F.C. Meral, T.J. Royston, R.L. Magin: *Fractional calculus in viscoelasticity: An experimental study.* Commun. Nonlinear Sci. Numer. Simul. 2010, 15, 939-945

[19] I. Podlubny: *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[20] A.C. Scott: *A nonlinear Klein-Gordon equation.* Am. J. Phys. 37(1), 52-61, 1969.

[21] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y.Q. Chen: *A new collection of real world applications of fractional calculus in science and engineering.* Commun Nonlinear Sci Numer Simulat 64 (2018) 213-231.

[22] H. Thabet, S. Kendre, D. Chalishajar: *New analytical technique for solving a system of nonlinear fractional partial differential equations*. Mathematics, 5(4), 2017.