Int. J. Open Problems Compt. Math., Vol. 18, No. 1, March 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Local existence of solution for the delayed sixth-order Boussinesq equation with dynamic boundary conditions

Erhan Pişkin[1], Zeynep Zehra Çeçen[2]
[1]Dicle University, Department of Mathematics, Diyarbakır, Turkey
[2]Dicle University, Institute of Natural and Applied Sciences, Department of Mathematics, Diyarbakır, Turkey
e-mails:episkin@dicle.edu.tr, 68zeynep86@gmail.com

Received 28 October 2024; Accepted 5 February 2025

Abstract

This work is concerned with the study of the delayed sixth-order Boussinesq equation with dynamic boundary conditions in a bounded domain. The local existence is investigated under some under suitable conditions by using Faedo-Galerkin method.

Keywords: *Boussinesq equation, delay, dynamic boundary, local existence.*

2010 Mathematics Subject Classification: 35A01; 35G31.

1 Introduction

1.1 Setting of the problem:

In this work, we consider the following delayed sixth-order Boussinesq equation with dynamic boundary conditions

$$\begin{cases} u_{tt} - u_{xxxxxx} + \mu_1 u_{xxxxt} + \mu_2 u_t\left(x, t-\tau\right) = \left|u\right|^{q-2} u & \text{in } \left(x,t\right) \in \left(0,l\right) \times \left(0,T\right), \\ u\left(0,t\right) = u_x\left(0,t\right) = u_{xx}\left(0,t\right) = 0 & \text{in } t \in \left(0,T\right), \\ \left(u_{xx} + u_{xxx}\right)\left(l,t\right) = 0 & \text{in } \left(x,t\right) \in \left(0,l\right) \times \left(0,T\right), \\ \left(u_{xtt} - u_{xxxx} + \mu_1 u_{xxt}\right)\left(l,t\right) = 0 & \text{in } \left(x,t\right) \in \left(0,l\right) \times \left(0,T\right), \\ \left(u_{tt} - \mu_1 u_{xxxt} + u_{xxxxx}\right)\left(l,t\right) = 0 & \text{in } \left(x,t\right) \in \left(0,l\right) \times \left(0,T\right), \\ u_t\left(x,t-\tau\right) = f_0\left(x,t-\tau\right) & \text{in } \left(x,t\right) \in \left(0,l\right) \times \left[0,\tau\right), \\ u\left(x,0\right) = u_0\left(x\right),\ u_t\left(x,0\right) = v_0\left(x\right) & \text{in } x \in \left(0,l\right) \end{cases} \tag{1}$$

where μ_1 is a positive constant, μ_2 is a real number, $q \geq 2$, $\tau > 0$ denotes the time delay, the initial value u_0, v_0, f_0 are given functions, $l > 0$.

- Time delays often appear in many practical problems such as biological, chemical and physical, thermal and economic phenomena [6].
- In 1872, Boussinesq [2] derived the following classical Boussinesq equations
$$u_{tt} - u_{xx} + \mu u_{xxxx} = \left(u^2\right)_{xx},$$
$$u_{tt} - u_{xx} - u_{xxtt} = \left(u^2\right)_{xx}.$$
The Boussinesq equation to describe the propagation of small amplitude long waves on the surface of shallow water. In [4] Daripa derived the higher-order Boussinesq equation
$$u_{tt} - u_{xx} - \alpha(u^2)_{xx} \mp \alpha u_{xxxx} - \varepsilon^2 \alpha u_{xxxxxx} = 0$$
for two-way propagation of shallow water waves.
- Dynamic boundary condition are not only important from the theoretical views but also arising in various practical problems [14].

1.2 Literature overview:

In [3], Dalsen (1994) considered the damped beam problem as follows:

$$\begin{cases} u_{tt} - 2\rho u_{xxt} + u_{xxxx} = 0 & \text{in } (0,l) \times (0,\infty), \\ u(0,t) = u_t(0,t) = u_x(0,t) = u_{xt}(0,t) = 0 & \text{in } (0,\infty), \\ (u_x + u_{xx})(l,t) = 0, (u_{xx} - \alpha_1 u_{xxx})(0,t) = 0 & \text{in } (0,\infty), \\ (u_{tt} + 2\rho u_{xt} - u_{xxx})(l,t) = 0 & \text{in } (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = v_0(x) & \text{in } (0,l), \\ u(l,0) = \eta, \ u_t(l,0) = \mu. \end{cases}$$

He established the existence and uniqueness.

In [14], Zhang and Liu (2020) studied the problem as follows:

$$\begin{cases} u_{tt} - u_{xxxxxx} + \mu_1 u_{xxxxt} + \mu_2 u_t(x,t-\tau) = |u|^{q-2} u & \text{in } (0,l), \\ u(0,t) = u_x(0,t) = 0 & \text{in } t \in (0,T), \\ (u_x + u_{xx})(l,t) = 0 & \text{in } (x,t) \in (0,l) \times (0,T), \\ (u_{tt} + \mu_1 u_{xt} - u_{xxx})(l,t) = 0 & \text{in } (x,t) \in (0,l) \times (0,T), \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } (x,t) \in (0,l) \times (0,\tau), \\ u(x,0) = u_0(x), \ u_t(x,0) = v_0(x) & \text{in } (0,l). \end{cases}$$

They proved the existence and blows up in finite time of solutions.

In [5], Esfahani et al. (2012) studied the problem as follows:

$$u_{tt} = u_{xx} + \beta u_{xxxx} + u_{xxxxxx} + u_{xxxxxx} + (|u|^{\alpha} u)_{xx}.$$

They proved the local existence and well-posedness of solutions.

In [11], Polat and Pişkin (2015) studied the problem as follows

$$u_{tt} - u_{xx} - u_{xxtt} - u_{xxxxxx} - \alpha u_{xxt} = f(u)_{xx}.$$

They proved the lokal existence, global existence and asymptotic behaviour of solution.

In [8], Pişkin (2013) studied the problme as follows:

$$u_{tt} - u_{xx} - u_{xxtt} - u_{xxxxxx} - k u_{xxt} = f(u)_{xx},$$

studied the blow up of solutions of the equation using the generalized concavity method.

In [12], Wang (2017) studied the problem as follows:

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxx} - u_{xxxxxx} - \gamma u_{xxt} = (f(u))_{xx}$$

studied the existence of global solutions of the equation and its asymptotic behavior.

In [10], Pişkin and Irkil (2019) studied the problem as follows:

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxxtt} + u_{xxxx} + u_{xxxxxx} + \left(u_x \log |u_x|^k\right)_x = 0$$

studied the global existence and blow up of the sixth-order logarithmic Boussinesq equation.

In addition to the introduction, this paper has two other parts. In part 2, we give some notations. In part 3, we prove the local existence of solutions to (1) using Faedo-Galerkin method.

2 Preliminaries

In this part, we present some materials needed in this work. We use the standard Sobolev spaces $L^p(0,l)$ and $H^s(0,l)$ with the usual norms $\|.\|_{L^p}$ and $\|.\|_{H^s}$, respectively. Usually, $\|.\|$ denotes the norm $\|.\|_{L^2(0,l)}$. Next, we define $(u,v) = \int_0^l u(x)\, v(x)\, dx$ the scaler product in $L^2(0,l)$ (see [1, 9], for details). We introduce by

$$\mathcal{W} = \left\{ u \in H^3(0,l) : \quad u(0) = u_x(0) = u_{xx}(0) = 0 \right\}$$

the closed subspace of $H^3(0,l)$ equipped with the norm equivalent to the usual norm in $H^3(0,l)$. The poincare inequality holds in $\mathcal{W}$; i.e., there exist some positive constant $B = l^{\frac{q+2}{2q}}$ such that

$$\|u\|_q \leq B\|u_{xx}\|, \ q \geq 2. \tag{2}$$

For studying problem (1), we introduce as in [7] a new function z as follows:

$$z(x,\rho,t) = u_t(x,t-\rho\tau), \ \ (x,\rho,t) \in (0,l)\times(0,1)\times(0,T). \tag{3}$$

Thus, we get

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \ (x,\rho,t) \in (0,l)\times(0,1)\times(0,T). \tag{4}$$

Then, the problem (1) takes the form

$$\begin{cases} u_{tt} - u_{xxxxxx} + \mu_1 u_{xxxxt} + \mu_2 z(x,1,t) = |u|^{q-2}u & \text{in } (x,t)\in(0,l)\times(0,T), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } (x,\rho,t)\in(0,l)\times(0,1)\times(0,T), \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0 & \text{in } t\in(0,T), \\ (u_{xx}+u_{xxx})(l,t) = 0 & \text{in } (x,t)\in(0,l)\times(0,T), \\ (u_{xtt} - u_{xxxxx} + \mu_1 u_{xxt})(l,t) = 0 & \text{in } (x,t)\in(0,l)\times(0,T), \\ (u_{tt} - \mu_1 u_{xxxt} + u_{xxxxx})(l,t) = 0 & \text{in } (x,t)\in(0,l)\times(0,T), \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } (x,\rho)\in(0,l)\times(0,1), \\ u(x,0) = u_0(x), \ u_t(x,0) = v_0(x) & \text{in } x\in(0,l). \end{cases} \tag{5}$$

3 Local existence

In this part, we will give the following local existence of solution for the system (5).

Theorem 3.1. *Assume that* $u_0 \in H^6(0,l)\cap\mathcal{W}$, $v_0 \in \mathcal{W}$, $f_0 \in L^2((0,l)\times(0,1))$ *and* $\mu_1 > B^2|\mu_2|$ *then there exist a unique solution* (u,z) *of system (5) defined on* $(0,l)\times(0,T)$ *for some constant* $T>0$ *satisfying*

$$u \in L^\infty\left(0,T; H^6(0,l)\cap\mathcal{W}\right), \ \ u_t \in L^\infty(0,T;\mathcal{W}).$$

Proof. We divide the proof into the following two steps.

Step 1. (Approximate solution)

Let $\{\omega_m\}_{m=1}^\infty$ a complate orthogonal bases of $\mathcal{W}$. We consider $\mathcal{W}_n = span\{\omega_1,...,\omega_n\}$, for all $n\in N$. Also, we can find a set of bases $\{\varphi_j(x,\rho)\}_{j=1}^n$ which is subset of $L^2((0,l)\times(0,1))$ such that

$$\varphi_j(x,0) = \omega_j(x), \quad 1\leq j\leq n$$

and we define $V_n = span\{\varphi_1, ..., \varphi_n\}, \; n \in N.$

Choosing $\{u_{0n}\}$, $\{v_{0n}\}$ in $\mathcal{W}_n$ and $\{z_{0n}\}$ in V_n such that $u_{0n} \to u_0$ strongly in $\mathcal{W}$, $v_{0n} \to v_0$ strongly in $\mathcal{W}$, and $z_{0n} \to f_0$ strongly in $L^2\left((0,l) \times (0,1)\right)$. We define approximates

$$u_n\left(x,t\right) = \sum_{j=1}^{n} \omega_j\left(x\right) g_{jn}\left(t\right),$$

$$z_n(x,\rho,t) = \sum_{j=1}^{n} \varphi_j(x,\varphi) h_{jn}(t),$$

where $\left(u_n\left(t\right), z_n\left(t\right)\right)$ are solutions of the following system:

$$\begin{cases} \int_0^l u_{ntt}\omega_j dx - \int_0^l u_{nxxxxxx}\omega_j dx + \mu_1 \int_0^l u_{nxxxxt}\omega_j dx + \int_0^l \mu_2 z_n\left(x,1,t\right)\omega_j dx \\ = \int_0^l \left|u_n\right|^{q-2} u_n \omega_j dx, \\ z_n\left(x,0,t\right) = u_{nt}\left(x,t\right), \\ u_n\left(0,t\right) = u_{nx}\left(0,t\right) = u_{nxx}\left(0,t\right) = 0, \\ \left(u_{nxx} + u_{nxxx}\right)\left(l,t\right) = 0, \\ \left(u_{nxtt} - u_{nxxxx} + \mu_1 u_{nxxt}\right)\left(l,t\right) = 0, \\ \left(u_{ntt} - \mu_1 u_{nxxxt} + u_{nxxxxx}\right)\left(l,t\right) = 0, \\ u_n\left(x,0\right) = u_{0n}\left(x\right), \; u_{nt}\left(x,0\right) = v_{0n}\left(x\right), \\ \int_0^l \left[\tau z_{nt}\left(x,\rho,t\right) + z_{n\rho}\left(x,\rho,t\right)\right]\varphi_j dx = 0, \\ z_n\left(x,\rho,0\right) = f_{0n}(x,-\rho\tau) = z_{0n}. \end{cases} \tag{6}$$

by using the theories of ordinary differantial equation, we have (6) has a unique solution $\left(g_{jn}\left(t\right), h_{jn}(t)\right)_{j=1}^{n}$ defined on $(0,T)$.

Step 2. (A priori estimate)

By multiplying the first equation of (6) by $g'_{jn}\left(t\right)$, inregrating over $\left(0,t\right)$, using integration by parts, and considering the initial boundary value conditions, we get

$$\begin{aligned} & \frac{1}{2}\int_0^l \left|u_{nt}\right|^2 dx + \frac{1}{2}\int_0^l \left|u_{nxxx}\right|^2 dx + \mu_1 \int_0^t \left\|u_{nxxt}\right\|^2 dt \\ & +\mu_2 \int_0^t \int_0^l z_n(x,1,t) u_{nt}\left(t,x\right) dx dt - \frac{1}{q}\int_0^l \left|u_n\right|^q dx \\ & +\frac{1}{2}u_{nxt}^2\left(l,t\right) + \frac{1}{2}u_{nt}^2\left(l,t\right) + \frac{1}{2}u_{nxx}^2\left(l,t\right) \\ = & \frac{1}{2}\left\|v_{0n}\right\|^2 + \frac{1}{2}\left\|u_{0nxxx}\right\|^2 - \frac{1}{q}\left\|u_{0n}\right\|_q^q \\ & +\frac{1}{2}u_{nxt}^2\left(l,0\right) + \frac{1}{2}u_{nt}^2\left(l,0\right) + \frac{1}{2}u_{nxx}^2\left(l,0\right). \end{aligned} \tag{7}$$

Let constant $\xi > 0$, multiplying eighth equation of (6) by $\left(\frac{\xi}{\tau}\right) h_{jn}(t)$ and integrating over $(0,1)\times(0,t)$, we get

$$\begin{aligned} & \frac{\xi}{2}\int_0^l\int_0^1 z_n^2(x,\rho,t)\,d\rho dx + \frac{\xi}{\tau}\int_0^t\int_0^l\int_0^1 z_{n\rho}z_n(x,\rho,t)\,d\rho dxdt \\ = & \frac{\xi}{2}\|z_{0n}\|^2_{L^2((0,l)\times(0,1))}. \end{aligned} \tag{8}$$

Handling the second term in the left-hand side of (8), we have

$$\begin{aligned} \int_0^t\int_0^l\int_0^1 z_{n\rho}z_n(x,\rho,t)\,d\rho dxdt &= \frac{1}{2}\int_0^t\int_0^l\int_0^1 \frac{\partial}{\partial\rho}z_n^2(x,\rho,t)\,d\rho dxdt, \\ &= \frac{1}{2}\int_0^t\int_0^l\left[z_n^2(x,1,t)-z_n^2(x,0,t)\right]dxdt. \end{aligned} \tag{9}$$

Adding (7) and (8), using (9), we obtain

$$\begin{aligned} E_n(0) &= E_n(t)+\mu_1\int_0^t\|u_{nxxt}\|^2dt+\mu_2\int_0^t\int_0^l z_n(x,1,t)u_{nt}(t,x)\,dxdt \\ &+\frac{\xi}{2\tau}\int_0^t\int_0^l\left[z_n^2(x,1,t)-z_n^2(x,0,t)\right]dxdt, \end{aligned} \tag{10}$$

here

$$\begin{aligned} E_n(t) &= \frac{1}{2}\int_0^l|u_{nt}|^2dx+\frac{1}{2}\int_0^l|u_{nxxx}|^2dx+\frac{1}{2}u_{nxt}^2(l,t)+\frac{1}{2}u_{nt}^2(l,t) \\ &+\frac{1}{2}u_{nxx}^2(l,t)-\frac{1}{q}\int_0^l|u_n|^q dx+\frac{\xi}{2}\|z_n\|^2_{L^2((0,l)\times(0,1))}. \end{aligned} \tag{11}$$

Thanks to the Young inequality and (2), we obtain

$$\begin{aligned} & E_n(t)+\left(\mu_1-\frac{|\mu_2|B^2}{2}-\frac{\xi B^2}{2\tau}\right)\int_0^t\|u_{nxxt}\|^2dt \\ & +\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t\int_0^l z_n^2(x,1,t)\,dxdt \\ \leq\ & E_n(0). \end{aligned}$$

Furthermore by choosing $\tau|\mu_2| < \xi < \frac{\tau}{B^2}\left(2\mu_1-|\mu_2|B^2\right)$, we have

$$\mathcal{Z}_0=\mu_1-\frac{|\mu_2|B^2}{2}-\frac{\xi B^2}{2\tau}>0,$$

$$\mathcal{Z}_1=\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}>0.$$

Thus, we get

$$E_n(t) + \mathcal{Z}_0 \int_0^t \|u_{nxxt}\|^2 \, dt + \mathcal{Z}_1 \int_0^t \int_0^l z_n^2(x, 1, t) \, dx dt \leq E_n(0) . \tag{12}$$

Since the sequence $(u_{0n}), (v_{0n})$ and (z_{0n}) are convergent, we can find some positive constant C_* independent of n such that

$$E_n(t) \leq C_*. \tag{13}$$

Combining (11) and (13), we have

$$\begin{aligned} &(u_n) \text{ is bounded in } L^\infty(0, T; \mathcal{W}), \\ &(u_{nt}) \text{ is bounded in } L^\infty(0, T; \mathcal{W}), \\ &(z_n) \text{ is bounded in } L^\infty\big(0, T; L^2(0, l) \times (0, 1)\big). \end{aligned}$$

Therefore, we conclude that

$$\begin{array}{rcl} (u_n) & \to & u \text{ weak star in } L^\infty(0, T; \mathcal{W}) \\ (u_{nt}) & \to & u_t \text{ weak star in } L^\infty(0, T; \mathcal{W}) \\ (z_n) & \to & z, \text{ weak star in } L^\infty\big(0, T; L^2(0, l\big) \times (0, 1)). \end{array}$$

Noting that the embedding $H^3(0, l) \hookrightarrow H^2(0, l) \hookrightarrow H^1(0, l) \hookrightarrow L^2(0, l)$ are compact, from Aubin- Lions theorem (see [13]), we conclude that there exist a subsequence (u_k) of (u_n) such that

$$(u_k) \to u \text{ strongly in } L^2(0, T; H^1(0, l)).$$

Thus, we have

$$(u_k) \to u \text{ strongly and a.e. on } (0, l) \times (0, T).$$

We complate the proof. □

4 Open problems

In the present work, we proved the local existence of solutions for problem (1) using the Faedo-Galerkin method. Under what conditions can the local solution be extended to a global solution? Moreover, features of the problem such as energy decay and attractors can be studied.

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