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On Certain Integral Operator Inequalities in Normed Spaces

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Abstract

A lot of researches have been carried out on inner product type integral transformers (IPTIT) with regard to various aspects including spectra, numerical ranges and operator inequalities. Consider M ***and*** N ***to be weakly*** μ***-measurable operator valued (OV) functions such that*** $M, N : \Omega \to B(X)$ ***for any*** $Q \in \mathcal{B}(H)$***. If*** M ***and*** N ***are integrable with respect to Gel'fand axiom, then we obtain a linear transformation arising from the inner product space as*** $Q \mapsto \int_\Omega M_t Q N_t \partial(t)$***. There exists an open problem regarding IPTIT while studying inequalities for IPTIT with spectra limited to the unit disc in complex domains. It has been pointed out that the inequalities, and in particular Cauchy-Schwarz (CS) and Cauchy-Buniakowski-Schwarz (CBS) inequalities, can only be attained for these IPTIT if only one of the operator*** M ***or*** N ***is normal. Therefore, in this note we solve this problem by obtaining CBS-inequalities for IPTIT in Banach spaces.***

Keywords: *Integral Operator, CBS-inequality, Norm, IPTIT.*
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1 Introduction

Studies on IPTIT have been carried out for sometime by researchers with interesting results obtained. In [1], the authors studied on the $CS-$inequalities of operators that are weak integrals. In their work, finite measures dominated the study with measurable families. They described the CS-inequalities for non-commutative operators in the Schatten P-ideals. The unitarily invariant norms for these weak integrals were also established after applying the arithmetic mean and the Young's inequality as described in [2]. The research also gave out the some definitions related to the operatot Valued (OV) functions and measurable functions. The work also had some applications involving the closed graph theorem. By use of the continuous orthogonal projections of subspaces, some preliminary results were obtained. Their study also had employed some knowledge of vectors that are integrable with respect to OV functions. Gelfand axioms also dominated this work making it to fully describe the weak integrals [4]. Their results had a lot of details particularly on the norms of operators in Banach spaces by utilizing the CS-inequalities. The CS-inequalities for the functions deemed operator valued and factorable were also addressed here and their results well put. They also went ahead and gave their relevant applications of their findings with regard to inequalities of operator norms as discussed also in [3].

The authors of [5] described the advances and reverse inequalities for the CS-inequalities. These reverses were mostly centered to the CS-inequalities for the inner product space and the C^*-modules. In their study, a comprehensive description was made on the CS-inequalities for the classical analysis. For instance, some proofs for the inner product spaces were well put forth especially those arising from the parallelogram law [10]. They had a principle argument for non-zero vectors attained by normalization process to attain results for the classical analysis case. They also employed the Wagner's inequality for the for the operator version, self adjoint and C^*-algebras [9]. For instance, the they talked about the positive unital linear maps acting between the the class of C^*-algebras for any commutative operator to be completely positive. This was actually attained by mainly by the Applications of the Choi inequality for unital normal operators. in the case of the C^*-modules, the semi-inner product notion was actually evidenced as a natural generalization. Their work also gave a good description and definitions of the C^*-algebras and C^*-modules. They also put forward some study on the operator reverses and operator CS-inequalities utilizing the Wielandt inequality [12].

The work of [6] described the CS-inequalities for the for the general elementary operators and also gave their applications of their work. In their article, they employed the unitarily invariant norms for positive scalars that are real. In doing this, they gave a strong relationship that exist between known versions of the CS-inequalities and utilized it to coin the norms. In [7], it was noticed that the symmetric gauge function is basic property for the operator singular values. The absolute singular values were also discussed here. the study also put forward several theorems describing the $CS-$inequalities for the operators that are semi-definite and positive.

The study by [11] discussed about the unitarily invariant norms involving the G_1 operators. In their study, they represented upper bounds for $|||g(M)X_f(N) + X|||$ to be $|||MXN| + |NXB|||$ provided M and N are G_1 operators, $|||.|||$ is a unitarily invariant norm. The workof [14] also discussed the norms for IPTIT with their spectrum in the unit discs. In their study, they introduced the weakly*-measurable families $\mathcal{B}(H)$, where $\triangle_{\mathcal{M}}$ is a defect operator on $\mathcal{M}$. In [13] the authors discussed extensively about norm estimates of operators in relation to their contractions and spectra. The study postulated that given any two operators A and B be normal contractions in Hilbert spaces, then their norms were attained in terms of partial summations involving column matrices. The study by [15] wrote extensively on applications of CSI for Hilbert modules to IPTIT. For example, for any Banach algebra,A, it can be described by $x \mapsto \sum_{j=1}^{n} a_j x b_j$. The study described this mapping $X \mapsto \int_\Omega A_t X B_t \partial\mu(t)$ as an inner product type integral transformer for any measure space (Ω, μ).

In [8], the author later on studied on norm interpolations for rows and columns elementary operators. In this study, described the the formula for Hilbert-Schmidt norm of IPTIT as,

$$\left\| X \mapsto \int_\Omega A_t X B_t d\mu(t) \right\|_{B(C_2,(H))}$$

$$=^{lim}_{l\to\infty} \sqrt[2n]{\int_{\Omega^{2n}} tr(\prod_{k=1}^{n} A^*_{t_{n+1}} A_{s_{n+1-k}}) tr(\prod_{k=1}^{n} B_{S_k} B^*_{t_k}) \prod_{k=1}^{n} d\mu(s_k) d\mu(t_k)}$$

wherever $\int_\omega \|A_t\|_p \|B_t\|_p d\mu(t) < \infty$ for some $p > 0$.

In [16] and [17] they discussed recently the Landau and Gross inequalities for IPTIT in general Hilbert spaces. The study gave some description of norm inequalities, with respect to unitarily invariant norms, in the space of all compact separable Hilbert space. The study by [18] also dealt with the operator valued function over a measure spaces $(\Omega, \mathcal{M}, \mu)$ and for a finite measure μ. He indicated that if (A_t), $(B_t)_{t\in\Omega}$ be weakly*-measurable in Hilbert spaces, and that $|\int |A_t|^2 d\mu(t)|^0$ and $|\int |B_t|^2 d\mu(t)|^0$ are in complex unitarily invariant normed Hilbert spaces, then it holds [19] that ,

$$|||| \int |A_t|^* B_t d\mu(t)||||^0|||| \leq |||| \int |A_t|^* A_t d\mu(t)||||^0||||^{\frac{1}{2}}$$

$$|||| \int |B_t|^* B_t d\mu(t)||||^0||||^{\frac{1}{2}}$$

. The same study gave the generalization of landau inequality for Gelfand integrals for operator valued functions in unitarily invariant norm ideals as

seen in [20]. To execute our tasks well, we needed an understanding of some basic concepts that are necessary in this study.
In [21] the authors studied on reverse inequalities of the CBS inequalities and the TI in the normed Hilbert space. Their results were simply a generalization of the work done by [22] on the same described spaces. The systematic generalization of these findings were also done in line with the 2-dimensional normed space in [23].
In [24] they studied the integral inequalities in relation to the CS-inequalities invoking the Cauchy-Buniakowski-Schwarz inequalities. In this study, they proposed some findings and CBS-inequality reverses. The study gave some detailed findings in any measure space on the issues of measure spaces [25]. The Hilbert space setting also dominated their study $L^2_w(\Omega, K|K = \mathcal{C}or\mathcal{R})$, Their main objective was achieved using basic elementary identities such as $\frac{v^2+u^2}{2} - uv = \frac{1}{2}uv(\sqrt{\frac{w}{v}} - \sqrt{v}u)^2$ for all $u, v > 0$.
The work of [26] studied on the additive reverses of the CBS-integral inequalities. In their study, additive inequalities with reversibility traits were put forward and some results obtained. They also gave out some applications for the variance and moments for a continuous random variable defining the Hilbert space setting with non-finite dimension and finite intervals [27]. For instance, they postulated that for any $g, f : [a, b] \to \mathcal{R}$, and that the functions f and g are Lebesgue measurable functions as stated and that f^2 and g^2 are Lebesque integrable over $[a, b]$ then their dot product is also integrable over the same interval. The same study also gave an overview of os well known reverse inequalities and the widely applied CBS-inequalities in probability and numerical analysis. The main results obtained in this work also gave some applications of the Holder's inequality in pursuit for the desired results. In [28] the author studied on the CBS-inequalities for the Hilbert space operators. The study had its focus on a major thematic concern as the symmetric shape of some inequalities ranging between real number sequences. The sequences that generate these inequalities as per the postulates of operator theory were also discussed at some length. The same study also gave some insight of CBS-inequalities for the Euclidean spaces. They did apply knowledge of Bhor's inequality and to some extent Bergostrom's inequalities were invoked. In addition to this, monotonicity of sequences also became an area of touch and widely applied in their executions of their major results. They gave their overview of the Aczel's inequality, as seen in [22], for any operator in Hilbert space that is bounded and linear.

2 Preliminaries

Here, we have described some definitions which are useful to this study.

Definition 2.1 *([19]) Let X be a complex Hilbert space with infinite dimension, then for $\mathcal{B}(X)$ representing operators on X which are bounded, we have a mapping S represented as $S_{a_i,b_i}(x) = \sum_{i=1}^{k} a_i x b_i$ called the elementary operator.*

Definition 2.2 *([22]) Consider a weakly μ-measurable operator valued functions $M, N : \Omega \to B(X)$ for any $Q \in \mathcal{B}(H)$, we can have a mapping $Q \mapsto \int_\Omega M_t Q N_t$ as weakly measurable too. If M and N are integrable with respect to Gel'fand axiom, then we get a linear transformation arising from the inner product space as $Q \mapsto \int_\Omega M_t Q N_t \partial(t)$ called an inner product type integral transformer denoted by $\int_\Omega M_t \bigotimes N_t \partial(t)$.*

Definition 2.3 *([27]) If a vector space V is described over any field of real scalars like $\mathcal{R}$, then the the function which is positive and real valued that takes vectors to real scalars represented by $\|.\|$ is referred to as a norm if it obeys the conditions below:*

(i). Positive definite: $\|a\| \geq 0, \ \forall \ a \in V$,

(ii). Zero property : $\|a\| = 0$ *iff* $a = 0, \ \forall \ a \in V$,

(iii). Homogeneity: $\|\beta a\| = |\beta| \|a\|, \ \forall \ a \in V$, *and* $\beta \in \mathcal{R}$,

(iv). Triangle inequality: $\|a + b\| \leq \|a\| + \|b\|, \ \forall \ a, b \in V$.

3 Literature review

We discuss literature on CBS-inequalities for IPTIT in this work. We consider various studies, their relevance and critical contributions to this study.

Theorem 3.1 *([9, Theorem 1.]) Fix any $a_m \in \mathcal{C}$ and also $b_m \in \mathcal{C}$, given that $m \in \mathcal{R}$ satisfying the axiom*

$$\sum_{m=1}^{n} a_m b_m = 0.$$

From Theorem 3.1, the study uses the constants that $\frac{1}{2}$ that cannot be replaced whatsoever. The study was giving details about the refinement of the CBS inequalities of operators especially elementary operators. Their proof involved taking the modulus of the partial sums as per the definitions of the CBS-inequality and utilizing the elementary inequality for real numbers to get their desired inequality as per the theorem. Clearly, this study focused on the inequalities for normal elementary operators but had no regard to the IPTIT.

In [4], they addressed the inequalities for upper bounds for normal integral operator from the maximal deviation and weighted mean. However, the study had its major focus on the elementary form but had no reference to the IPTIT that our study has addressed.

Theorem 3.2 *([9, Theorem 2.]) Let* X *and* Y *represent normal and bounded operators in complex Hilbert spaces and* x_m*,* y_m *also be a complex sequence and* p_m*,* y_m *be probability sequences i.e* $\sum_{m=1}^{n} p_m x_m y_m = 1 \; \forall \, m \in \mathcal{R}$*,*

$$\substack{max \\ i\in\{1,\ldots,n\}}\{p_i|x_i y_i|\} \leq \frac{1}{2}\sqrt{\left[\sum_{m=1}^{n} p_m|x_m|^2\right]\left[\sum_{k=1}^{n} p_m|y_m|^2\right]}$$

The theorem here addressed several issues arising from the probability sequences that are complex and for normal and bounded operators in Hilbert spaces. Their norm estimates were derived with restriction to the general elementary operators. Its clear to see that the work had no considerations to the special class of operators dubbed IPTIT which we have described and attained their norm bounds in this work.

Theorem 3.3 *([12],Theorem 1.) Suppose that we have the mappings*$M, N : [p, q] \to \mathcal{R}$ *being strictly increasing and monotonic on the interval, let* $H : [p, q] \to \mathcal{C}$ *be* $S-$*dominated by the pair* (M, N) *and the function* f *be shown as* $f : [p, q] \to \mathcal{C}$ *being a continuous function on interval chosen, then the Riemann-Steljtjes integrals (RSI)* $\int_p^q f(t)dh(t)$ *holds.*

This study tried to give some results of CBS inequalities for the Riemann-Steljtjes integrals. To achieve this, they considered the continuous functions on decreasing and monotonic sequences on the intervals dominated by the vectors u, v. to achieve this, partitioned the sequences and described the norms for any intermediate points in the interval $[a, b]$ and by CBS-weighted discrete inequality, they attained the desired results. Clearly, in this study focused on the Riemann integrals without any regard to the special type of integrals called IPTIT.
In [6] the study used integration by parts to achieve the the results. this theorem addressed the CBS for the the Riemann integrals. By integrating by parts and applying the trapezoidal rule, they solved the problem also incorporating the Chebyshev inequalities for the monotonic nondecreasing functions and then the desired inequality was obtained. Its therefore clear that the study only focused on the RSI and didn't address the CBS-inequalities for IPTIT.
Also in [18], the work described the CBS-inequality applying some aspects of Chebyshev inequalities. By taking x as a fixed point in the interval $[a, b]$ and making use of the definition of CBS on the Riemann integrals. Clearly, this

study focused on the RSI but not the special type of integrals, the IPTIT, which we have dealt with in this note.
The study of [13] considered monotone nondecreasing pair, (p, q) on $[a, b]$, and continuous nonnegative function. It was observed that p was positive and f, g are synchronous, then the function f was continuous on the rectangle as described in their earlier result above. They applied this assertions to achieve their results on the same integrals and their CBS inequalities described. Following through the proof for this claim, its true that the study only focused on the RSI but not special type of integrals known as the IPTIT considered in this work.
The work of [10] sought to address the inequalities with respect to self adjoint integral operators. By use of the properties of self adjoint operators, they attained their desired results. Its also important to note that the proposition considered the continuous functions in achieving their objectives. Moreover, the major aspect is emphasized on the Riemann-Stieltjes type integral. In [16] they considered the maximum and minimum eigenvalues and the synchronous functions on the $[m, M]$. The concept of self-adjointedness also dominated especially on Riemann-Stieltjes integrals.

Proposition 3.4 *([2], Proposition 4.) If $U \in B(H)$ is a unitary, then it follows closely that each complex-valued function that is continuous on a complex unit circle.*

This proposition here addressed the unitaries in Hilbert spaces. All functions here were deemed complex-valued and confined within the complex unit circle. The study utilized the knowledge of Schwartz inequalities for nonnegative operators, they attained their desired results. The results here were basically for Riemann-Stieltjes integrals.

Proposition 3.5 *([14],Proposition 5.) If $U \in B(H)$ is a unitary, Then it follows closely that each complex-valued function that is continuous on a complex unit circle denoted by $f : C(0,1) \to \mathcal{C}$ gives the inequality;*

$$|(f, g; (U)x, y)|^2 \leq \frac{1}{2} D(f, g; (U)x, y) \sqrt{[B(f, g; (U)x)][B(f, g; (U)x)]}$$

$\forall\, x, y \in H$

From this study, it was postulated that every Unitary operator acting on Hilbert space and the functions here were deemed complex-valued acting on a complex unit circle and their results attained. However, the study results here were geared towards Riemann-Stieltjes integrals.

Theorem 3.6 *([10],Theorem A.) Suppose that $f, g \in L_p^2([p,] : h)$, such that $L_p^2 \in H$ and h is function that is strongly measurable in that $h : [p, q] \to H$ giving $int_p^q \rho(t) \|h(t)\|^2 dt$ for any function ρ.*

From Theorem 3.6, their study puts L_p^2 as a Hilbert space containing some functions that are strongly measurable for which all integrals defined here are finite. They obtained the positive constants $\frac{1}{2}$ and $\frac{1}{4}$ in their final results of this theorem. Its clear from this work that they majorly gave much knowledge about reverses of CBS inequalities for the general integral operators. However, the study does not in any measure include in totality any aspect of IPTIT.
Pecaric in [10] theorem B applies the idea of strongly measurable functions to achieve the desired results. It was noted that the factor $\frac{1}{4}$ is the least possible. Having looked at this work keenly, the study only focused on obtaining the reverse inequalities for general form of integral operators but no regard to IPTIT.
The work of [27] postulates that for all strongly measurable functions with finite integrals, and all scalars were real complex numbers. The study describes the reverse inequalities of the general integral operators majorly. Indeed the study had very interesting proof here but failed to address the issue of IPTIT that has attracted our attention and other researchers as well. Our findings are well elaborative and found in this work.

Lemma 3.7 *([20], Lemma 1.) For any two vector functions f and g in a complex measurable space $L^2(\Omega, H)$ they satisfy the CBS-inequality.*

This lemma used the idea of vector functioning from the complex Hilbert spaces to describe the inequalities of CBS with respect to the general integral operators. By the method of integration of the functions over a measurable space and a complex valued function. Clearly, this result show that they only considered in general the integral operators but not specifically the main subject of this study.
In [10] theorem 2 the authors described the inequalities in terms of the quadratic arithmetic mean for integrals. Consequently, the reverse inequalities for CBS general integrals were obtained. This means that the specificity of IPTIT have not been exploited yet.

4 Research methodology

In this study, we used some fundamental principles and known results that are deemed useful to execute our tasks. Several technical approaches have been employed and the known results which are useful to our study utilized.

4.1 Known fundamental theorems

Lemma 4.1 *([5] Lemma 2.11) Let any two non empty sets W and S be bounded in the plane and $\alpha \in U$, $\beta \in V$ then $\exists\, t_0$ (positive number) then $t_0 \in bndry(W)$, $\beta|t_0 \in S$ or, $t_0 \in W$, $\beta|t_0 \in bndry(S)$.*

Lemma 4.2 *([5], Lemma 2.2.1) For some operators X A_n and B_n if bounded, it holds that* $\Pi_{i=1}^{k} s_i^p\left(\Sigma_{i=1}^{n} AXB\right) \leq \Pi_{i=1}^{k} s_i^p\left(A*\right) s_i(B^*) s^p_{i+[\frac{i-1}{N}]}(X)$

Theorem 4.3 *([3], Theorem 3.1d) For every $n \in \mathcal{N}$ and $\alpha > 0$, there holds,* $\sum_{k=1}^{n} s_k^{\alpha}\left(\int_{\Omega} CXD d\mu\right) \leq \sum^{\frac{\alpha}{2}}\left(\int_{\Omega} C|X^*|^{2-1\theta} C^* d\mu\right) s_k^{\frac{\alpha}{2}}\left(\int_{\Omega} D^*|X|^{2\theta} D^* d\mu\right)$.

Theorem 4.4 ***Fuglede-Putnam Theorem*** *([13], Theorem 1) Let $A \in \mathcal{C}^{mn}$, $Q \in \mathcal{C}^{nn}$ and $T \in \mathcal{C}^{nm}$ where $\mathcal{C}^{mn}$ is a set comprising of all complex matrices. If R and Q are normally represented and $RT = TQ$, then $R^*T = TQ^*$.*

Theorem 4.5 ***Fuglede-Putnam Theorem*** *([20], Theorem 2) $A \in \mathcal{C}^{mn}$ and $B \in \mathcal{C}^{nn}$, where $\mathcal{C}^{mn}$ is a set comprising complex matrices, then AB and BA are normal if and only if $A^*AB = BBA^*$ and $ABB^* = B^*BA$.*

Proposition 4.6 *([1] Proposition 2.1) If A, B, C, D are normal operators, then*

(i). The sum of normal operators is normal. I.e, $A + B$ is normal

(ii). The product of any normal operators is normal. I.e, AB is normal. Similarly $C \times D$ are normal operators.

Theorem 4.7 *([11], theorem 3.1) Any two definite sequences with m-tuples power have the property that $\sum_{k=1}^{m} a_k, b_k = 0$.*

In the next part we give the technical approaches which are very instrumental to this work.

4.2 Technical approaches

Tensor products: Any space containing all linear maps taking elements from the cross product of X and Y to another vector space Z is naturally isomorphic in relation to a space containing linear maps from the tensor product to another space. This is a construction; $X \otimes Y$ to Z which will be linear.

Theorem 4.8 *For any two vector spaces X and Y over a field $\mathcal{K}$ there exist a tensor product $X \otimes Y$ with a canonical bilinear homeomorphism distinguished up to an isomorphism by the following universal properties: Every bilinear homeomorphism $\phi : X \times Y \to Z$ lifts to a unique homeomorphism, and $\phi : X \otimes Y \to Z$.*

Orthogonal direct sum: Let$\{M_i\}_{i\in I}$ be a collection of closed subspaces of H such that $M_i \perp M_j$ whenever $i \neq j$. Then the orthogonal direct sum of the M_i is the smallest closed subspace which contains every M_i. This space is $\oplus_{i\in I} = \overline{span}(\cup_{i\in I} M_i)$

Splitting lemma: Let $A = A_1+A_2+...+A_g$ and $B = B_1+B_2+...+B_g$. If all A_i and B_i are symmetric positive semi-definite and if for each i, A_i is in the range of B_i then $\sigma(A,B) \leq max_i\sigma(A_i,B_i)$. The splitting lemma is normally used for decomposition of a matrix A into the sum of rank-one matrices with each corresponding to one off-diagonal and by decomposing B into path matrices. (The matrices that can be permuted symmetrically to a traditional form and which have only one non-zero irreducible block).

Example 4.9 *Let* $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$ *to have* U *and* V *complete to a triplet which is symmetric we shall have that* $W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ *then the 2-norm of* $\|W\|_2 = \sqrt{3}$. *by choosing* $\epsilon = \frac{1}{\sqrt{2}}$ *it yields,* $\|W\|_2 = 2$. *We use splitting to achieve* $\|\tilde{W}\|_2 = \|W\|_2 = \sqrt{3}$.

Let $S = \begin{pmatrix} \sqrt{\frac{1}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} \end{pmatrix}$ *So*

$$W = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & & \\ 0 & & \\ & 0 & \\ & \sqrt{3} & \\ & & \frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} \end{pmatrix}$$

The W *is shown without zeros to emphasize the structure of its columns.*

Frobenius Heuristic : Let W be a matrix and M a diagonal matrix s.t (i,j) the value in the j^{th}-matrix will be given by:

$$(D_j)_{i,j} = \frac{\sqrt{|W_{i,j}|}}{\sqrt{\sum_{c=1}^{m} |W_{i,c}|}}$$

Example 4.10 *Frobenius Heuristic minimizes* $\|\tilde{W}\|_F$ *over all fractional splittings of* W. *To illustrate this, we already know that each row is minimized to* $\|\tilde{W}\|_F$ $\forall i,j \in \tilde{W}$, *are non zero elements of the vectors* $D_+^j W_{i,j}$ *for*

$j = 1, ..., m$. The i^{th} element of this vector $(D_+^j)_{i,i}W_{i,j}$ is $(D_+^j)_{i,i}W_{i,j}$ given that $W_{i,j} = 0$, so we have that $(D_+^j)_{i,i}W_{i,j} = 0$, otherwise $(D_+^j)W_{i,i}W_{i,j} = \frac{W_{i,j}}{(D)_{i,j}}$. On the second part, it follows that by breaking the minimization problem into k-independent sub-problems. It can be seen that $\sum_{j=1}^{m} x^2 = 1$. Let $f(x_1, ..., x_m, \lambda) = \sum_{j=1}^{m} (\frac{c_j}{x_j})^2 + \lambda(\sum_{j=1}^{m} x^2 - 1)$. The minimizer satisfies

$$0 = \frac{\delta f}{\delta x_i} = -2.\frac{c^2}{x_j^3} + 2\lambda i,$$

$$0 = \frac{\delta f}{\delta \lambda} = \sum_{j=1}^{m} x^2 j - 1$$

then it follows that $C_i^2 = \lambda x^4 i$, therefore $x_i^2 = \frac{|c_i|}{\sqrt{\lambda}}$. Since $\sum_{j=1}^{m} x^2 j = 1$, it follows that

$$\sum_{j=1}^{m} \frac{|c_j|}{\sqrt{\lambda}} = 1.$$

and hence,

$$\frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m} |c_j| = 1$$

and so'

$$\lambda = \sum_{j=1}^{m} |c_j|.$$

Now we have that $x_i^2 = \frac{|c_i|}{\sqrt{\lambda}}$ so it follows that,

$$x_i^2 = \frac{|c_i|}{\sum_{j=1}^{m} |c_i|}$$

and thus

$$x_i = \frac{\sqrt{|c_i|}}{\sqrt{\sum_{j=1}^{m} = 1 |c_j|}}$$

5 Main results

In this section, we investigate certain inequalities for IPTIT. We carry out our investigation under different technical approaches.

5.1 Cauchy-Schwarz Inequalities for IPTIT

At this point, we have given the results on the CS-inequalities for integral operators of the class of Inner product transformers. In this part also, we refer to any integrability of these operators to be Gel'fand. Moreover, these integral are weak.

Theorem 5.1 *Let* $X \in B(H_1, H_2)$ *and* $\Uparrow \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$ *then* $\left\|\Uparrow_{S^*,S} X\right\|_q \leq \left\|\Uparrow_{S,q}|X|\right\|_q^{\frac{1}{2}-\frac{q-1}{2q}} \left\|\Uparrow_{S,q}|X|\right\|_q^{\frac{1}{2}-\frac{1}{2q}} \|X\|_q^{\frac{1}{q}}$, $\forall\, S, S^* \in B(H_1, H_2)$ *and* $q \in H$.

Proof. We can represent the polar decomposition of X as $U|X|$ such that $U|X| = |X^*|^{\frac{1}{2}} U|X|^{\frac{1}{2}}$ then we apply $|X|^{\frac{1}{2}} S, S|X^*|^{\frac{1}{2}}$ in this case to give

$$
\begin{aligned}
\left\|\Uparrow_{S^*,S} X\right\|_q &= \left\|\Uparrow_{S^*,S\sqrt{|X^*|},\sqrt{|X|}}(U)\right\|_q \\
&\leq \|(\Uparrow_{S\sqrt{|X^*|},S^*\sqrt{|X^*|}}(\Uparrow_{S^*\sqrt{|X^*|}\},\sqrt{|X|}S}(I))^{q-1})^{\frac{1}{2}} \\
&\quad XU(\Uparrow_{S\sqrt{|X^*|},S^*\sqrt{|X^*|}}(\Uparrow_{S^*\sqrt{|X^*|},S\sqrt{|X|}}(I))^{q-1})^{\frac{1}{2}}\| \\
&= \left\|(\Uparrow_{S\sqrt{|X^*|},S^*\sqrt{|X^*|}}(\Uparrow_{S^*\sqrt{|X^*|},S\sqrt{|X|}}(I))^{q-1})^{\frac{1}{2}}\right\|_{2q} \\
&\quad X\,\|U\| \left\|(\Uparrow_{S\sqrt{|X^*|},S^*\sqrt{|X^*|}}(\Uparrow_{S^*\sqrt{|X^*|},S\sqrt{|X|}}(I))^{q-1})^{\frac{1}{2}}\right\|_{2q} \\
&= \left\||X^*|^{\frac{1}{2}}\Uparrow_{S,S^*}(\Uparrow_{S^*,S}|X^*|^{q-1}|X^*|)^{\frac{1}{2q}}\right\|_1 X\|U\| \left\||X|^{\frac{1}{2}}\Uparrow_{S,S^*}(\Uparrow_{S,S^*}|X|^{q-1}|X|^{\frac{1}{2}})\right\|^{\frac{1}{2}} \\
&\leq \left\||X|^{\frac{1}{2}}\right\|_{2q}^{\frac{2}{2q}} \left\|\Uparrow_{S,S^*}(\Uparrow_{S^*,S}|X^*|^{q-1})\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \left\||X|^{\frac{1}{2}}\right\| X \left\|\Uparrow_{S,S^*}(\Uparrow_{S,S^*}|X|)^{q-1}\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \\
&= \left\|(\Uparrow S, q|X^*|)^{q-1}\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \left\|(\Uparrow_{S,q}|X|^{q-1})\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \|X\|_q^{\frac{1}{q}} \\
&\leq \left\|\Uparrow_{S,q}|X|\right\|_q^{\frac{1}{2}-\frac{q-1}{2q}} \left\|\Uparrow_{S,q}|X|\right\|_q^{\frac{1}{2}-\frac{1}{2q}} \|X\|_q^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 5.2 *Suppose that* $X \in B(H_1, H_2)$ *and* $\Uparrow \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$ *then* $\forall\, 1 \leq l \leq m$, *we have that* $\left\|\Uparrow_{S,q}^{l} X\right\|_q \leq \left\|\Uparrow_{S,q}^{m} X\right\|_q^{\frac{l}{m}} \|X\|_q^{1-\frac{l}{m}}$, $\forall q \in H$.

Proof. By Theorem 5.1, we use S^* as S and $\frac{q}{q-1}$ as q. Then we have that, $\left\|\Uparrow_{S,S^*} X\right\|_{\frac{q}{q-1}} \leq \left\|\Uparrow_{S^*,\frac{q}{q-1}}|X^*|\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \left\|\Uparrow_{S^*,\frac{q}{q-1}}|X^*|\right\|_{\frac{q}{q-1}}^{\frac{1}{2q}} \|X\|_{\frac{q}{q-1}}^{1-\frac{1}{q}}$. This implies that

$$
\begin{aligned}
\left\|\Uparrow_{S^*,q} X\right\|_q &= \left\|\left(\Uparrow_{S,S^*}\left(\Uparrow_{S^*,S} X\right)^{q-1}\right)^{\frac{1}{q-1}}\right\|_q \\
&= \left\|\Uparrow_{S,S^*}\left(\Uparrow_{S^*,S} X\right)^{q-1}\right\|_{\frac{q}{q-1}}^{\frac{1}{q-1}} \\
&= \left(\left\|\Uparrow_{S^*,\frac{q}{q-1}}\left(\Uparrow_{S^*,S} X\right)^{q-1}\right\|_{\frac{q}{q-1}}^{\frac{1}{q}} \left\|\left(\Uparrow_{S^*,S} X\right)^{q-1}\right\|_{\frac{q}{q-1}}^{1-\frac{1}{q}}\right)^{\frac{1}{q-1}} \\
&= \left\|\left(\Uparrow_{S^*,S}\Uparrow_{S^*,q} X\right)^{q-1}\right\|_{\frac{q}{q-1}}^{\frac{1}{q}} \left\|\Uparrow_{S^*,S} X\right\|_q^{1-\frac{1}{q}}
\end{aligned}
$$

$$\begin{aligned}
&= \left\|\mathcal{T}_{S^*,S}\,\mathcal{T}_{S,q}X\right\|_q^{\frac{q-1}{q}}\left\|\mathcal{T}_{S^*,S}X\right\|_q^{1-\frac{1}{q}}\\
&= \left(\left\|\mathcal{T}^{\in}{}_{S,q}X\right\|_q^{\frac{q-1}{q}}\left\|\mathcal{T}_{S,q}X\right\|\right)^{\frac{1}{q}}\left(\left\|\mathcal{T}_{S,q}X\right\|_q^{\frac{q-1}{q}}\|X\|_q^{\frac{1}{q}}\right)^{1-\frac{1}{q}}\\
&= \left\|\mathcal{T}^{\in}{}_{S,q}X\right\|_q^{\frac{q-1}{q}}\left\|\mathcal{T}_{S,q}X\right\|_q^{\frac{1}{q}+\frac{(q-1)^2}{q^2}}\|X\|_q^{\frac{q-1}{q}}.
\end{aligned}$$

Dividing through by $\left\|\mathcal{T}_{S,q}X\right\|_q^{\frac{1}{q}+\frac{(q-1)^2}{q^2}}$, we get the earlier supposition that

$$\begin{aligned}
\left\|\mathcal{T}_{S,q}X\right\|_q^2 &= \left\|\mathcal{T}_{S,q}\mathcal{T}_{S,q}^{m-1}X\right\|_q^2 &(1)\\
&\le \left\|\mathcal{T}_{S,q}^{m+1}X\right\|_q\left\|\mathcal{T}_{S,q}^{m-1}X\right\|_q &(2)\\
&\le \left\|\mathcal{T}_{S,q}^{m+1}X\right\|_q\left\|\mathcal{T}_{S,q}X\right\|_q^{m-1}\|X\|_q^{\frac{1}{m}}. &(3)
\end{aligned}$$

This implies that $\left\|\mathcal{T}_{S,q}^{m}X\right\|_q \le \left\|\mathcal{T}_{S,q}^{m+1}X\right\|_q^{\frac{m}{m+1}}\|X\|_q^{\frac{1}{m+1}}$ for all $1 \le l \le m$, we have by 5.8 and 3 that $\left\|\mathcal{T}_{S,q}^{l}X\right\|_q \le \left\|\mathcal{T}_{S,q}^{m}X\right\|_q^{\frac{l}{m}}\|X\|_q^{1-\frac{l}{m}}$.

Theorem 5.3 *Let* $\mathcal{T}_{S_1,S_2} \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$, $S_1, S_2 \in B(H_1, H_2)$ *and for all* $p, t, k, q \ge 1$ *such that* $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} + \frac{1}{k}$, *then the following inequality holds.* $\left\|\frac{1}{M}\sum_{m=1}^{m} S_1XS_2\right\|_q \le \left\|\overline{\frac{1}{M}\sum_{m=1}^{m} S_1S_1^*}\right\|_p\left\|\frac{1}{M}\sum_{m=1}^{m} S_2S_2^*\right\|$, *for all* X *fixed in* $B(H_1, H_2)$.

Proof. From Lemma 4.2 we have an implication that for all $n = 1, 2, ...$, then it holds that

$$\prod_{i=1}^{n} k_i^q\left(\sum_{m=1}^{m} S_1XS_2\right) \le \prod_{i=1}^{n} k_i^q\left(S_1^*\right)k_i^q(S_1^*)k^q_{i+[\frac{i-1}{M}]}(X).$$

Now, there exist a sequence $\{ln(k_i^q(\sum_{i=1}^{m} S_1XS_2))\}_i$ that is weakly majorized by the sequence $\{ln(k_i^q(S_1^*)K_i^q(S_2)S^q_{i+[\frac{i-1}{M}]}(X))\}_i$ to give

$$\sum_{i=1}^{n} k_i^q\left(\sum_{m=1}^{m} S_1XS_2\right) \le \sum_{i=1}^{n} k_i^q\left(S_1^*\right)k_i^q\left(S_1^*\right)k^q_{i+[\frac{i-1}{M}]}(X).$$

Applying the Hölders inequality for real numbers we get

$$\begin{aligned}
\left\|\sum_{m=1}^{m} S_1XS_2\right\|_q^q &= \left\|\sum_{i=1}^{\infty} k_i^q\left(\sum_{m=1}^{m} S_1XS_2\right)\right\|\\
&\le \left(\sum_{i=1}^{\infty} k_i^q\left(S_1^*\right)^{\frac{q}{p}}\sum_{i=1}^{\infty} k_i^t\left(S_1^*\right)^{\frac{q}{t}}\sum_{i=1}^{\infty} k^k_{i+[\frac{i-1}{M}]}(X)\right)^{\frac{q}{k}}\\
&= M^{\frac{q}{t}}\|S_1^*\|_p^q\|S_2\|_t^q\|X\|_k^q.
\end{aligned}$$

As $M^{\frac{q}{p}-p-\frac{q}{t}} = M^{\frac{q}{t}}$ we get the equivalence as stated in this theorem.

Theorem 5.4 *Let* $\pitchfork \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$ *such that* $S_1, S_2 \in B(H_1, H_2)$ *and for all* $X \in B(H_1, H_2)$ *then* $\pitchfork$ *can represented as* ,$\pitchfork_{S_1,S_2}(X) = \sum_N^{n=1} S_1 X S_2$ *and we have that*

$$\left\|\pitchfork_{S_1,S_2}\right\|_{B(H_1)\to B(H_2)} \leq \left\|\left\{t_i^{\frac{1}{2}}\left(\sum_{n=1}^{N} S_1 S_1^*\right) t_i^{\frac{1}{2}}\left(\sum_{n=1}^{N} S_2^* S_2\right)_{i=1}\right\}^{\infty}\right\|. \qquad (4)$$

Thus,

$$\left\|\pitchfork_{S_1,S_2}\right\|_{B(H)\to B(H)} \leq q\overline{\sum_{i=1}^{\infty} t_1^{\frac{1}{2}}\left(\sum_{n=1}^{N} S_1^*\left(\sum_{n=1}^{N} S_1, S_1^*\right)^{q-1} S_1\right) t_i^{\frac{1}{2}}\left(\sum_{n=1}^{N} S_2\left(\sum_{n=1}^{N} S_2^*, S_2\right)^{q-1} S_2^*\right)}.$$

In particular, if $S_2^* = S_1\ \forall\, n = 1, \ldots, N$ *then we have that*

$$\left\|\pitchfork_{S_1,S_2}\right\|_{B(H)\to B(H)} = \left\|\pitchfork_{S_1,S_2}(I)\right\|_q = \left\|\sum_{i=1}^{N} S_1 X S_2^*\right\|_q.$$

Proof. From Equation 4 and Theorem 5.3, we have that

$$\left\|\left(\sum_{m=1}^{m} S_1^*\left(\sum_{m=1}^{m} S_1 S_1^*\right)^{q-1} S_1\right) X\left(\sum_{m=1}^{m} S_2^*\left(\sum_{m=1}^{m} S_2 S_2^*\right)^{q-1} S_1\right)^{\frac{1}{2q}}\right\|_q$$

$$\leq \left\|\left\{t_i^{\frac{1}{2q}}\left(\sum_{m=1}^{m}(S_1^*)\left(\sum_{m=1}^{m} S_1 S_1^*\right)^{q-1} S_1\right) t_i^{\frac{1}{2q}}\left(\sum_{m=1}^{m}(S_2^*)\left(\sum_{m=1}^{m} S_2 S_2^*\right)^{q-1} S_2\right)\right\}\right\|_q \|X\|$$

$$= q\overline{\sum t_i^{\frac{1}{2}}\left(\sum_{m=1}^{m}(S_1^*)\left(\sum_{m=1}^{m} S_1 S_1^*\right)^{q-1} S_1\right) t_i^{\frac{1}{2q}}\left(\sum_{m=1}^{m}(S_2^*)\left(\sum_{m=1}^{m} S_2 S_2^*\right)^{q-1} S_2\right)} \|X\|.$$

Again, we have that

$$\begin{aligned}
\|\pitchfork_{S_1,S_2}\|^q &\leq \left(\sum_{i=1}^{\infty} t_i\left(\sum_{m=1}^{m} S_1, S_1^*\right)^{q-1} S_1\right) \\
&\leq tr\left(\sum_{i=1}^{\infty} S_1\left(\sum_{m=1}^{m} S_1, S_1^*\right)^{q-1} S_1\right) \\
&= \|\left(\sum_{m=1}^{m} S_1, S_1^*\right)\|_q^q \\
&= \|\pitchfork_{S_1,S_2}(I)\|_q^q \\
&\leq \|\pitchfork_{S_1,S_2}\|^q.
\end{aligned}$$

This gives the desired results.

Theorem 5.5 *Let* $\Upsilon \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$ *such that* $S_1, S_2 \in B(H_1, H_2)$ *and also let each of the sequences* $\{S_{1_n}\}_{n=1}^{\infty}$, $\{S_{2_n}\}_{n=1}^{\infty} \in B((H_1), B(H_2))$ *be consisting of normal operators* S_1 *and* S_2. *Let also* f *be an analytic function with positive Taylor coefficients such that* $f\left(\left\|\sum_{n=1}^{\infty} S_{1_n}^* S_{1_n}\right\|\right) < \infty$ *and* $f\left(\left\|\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}\right\|\right) < \infty$ *thus for all symmetrical normal functions* Φ *and for all* $X \in C_\Phi(H)$ *we have* $\left\|\Upsilon_{S_1,S_2} X\right\| \leq \left\|\sqrt{f\left(\sum_{n=1}^{\infty} S_{1_n}^* S_{1_n}\right)} X \sqrt{f\left(\sum_{n=1}^{\infty} S_{1_n}^* S_{1_n}\right)}\right\|_\Phi$.

Proof. We know that $f(z) = \sum_{n=0}^{\infty} C_n Z^n$, where $C_n = \frac{f^{(n)}(0)}{n!} \geq 0$, we have that,

$$\begin{aligned}
f\left(\sum_{n=1}^{\infty} S_{1_n} \bigotimes S_{2_n}\right) X &= \sum_{n=0}^{\infty} C_n \left(\sum_{n=1}^{\infty} S_{1_n} \bigotimes S_{2_n}\right)^n X \\
&= C_0 X + \sum_{n=0}^{\infty} C_n \sum_{n_1^n,\dots,n_n^n=1}^{\infty} S_{1_{n_n^n}} \dots S_{1_{n_n^n}} X S_{2_{n_n^n}} \\
&= C_0 X + \sum_{1 \leq k \leq n}^{\infty} C_n S_{1_{n_1^n}} \dots S_{1_{n_1^n}} \dots S_{2_{n_1^n}} \\
&= C_0 X + \sum_{n=1}^{\infty} C_n S_{1_{n_1'}} X S_{2_{n_1'}} + \left(\sum_{n_1'',n_1''}^{\infty} S_{2_{n_2''}} S_{1_{n_1''}} X S_{2_{n_1''}} S_{2_{n_2''}} + \dots\right) \\
& f\left(\sum_{n=1}^{\infty} S_{1_n} \bigotimes S_{2_n}\right) \\
&= C_0 I \bigotimes I + \sum_{1 \leq k \leq n}^{\infty} C_n S_{1_{n_n^n}} \dots S_{1_{n_1^n}} \bigotimes S_{2_{n_1^n}} \dots S_{2_{n_n^n}}.
\end{aligned}$$

Now applying ([15], Theorem 3.2) on $\{\sqrt{C_0 I}\} \bigcup \{\sqrt{C_n} S_{1_{n_n^n}} \dots S_{1_{n_1^n}}\}_{1 \leq k \leq n}$ and $\{\sqrt{C_0 I}\} \bigcup \{\sqrt{C_n} S_{2_{n_n^n}} \dots S_{2_{n_1^n}}\}_{1 \leq k \leq n}$ in stead of $\{S_{1_n}\}_{n=1}^{\infty}$ and $\{S_{2_n}\}_{n=1}^{\infty}$ respectively and by the fact that

$$\|\sqrt{C_0^n}\|^2 + \sum_{1 \leq k \leq n} \left\|C_n S_{2_{n_n^n}} \dots S_{2_{n_1^n}} h\right\|^2 \leq f\left(\left\|\sum_{n=1}^{\infty} S_{1_n}^* S_{1_n}\right\|\right) \|h\|^2$$

and,

$$\|C_0 h\|^2 + \sum_{1 \leq k \leq n} \left\|C_n S_{2_{n_n^n}} \dots S_{2_{n_2^n}} h\right\|^2 \leq f\left(\left\|\sum_{n=1}^{\infty} S_{1_n}^* S_{1_n}\right\|\right) \|h\|^2$$

$\forall\, h \in H$, so we get,

$$\left\|f\left(\sum_{n=1}^{\infty} S_{1_n} \bigotimes S_{2_n}\right) X\right\| = \left\|C_0 X + \sum_{1 \leq k \leq n}^{\infty} C_n S_{1_{n_n^n}} \dots S_{1_{n_1^n}} X S_{2_{n_1^n}} \dots S_{2_{n_n^n}}\right\|_\Phi$$

$$\leq \left\| \left(C_0 I \bigotimes I + \sum_{1\leq k\leq n}^{\infty} C_n S^*_{1_{n_n^n}} ... S^*_{1_{n_1^n}} S_{2_{n_1^n}} ... S_{2_{n_n^n}} \right)^{\frac{1}{2}} \right\| |X|$$

$$\left\| \left(C_0 I \bigotimes I + \sum_{1\leq k\leq n}^{\infty} X C_n S^*_{2_{n_n^n}} ... S^*_{2_{n_1^n}} S_{2_{n_1^n}} ... S_{2_{n_n^n}} \right)^{\frac{1}{2}} \right\|_{\Phi}$$

$$= \left\| \left(C_0 I + \sum_{n=1}^{\infty} C_n \left(\sum_{n=1} S^*_{1_n} S_{1_n} \right)^n \right)^{\frac{1}{2}} \right\| |X|$$

$$\left\| \left(C_0 I + \sum_{n=1}^{\infty} C_n \left(\sum_{n=1} S^*_{2_n} S_{2_n} \right)^n \right)^{\frac{1}{2}} \right\|_{\Phi}$$

$$= \left\| \sqrt{f\left(\sum_{n=1}^{\infty} S^*_{1_n} S_{1_n} \right)} X \sqrt{f\left(\sum_{n=1}^{\infty} S^*_{1_n} S_{1_n} \right)} \right\|_{\Phi}.$$

Theorem 5.6 *Let* $\Upsilon_{S_1,S_2} \in \mathcal{B}_{IPTIT}(B(H_1, B(H_2))$ *and* $\forall\, S_1, S_2 \in B(H_1, H_2)$, *we have some strong square summable families of sequences* $\{S^*_{1_n}\}_{n=1}^{\infty}$, $\{S_{1_n}\}_{n=1}^{\infty}$, $\{S_{2_n}\}_{n=1}^{\infty}$ *and* $\{S^*_{2_n}\}_{n=1}^{\infty}$. *For any arbitrary symmetric normed function* Q *and* $q \geq 2$, *we have that*

$$\left\| \Upsilon_{S_1,S_2}(X) \right\|_{Q^{(q)}} \leq \sqrt{\left\| \sum_{n=1}^{\infty} S_{1_n} S^*_{1_n} \right\|} max \left\{ \sqrt{\left\| \sum_{n=1}^{\infty} S^*_{2_n} S_{2_n} \right\|}, \sqrt{\left\| \sum_{n=1}^{\infty} S_{2_n} S^*_{2_n} \right\|} \right\} \|X\|_{Q^{(q)}}. \tag{5}$$

Proof. Since $\{S_{1_n}\}_{n=1}^{\infty}$ and $\{S_{2_n}\}_{n=1}^{\infty}$ are mutually commuting normal operators, then for any $X \in B(H)$ we have that,

$$\left\| \sum_{n=1}^{N} S_{1_n} X S_{2_n} \right\|_{Q^{(q)}} \leq \left\| \sum_{n=1}^{N} S_{1_n} S^*_{2_n} \right\|^{\frac{1}{2}} \left\| \sum_{n=1}^{N} S^*_{2_n} X^* X S_{2_n} \right\|_{Q^{(\frac{q}{2})}}. \tag{6}$$

This implies that,

$$\left\| \sum_{n=1}^{N} S_{1_n} X S_{2_n} \right\|_{Q^{(q)}} \leq \sqrt{\left\| \sum_{n=1}^{N} S_{1_n} S^*_{1_n} \right\|} \max \left\{ \left\| \sum_{n=1}^{N} S^*_{2_n} S_{2_n} \right\|^{\frac{1}{2}}, \left\| \sum_{n=1}^{N} S_{2_n} S^*_{2_n} \right\|^{\frac{1}{2}} \right\} \|X^* X\|^{\frac{1}{2}}_{Q^{(\frac{q}{2})}}$$

$$= \sqrt{\left\| \sum_{n=1}^{N} S_{1_n} S^*_{1_n} \right\|} \max \left\{ \sqrt{\left\| \sum_{n=1}^{N} S^*_{2_n} S_{2_n} \right\|} \right\} \|X\|_{Q^{(q)}}$$

$$\leq \sqrt{\left\| \sum_{n=1}^{\infty} S_{1_n} S^*_{1_n} \right\|} \max \left\{ \sqrt{\left\| \sum_{n=1}^{\infty} S^*_{2_n} S_{2_n} \right\|}, \sqrt{\left\| \sum_{n=1}^{\infty} S_{2_n} S^*_{2_n} \right\|} \right\} \|X\|_{Q^{(q)}}.$$

Theorem 5.7 *Suppose that* $\Upsilon_{S_1,S_2} \in \mathcal{B}_{IPTIT}(B(H_1, B(H_2))$ *and that* $S_1, S_2 \in B(H_1, H_2)$. *We can also have* $\forall\, f$ *and* $X \in H$, *that* $\sum_{n=1}^{\infty}\left(\|S_{1_n}f\|^2 + \left\|S_{1_n}^* f\right\|^2 + \|S_{2_n}f\|^2 + \left\|S_{2_n}^* f\right\|^2\right.$ ∞. *Then* $\left\|\Upsilon_{S_1,S_2}(X)\right\| \leq \overline{\|\sum_{n=1}^{\infty} S_{1_n}S_{1_n}^*\| \, \|\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}\|}\|X\|$.

Proof. By definition, $\Upsilon\, S_1$ and S_2 are bounded operators and hence the functions f and g too are bounded in H. This implies that $\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}$ and $\sum_{n=1}^{\infty} S_{1_n}S_{2_n}^*$ too are bounded operators so we have that

$$\begin{aligned}
\left|\left\langle\left(\sum_{n=1}^{\infty} S_{1_n}XS_{2_n}^*\right)f, g\right\rangle\right| &\leq \sum_{n=1}^{\infty}\|X\|\,\|S_{2_n}f\|\left\|S_{1_n}^* g\right\| \\
&\leq \overline{\|X\|\sum_{n=1}^{\infty}\|S_{2_n}f\|^2\sum_{n=1}^{\infty} S_{1_n}S_{1_n}^* g, g} \\
&= \|X\|\overline{\left\langle\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}f, f\right\rangle\left\langle\sum_{n=1}^{\infty} S_{1_n}S_{2_n}^* g, g\right\rangle} \\
&\leq \overline{\left\|\sum_{n=1}^{\infty} S_{1_n}S_{1_n}^*\right\|\left\|\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}\right\|}\|X\|\|f\|\|g\| \\
&\leq \overline{\left\|\sum_{n=1}^{\infty} S_{1_n}S_{1_n}^*\right\|\left\|\sum_{n=1}^{\infty} S_{2_n}^* S_{2_n}\right\|}\|X\|.
\end{aligned}$$

Proposition 5.8 *Let* $\Upsilon \in \mathcal{B}_{IPTIT}(B(H_1), B(H_2))$ *and for all* $q \geq 1$, *its necessary that for* $\Upsilon_{S^*,S} : B(H_1) \to B(H_2)$ *to be bounded, then* $R_q = \underset{n,m\in\mathcal{N}\; q\in\Gamma}{Max}\left\|\Upsilon_{S,q}\left(\frac{q}{q\sqrt{n}}\right)\right\|^{\frac{q-1}{mq}}$, *where* Γ *is a rank two operator, for all* $R_q \leq \|\Upsilon_{S^*,S}\|_{q\,B(H)\to B(H)}$. *It is also sufficient that for* $q = 2$, *we have* $\left\|\Upsilon_{S^*,S}\right\|_{q\,B(H)\to B(H)} =_{m\in\mathcal{N},\, q\in\Gamma}^{\max}\left\|\Upsilon_{S,S^*}\,\Upsilon_{S^*,S}(X)\right\|^{\frac{1}{2m}}$.

Proof. Its clear without loss of generality that $\forall\, q > 1$ and $X \in B(H_1, H_2)$ we have that

$$\begin{aligned}
\left\|\Upsilon_{S,q}X\right\|_q &= \left\|\left(\Upsilon_{S,S^*}\left(\Upsilon_{S^*,S}X\right)^{q-1}\right)^{\frac{1}{q-1}}\right\|_q \\
&= \left\|\Upsilon_{S,S^*}\left(\Upsilon_{S^*,S}X\right)^{q-1}\right\|_{\frac{1}{q-1}}^{\frac{1}{q-1}} \\
&\leq \left\|\Upsilon_{S,S^*}\right\|_{B(H)\to B(H)}^{\frac{1}{q-1}}\left\|\left(\Upsilon_{S^*,S}X\right)^{q-1}\right\|^{\frac{1}{q-1}}\|X\|_q \\
&\leq \left\|\Upsilon_{S,S^*}\right\|_{B(H)\to B(H)}^{\frac{1}{q-1}}\left\|\Upsilon_{S^*,S}X\right\|_q \\
&= \left\|\Upsilon_{S,S^*}\right\|_{B(H)\to B(H)}^{\frac{1}{q-1}}\|X\|_q.
\end{aligned}$$

By mathematical induction, we have that

$$\begin{aligned}\left\|\mathcal{T}^m_{S,q}\frac{X}{\|X\|_q}\right\|^{\frac{q-1}{m}} &\leq \left(\left\|\mathcal{T}_{S,S^*}\right\|^{\frac{q-m}{mq}}_{B(H)\to B(H)}\left\|\frac{X}{\|X\|_q}\right\|_q\right)^{\frac{q-1}{mq}}\\ &= \left\|\mathcal{T}_{S,S^*}\right\|_{B(H)\to B(H)}.\end{aligned}$$

Taking X as Q, we have that $\|Q\|_q = \sqrt[q]{Q}$ for all $X \in \Gamma_n$ for all $q = 2$ we have that $\mathcal{T}_{S,q} = \mathcal{T}_{S,S^*}\mathcal{T}_{S^*S}$. So by the statement of Proposition 5.8, we have that

$$\begin{aligned}\left\|\mathcal{T}_{S^*,S}X\right\| &= \left\|\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S}|X^*|\right\|_2^{\frac{1}{4}}\|X\|_2^{\frac{1}{2}}\\ &\leq \left\|\left(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S}\right)^m|X^*|\right\|_2^{\frac{1}{4m}}\left\|\left(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S}\right)^m|X^*|\right\|_2^{\frac{1}{4m}}\|X\|_2^{1-\frac{1}{2m}}.\end{aligned}$$

Assuming that X be a singular value after expansion of X takes the form $X = \sum_{n=1}^{\infty} a_n(X)e_n^*\otimes f_n$. By choosing the orthogonal systems $\{e_m\}$ and $\{f_m\}$, Let also $Q \in \sum_{l=1}^{n}\{e_l^*\}\otimes\{e_l\}$ and $R = \sum_{l=1}^{n} f_l^*\otimes f_l$. So $R, Q \in \Gamma$ satisfying $RXQ = \sum_{l=1}^{n} a_k(X)e_l^*\otimes f_l$. Hence, $|RXQ| = \sum_{k=1}^{n} e_l(X)e_l^*\otimes e_l = |X|Q = Q|X|$. and $|(RXQ)^*| = \sum_{l=1}^{n} e_l(X)f_l^*\otimes f_l = |X^*|Q = Q|X^*|$. Now, a special case arises that $(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^m$ is an IPTIT. Thus, it takes the form, $\mathcal{T}_{S,S^*}X = \int_\Omega S^*XSd\mu$ and this implies that

$$(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^m = \int_{(nXn)^m}\left(\Pi_{l=1}^m S_{tl}S_{t_l}^*\right)^* X\Pi_{l=1}^m S_{tl}S_{t_l}^* d\mu(t-l)d\mu(t_l).$$

Since $q = 2$, we have

$$\begin{aligned}\|(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})(Q|X|Q)\|_2 &= \mathcal{T}_{S^*,S})(Q|X|Q)\|_2\\ &= \mathcal{T}_{S^*,S})|X|\|_2\\ &\leq (\mathcal{T}_{S^*Q,QS}\mathcal{T}_{SQ,S^*Q}(I)^{\frac{1}{4}}|X|)(\mathcal{T}_{S^*Q,QS}\mathcal{T}_{SQ,S^*Q}(I)^{\frac{1}{4}}|X|)\|_2\\ &\leq \|\mathcal{T}_{S^*Q,QS}\mathcal{T}_{SQ,S^*Q}(I)^{\frac{1}{2}}|X|\|\\ &\leq Q\mathcal{T}_{S^*,S}(\mathcal{T}_{S,S^*}(Q))(I)^{\frac{1}{2}}\|X\|_2\\ &\leq \|(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^2 Q\|^{\frac{1}{2}}\|X\|_2\end{aligned}$$

Likewise, $\|\mathcal{T}_{S,S^*}(R|X^*|R)\|_2 \leq \|(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^{2m}(R)\|^{\frac{1}{2}}\|X\|_2$.

Taking $L_2' =^{Max}_{m\in\mathcal{N},q\in\Gamma_n} \|(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^n(R)\|^{\frac{1}{4n}}$. Then we apply to RXQ in place of X to get

$$\begin{aligned}\|\mathcal{T}_{S,S^*}(Q|X^*|R)\|_2 &= \|(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^{2m}(Q)\|^{\frac{1}{8n}}(\mathcal{T}_{S,S^*}\mathcal{T}_{S^*,S})^{2m}(R)\|^{\frac{1}{8n}}\|X\|_2\\ &= n^{\frac{1}{2n}}L_2'\|X\|_2.\forall\, n\in\mathcal{N}\; and\, n\to\infty.\end{aligned}$$

We see that $\|\mathcal{T}_{S^*,S}(RXQ)\|_2 \leq L_2'\|X\|_2 \forall\, n \in \mathcal{N}$ and $Q, R \in \Gamma_n$. Therefore, the lower semi-continuity of $\|.\|$, assumes that $\|X\|_2 =_{n\to\infty}^{lim} \|R_nXQ_n\|_2$. Hence, $\|_{n\to\infty}^{lim}\|R_nXQ_n\|_2 - X\|_2^2 -_{n\to\infty}^{lim}\|R_nXQ_n\|_2 = 0$, for all $X \in B(H)$ a contradiction to that $\|\mathcal{T}_{S^*,S}X\|_2 \leq L_2'\|X\|_2$, for all $X \in B(H)$. Hence, a required reverse inequality $\|\mathcal{T}_{S^*,S}X\|_2 \leq L_2'$.

5.2 Cauchy-Buniakowski-Schwarz Inequalities for IPTIT

At this point we consider the CBS-inequality for IPTIT. We begin with the following proposition.

Proposition 5.9 *Let $\hat{\mathcal{T}}_n \in \mathcal{B}_{IPTIT}(B(H_n)$ and given $\alpha = \alpha_1, ..., \alpha_n \in \mathcal{C}^+$ then we have the $CBS-$inequality as*

$$\left(\sum_{i=1}^{p} |\hat{\mathcal{T}}_n|\right)\left\{\sum_{i=1}^{p} |\alpha|^2\right\} - \left|\sum_{i=1}^{p} \alpha\hat{\mathcal{T}}_n\right|^2 \leq \sum_{i=1}^{p} |\hat{\mathcal{T}}_n - \bar{\alpha}W|^2 \left\{\sum_{i=1}^{p} |\alpha|^2\right\}, \tag{7}$$

where

$$W = \left(\sum_{i=1}^{p} \alpha\hat{\mathcal{T}}_n\right)\left[\sum_{i=1}^{p} |\alpha|^2\right]^{-1}. \tag{8}$$

Proof. Consider $\sum_{i=1}^{p} |\alpha|^2 \neq 0$, then we see that

$$\begin{aligned}
\sum_{i=1}^{p} |\hat{\mathcal{T}}_n - \alpha W|^2 &\leq \sum_{i=1}^{p} (\hat{\mathcal{T}}_n - \alpha W^*)(\hat{\mathcal{T}}_n - \alpha W) \\
&\leq \sum_{i=1}^{p} |\hat{\mathcal{T}}_n|^2 - \sum_{i=1}^{p} \bar{\alpha}\hat{\mathcal{T}}_n^* W - \sum_{i=1}^{p} \alpha W^* \hat{\mathcal{T}}_n + \left\{\sum_{i=1}^{p} |\alpha|^2\right\}|W|^2.
\end{aligned}$$

Since W is well defined in Equation 8, we have that

$$\sum_{i=1}^{p} |\hat{\mathcal{T}}_n - \bar{\alpha}_n W|^2 = \sum_{i=1}^{p} |\hat{\mathcal{T}}_n|^2 - \frac{\left|\sum_{i=1}^{p} \alpha\hat{\mathcal{T}}_n\right|^2}{\sum_{i=1}^{p} |\alpha|^2}.$$

Theorem 5.10 *Let $\hat{\mathcal{T}}_n \in \mathcal{B}_{IPTIT}(B(H_n)$ and given $\alpha = \alpha_1, ..., \alpha_n \in \mathcal{R}^+$ and $k \geq 2$, we have that $\sum_{i=1}^{p} \frac{|\alpha_1 \mathcal{T}_i - \alpha_2\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_2} \leq \{\sum_{i=1}^{p} \alpha_1\} \sum_{i=1}^{p} \frac{|\hat{\mathcal{T}}_i|}{\alpha_1} - \left|\sum_{i=1}^{p} \hat{\mathcal{T}}_i\right|^2$.*

Proof. To achieve this, mathematical induction is applied to the second part and show that it coincides with our inequality. So

$$P(n) : \left(\sum_{i=1}^{p} \alpha_1\right) \sum_{i=1}^{p} \frac{|\hat{\mathcal{T}}_i|^2}{\alpha_1} - \left|\sum_{i=1}^{p} \hat{\mathcal{T}}_i\right|^2 \leq \sum_{i=1}^{p} \frac{|\alpha_1 \mathcal{T}_i - \alpha_2\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_2},$$

for all $n \geq 2$. Assuming $P(k)$ as for $k \geq 2$ is true, we prove for $P(k+1)$.

$$\begin{aligned} P(n): \left(\sum_{i=1}^{k+1}\alpha_1\right)\sum_{i=1}^{k+1}\frac{|\hat{\mathcal{T}}_i|^2}{\alpha_1} - \left|\sum_{i=1}^{k+1}\hat{\mathcal{T}}_i\right|^2 &= \left(\sum_{i=1}^{p}\alpha_1\right)\sum_{i=1}^{p}\frac{|\hat{\mathcal{T}}_i|^2}{\alpha_1} + \alpha_{k+1}\frac{|\hat{\mathcal{T}}_i|^2}{\alpha_1} + ... + \\ & \alpha_{k+1}\frac{|\hat{\mathcal{T}}_i|^2}{\alpha_1} + |\hat{\mathcal{T}}_{k+1}|^2 - \left|\sum_{j=1}^{p}\hat{\mathcal{T}}_k\right|^2 \\ &- (\hat{\mathcal{T}}_k^* + ... + \hat{\mathcal{T}}_k^*)\hat{\mathcal{T}}_{k+1} - (\hat{\mathcal{T}}_k + ... + \hat{\mathcal{T}}_k)\hat{\mathcal{T}}_{k+1}^* - |\hat{\mathcal{T}}_{k+1}|^2 \\ &= \sum_{i=1}^{p}\frac{|\alpha_1\hat{\mathcal{T}}_i - \alpha_2\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_2} + \sum_{i=1}^{p}\frac{|\alpha_1\hat{\mathcal{T}}_{k+1} - \alpha_{k+1}\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_{k+1}} \\ &\geq \sum_{i=1}^{p}\frac{|\alpha_1\hat{\mathcal{T}}_i - \alpha_2\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_2}. \end{aligned}$$

Hence, the reverse deduction and so our earlier supposition is true that

$$\sum_{i=1}^{p}\frac{|\alpha_1\hat{\mathcal{T}}_i - \alpha_2\hat{\mathcal{T}}_j|^2}{\alpha_1\alpha_2} \leq \left\{\sum_{i=1}^{p}\alpha_1\right\}\sum_{i=1}^{p}\frac{|\hat{\mathcal{T}}_i|}{\alpha_1} - \left|\sum_{i=1}^{p}\hat{\mathcal{T}}_i\right|^2.$$

Theorem 5.11 *If $\hat{\mathcal{T}}_a$ and $\hat{\mathcal{T}}_b$ are IPTITs, we can have the CBS-inequality as* $|\alpha_1\hat{\mathcal{T}}_a + \alpha_2\hat{\mathcal{T}}_b|^2 \leq (\alpha_1 - \alpha_2)(\alpha_1|\hat{\mathcal{T}}_a|^2 - \alpha_2|\hat{\mathcal{T}}_b|^2) + \alpha_1\alpha_2|\hat{\mathcal{T}}_a + \hat{\mathcal{T}}_b|^2$, $\forall\, \alpha_1, \alpha_2 \in \mathcal{R}^+$.

Proof. The right hand side shows that

$$\begin{aligned} &(\alpha_1^2 - \alpha_1\alpha_2)|\hat{\mathcal{T}}_a|^2 + (\alpha_2^2 - \alpha_1\alpha_2|\hat{\mathcal{T}}_b|^2) + \alpha_1\alpha_2(\hat{\mathcal{T}}_a^* + \hat{\mathcal{T}}_b^*)(\hat{\mathcal{T}}_a + \hat{\mathcal{T}}_b) \\ &= (\alpha_1^2 - \alpha_1\alpha_2)(|\hat{\mathcal{T}}_a|^2 + (\alpha_2^2 - \alpha_1\alpha_2)|\hat{\mathcal{T}}_b|^2)\alpha_1\alpha_2|\hat{\mathcal{T}}_a|^2 + \alpha_1\alpha_2\hat{\mathcal{T}}_a^*\hat{\mathcal{T}}_b^* + \alpha_1\alpha_2 \\ &\leq (\alpha_1^2|\hat{\mathcal{T}}_a|^2 + \alpha_2^2|\hat{\mathcal{T}}_b|^2) + \alpha_1\alpha_2\hat{\mathcal{T}}_a^*\hat{\mathcal{T}}_b^*\alpha_1\alpha_2\hat{\mathcal{T}}_b^*\hat{\mathcal{T}}_a^* \end{aligned}$$

Hence, the statement is true.

Theorem 5.12 *Given $\hat{\mathcal{T}}_a, \hat{\mathcal{T}}_b \in B_{IPTIT}(B(H_1, H_2))$ and that $0 \leq \alpha \geq 1$ then it holds that*

$$\begin{aligned} 0.5\bigvee\{\alpha, 1-\alpha\}|\hat{\mathcal{T}}_a - \hat{\mathcal{T}}_b|^2 &\leq \alpha|\hat{\mathcal{T}}_a|^2 + (1-\alpha)|\hat{\mathcal{T}}_b| - |\alpha\hat{\mathcal{T}}_a + (1-\alpha)\hat{\mathcal{T}}_b|^2 \\ &\leq 0.5\bigwedge\{\alpha, 1-\alpha\}|\hat{\mathcal{T}}_a - \hat{\mathcal{T}}_b|^2 \end{aligned}$$

only when $\alpha = 0.5$ or $\hat{\mathcal{T}}_a = \hat{\mathcal{T}}_b$ and $a, b \in \mathcal{R}^+$.

Proof. Since we already have the inequality $0.5\bigvee\{\alpha, 1-\alpha\} \leq 0.5\bigwedge\{\alpha, 1-\alpha\}$, for all $0 \leq \alpha \leq 1$ and that $\hat{\mathcal{T}}_a, \hat{\mathcal{T}}_b \in B_{IPTIT}(B(H_1, H_2))$, then we have that $\alpha_1\hat{\mathcal{T}} \leq \alpha_2\hat{\mathcal{T}} \leq \alpha_3\hat{\mathcal{T}}$, where $\hat{\mathcal{T}} = |\hat{\mathcal{T}}_a - \hat{\mathcal{T}}_b|^2 \geq 0$. By letting $\alpha = 0.5$, the inequality for $\hat{\mathcal{T}}_a = \lambda\hat{\mathcal{T}}_b$ follows that

$$\begin{aligned} 0.5\bigvee\{\alpha, 1-\alpha\}(\kappa - 1)^2 &\leq \alpha(1-\alpha)(\kappa - 1)^2 \\ &\leq 0.5\bigwedge\{\alpha, 1-\alpha\}(\kappa - 1)^2, \end{aligned}$$

for any $\kappa \in \mathcal{R}^+$.

Theorem 5.13 *Given $\hat{\mathcal{T}}_a, \hat{\mathcal{T}}_b \in B_{IPTIT}(B(H_1, H_2))$ be any positive operator and that $\alpha_1, ..., \alpha_n \in \mathcal{R}^+$, then we can have*

$$\left(\alpha^2 - \sum_{i=1}^{p} \alpha_i^2\right) \frac{|\alpha_j \hat{\mathcal{T}}_i - \alpha_i \hat{\mathcal{T}}_j|^2}{\alpha^2 - \alpha_j^2} \leq \left|\alpha \hat{\mathcal{T}} - \sum_{i=1}^{p} \alpha_i \hat{\mathcal{T}}_i\right|^2 - \left(\alpha^2 - \sum_{i=1}^{p} \alpha_i^2\right)\left(|\hat{\mathcal{T}}|^2 - \sum_{i=1}^{p} |\hat{\mathcal{T}}_i|^2\right) \geq 0,$$

for all $i, j \in \{1, ..., k\}$.

Proof. We choose any arbitrary sequence $X - n$ then by monotonicity we have that

$$X_n = \frac{\left|\alpha \hat{\mathcal{T}} - \sum_{i=1}^{p} \alpha_i \hat{\mathcal{T}}_i\right|}{\alpha^2 - \sum_{i=1}^{p} \alpha_i^2} - |X|^2 + \sum_{i=1}^{p} |\hat{\mathcal{T}}_i|^2,$$

for all $n \leq k$ we can have a representation as

$$X_{n+1} - X_n = \frac{\left|\alpha \hat{\mathcal{T}} - \sum_{i=1}^{p} \alpha_i \hat{\mathcal{T}}_i - \alpha_{n+1} \hat{\mathcal{T}}_{n+1}\right|^2}{\alpha^2 - \sum_{i=1}^{p} \alpha_i^2 - \alpha_{n+1}^2} + |\hat{\mathcal{T}}_{n+1}|^2 - \frac{\left|\alpha \hat{\mathcal{T}} \sum_{i=1}^{p} \alpha_i \hat{\mathcal{T}}_i\right|}{\alpha^2 - \sum_{i=1}^{p} \alpha_i^2}.$$

Applying the inequality of Bergetrom, for any two operators, we have

$$\left|\alpha \hat{\mathcal{T}} - \sum_{i=1}^{k} \alpha_i \hat{\mathcal{T}}_i\right|^2 \leq \frac{|\alpha_{n+1} \hat{\mathcal{T}}_{n+1}|^2}{\alpha_{n+1}^2} + \left|\alpha \hat{\mathcal{T}} - \sum_{i=1}^{p} \hat{\mathcal{T}}_i - \alpha_{n+1} \hat{\mathcal{T}}_{n+1}\right|^2.$$

This implies that $X_{n+1} - X_n \geq 0$ hence increasing sequence. Therefore, we get $X_k \geq X_{k-1} \geq ... \geq X_i$. We note that

$$\begin{aligned} \frac{|\alpha_i \hat{\mathcal{T}}_i - \alpha_j \hat{\mathcal{T}}_j|^2}{\alpha_i^2 - \alpha_j^2} &= \frac{|\alpha_i \hat{\mathcal{T}}_i - \alpha_j \hat{\mathcal{T}}_j|}{\alpha_i^2 - \alpha_j^2} - |\hat{\mathcal{T}}_i|^2 + |\hat{\mathcal{T}}_j|^2 \\ &= X_i. \end{aligned}$$

Rearranging the terms of the sequence, we have the inequality $X_n = \frac{\alpha_j \hat{\mathcal{T}}_j - \alpha_i \hat{\mathcal{T}}_i}{\alpha_i^2 - \alpha_j^2}$. Multiplying through by $\alpha_i^2 - \sum_{i=1}^{p} \alpha_j^2 > 0$ we get the initial supposition of the inequality.

6 Open Problems

A lot of researches have been carried out on inner product type integral transformers (IPTIT) with regard to various aspects including spectra, numerical ranges and operator inequalities. Consider M and N to be weakly μ-measurable operator valued (OV) functions such that $M, N : \Omega \to B(X)$ for any $Q \in \mathcal{B}(H)$. If M and N are integrable with respect to Gel'fand axiom, then we obtain a linear transformation arising from the inner product space as

$Q \mapsto \int_\Omega M_t Q N_t \partial(t)$. There still exists an open problem regarding IPTIT while studying inequalities for IPTIT with spectra limited to the unit disc in complex domains. It has been pointed out that the inequalities, and in particular CBS-inequality, can only be attained for these IPTIT if only one of the operator M or N is normal. Therefore, in this note we solve this problem by obtaining CBS-inequalities for IPTIT in Banach spaces. This leaves one natural open problem that should be tackled. **Problem :** Can the CBS-inequalities be deduce in the space of norm-attainable operators?

References

[1] Aldaz, J. M., Barza, S., Fujii, M., & Moslehian M.J. (2015). Advances in Operator Cauchy-Schwarz inequalities and their reverses. *Ann.Func.Anal.* , 6(3), 275-295.

[2] Arfken, G. (2015). Triple scalar product. *Mathematics methods for physics, Orlando academic press,* 32(3), 13-18.

[3] Bernet, N. (2010). On some integral inequalities related to the CBS. *Proc. AMS*, 26(10), 200-217.

[4] Bernet, N., & Dragomor, S. (2008). On additive reverses of the CBS integral inequalities. *Applied Math Letters*, 21(3), 13-35.

[5] Danko, R., Stefan, M., & Viladmir, D. (2015). Norm Inequalities for Elementary operators and Other inner product type integral Transformers with the Spectra Contained in the unit Disc. *Serbia Mathematics Journal*, 31(2), 176-206.

[6] Danko, R. (1997). Norm Inequalities for operators of the class of self adjoint absolute value generalied derivations. *Mathematical analysis and applications,* 49(3), 151-162.

[7] Danko, R. (2005). Cauchy-Shwarz Norm Inequalities for $weak^* - integral$ of operators valued functions. *J. Func. analysis,* 218(3), 318-346.

[8] Dragoljub, T. (2010). Applications of Cauchy-Schwartz inequality for hilbert modules to elementary operators and IPTI transformers. *Serbia mathematical society Classification*, 27(11), 1-24.

[9] Dragomir, S., & Sofo, A. (2003). A survey on Cauchy-Bumiakowski-Schwartz type discrete inequalities and their applications. *Journal of Pure and applied Math*, 4(3), 110-118.

[10] Dragomir, S., Brnetic, S., & Pecaric, J. (2008). Reverse of the Cauchy-Bumiakowski-Schwartz integral inequalities in Hilbert spaces. *Journal of Mathematics Analysis and applications*, 3(3), 23-37.

[11] Dragomir, S., & Sofo A. (2008). On some inequalities of Cauchy-Bumiakowski-Schwartz type and their applications. *TamKang Journal of Math*, 39(4), 56-78.

[12] Dragomir, S. (2015). Inequalities for the Riemann integral of the S-dominated integrators with their applications. *Probl. Anal. Issues Anal.*, 22(4), 11-37.

[13] Dragomir, S. (2016). Some Vector inequalities for two operators in Hilbert spaces with their applications. *Acta. Univ. Spiantiae, Mathematica*, 8(1), 75-92.

[14] Driis, M., & Hodaibi M. (2007). Spectra of quasisimilar operators. *Bull. Belg. Math. Soc. Simon Stevin.*, 14(5), 723-733.

[15] Du, H. K., Wang, Y. Q., & Gao, G. B. (2008). Norms of elementary operators. *Proceedings of the American mathematical society*, 136(4), 1337-1348.

[16] Milosevic, S. (2016). Norm inequalities for elementary operators related to contractions and their operators with spectra contained in the unit disck in norn ideals. *Serbia Mathematics Journal,* 23(2), 147-159.

[17] Okelo, N. B. (2020). On Properties of Inner Product Type Integral Transformers. *Open J. Math. Anal.,* 4(2), 160-169.

[18] Okelo, N. B., & Mogotu, P. O. (2020). On Norm Inequalities and Orthogonality of Commutators of Derivations. *Italian Journal of Pure and Applied Mathematics,* 44(2), 163-169.

[19] Okelo, N. B. (2020). Norms of Self-Adjoint Two-Sided Multiplication Operators in Norm-Attainable Classes. *MATLAB Jourmal,* 6(1), 77-79.

[20] Okelo, N. B., Agure, J. O., & Ambogo, D. O. (2010). Norms of elementary operators and characterization of norm-attainable operators. *Int. J. Math. Anal.,* 24(3), 1197-1204.

[21] Okelo, N. B. (2013). The norm-attainability of some elementary operators. *Appl. Math. E-Notes,* 13(2), 1-7.

[22] Okelo, N. B., Agure, J. O., & Oleche, P. O. (2013). Various notions of orthogonality in normed spaces, *Acta Mathematica Scientia,* 33(5), 1387-1397.

[23] Okelo, N. B. (2020). On Norm-attainable operators in Banach Spaces. *J. Function Spaces,* 4(1), 1-6.

[24] Rajendra, B., & Cnandler, D. (1995). A Cauchy Schwarz Inequality for operators with Applications. *Linear Algebra and its Applications,* 6(2), 119-129.

[25] Renu, C. (2020). Reverses of some inequalities in n-linear product spaces. *International Journal Of Mathematics and Mathematical Sciences,* 3(12), 11-21.

[26] Wagner, S. (1965). On Wagners inequality inequality and its applications, *Amer. Math. Soc., Notices*, 220(12), 22-30.

[27] Wickstead A. W. (2001). Norms of basic Elementary Operators on Algebras of regular operators. *Proceedings of the American mathematical society*, 16(2), 223-227.

[28] Wu, S. (2007). A further generalization of Aczel's inequality and Popovicius inequality. *Comp. Math Applications,* 105(3), 569-573.