Int. J. Open Problems Compt. Math., Vol. 18, No. 4, December 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376

On the involutes of a regular curve according to lightcone frame

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Received 10 July 2025; Accepted 22 October 2025

Abstract

In this study, we introduce a new definition of the involute curve $\widetilde{\gamma}$ of a regular curve γ in Minkowski 3-space using the lightcone frame $\{L_\theta^+, L_\theta^-, M_\theta\}$. First, we determine the distance between the involute-evolute pair $(\gamma, \widetilde{\gamma})$ with respect to the lightcone frame. Then, we derive the Frenet apparatus of the involute curve $\widetilde{\gamma}$ and establish the necessary condition for $\widetilde{\gamma}$ to be the involute of γ. Finally, to enhance understanding, we illustrate the proposed method with a representative example.

Keywords: *Lightcone frame, mixed-type curves, involute.*
2010 Mathematics Subject Classification: 51A35, 53B30.

1 Introduction

In mathematical physics, Minkowski space is considered the most suitable mathematical framework for modeling Einstein's special theory of relativity. In this space, the spacetime interval between two events can be spacelike, lightlike (null), or timelike. Depending on the causal character of a curve in Minkowski space, a non-lightlike curve is usually studied using the Frenet frame, while a lightlike curve is analyzed with the Cartan frame. However, these classical frames are not sufficient for curves that contain both lightlike and non-lightlike points. Recently, developing new frames for such mixed-type curves has attracted significant interest among researchers. In this context, Izumiya et al. [1] introduced the lightcone frame for curves without inflection points in the Lorentz-Minkowski plane and investigated the geometric properties of their evolutes. Following this, Liu and Pei [2] defined mixed-type curves and their lightcone frames in Minkowski 3-space. For further recent studies on these new frames, see [3-7].

The involute curve, frequently defined in geometry textbooks, is the path traced by a point as a string unwinds from the original curve along its tangent lines. The original curve and its involute form what is known as an involute-evolute pair, a well-known concept in Euclidean space [8–14]. Bilici [15] introduced three types of involutes for non-lightlike curves in 3-dimensional Minkowski space (for more details, see [16–18]). The involutes of both lightlike and non-lightlike curves in Minkowski space have been studied extensively by various authors [19–26].

In this paper, inspired by the studies mentioned above, we define the involute curve $\widetilde{\gamma} = inv_L(\gamma)$ of a regular curve γ with lightcone semipolar coordinates (α, β, θ) according to lightcone frame $\left\{L_\theta^+, L_\theta^-, M_\theta\right\}$. Also we show that for $\widetilde{\gamma}$ to be the involute of γ, the condition $\alpha\beta$=constant must be satisfied. At the end of this article, we illustrate the main idea using an example.

2 Preliminaries

Let $\mathbb{R}^3$ be a vector space equipped with the inner product

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, then we call $(\mathbb{R}^3, \langle\ ,\ \rangle)$ the Minkowski 3-space and denote by $\mathbb{R}^3_1$, [27]. The Lorentzian vectorial product is defined by

$$x \times y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

A vector $x \in \mathbb{R}^3_1$ is called timelike, spacelike or lightlike (null) if

$$\begin{cases} \langle x, x\rangle < 0, \\ \langle x, x\rangle > 0 \text{ or } x = \overrightarrow{0}, \\ \langle x, x\rangle = 0, \end{cases}$$

respectively. The norm of a vector x is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ [27]. Likewise, the causal character of a curve in Minkowski 3-space can be determined by its tangent vector field.

Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a non-lightlike curve parametrized by arc-length, we suppose that $\left\langle \ddot{\gamma}, \ddot{\gamma}\right\rangle \neq 0$. We denote $\{T, N, B\}$ the moving Frenet frame along the curve γ. The vector fields, T, N and B are the tangent, the principal normal, and the binormal vector of γ, respectively. Then the following Frenet equations are satisfied

$$\begin{cases} \dot{T} = \kappa N, \\ \dot{N} = -\varepsilon_1 \varepsilon_2 \kappa T + \tau B, \\ \dot{B} = \varepsilon_1 N, \end{cases} \tag{1}$$

where $\varepsilon_1 = \langle T, T \rangle$, $\varepsilon_2 = \langle N, N \rangle$. Here κ and τ are the curvatures of γ [28]. From [25], we know that if γ be a non lightlike curve, then we can write

$$\begin{cases} T = \frac{\dot{\gamma}}{\left\|\dot{\gamma}\right\|}, N = \varepsilon_2 \left(B \wedge T\right), B = -\varepsilon_1 \varepsilon_2 \frac{\dot{\gamma} \wedge \ddot{\gamma}}{\left\|\dot{\gamma} \wedge \ddot{\gamma}\right\|}, \\ \kappa = \frac{\left\|\dot{\gamma} \wedge \ddot{\gamma}\right\|}{\left\|\dot{\gamma}\right\|^3}, \ \tau = \frac{\det\left(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}\right)}{\left\|\dot{\gamma} \wedge \ddot{\gamma}\right\|^2}. \end{cases} \tag{2}$$

Let $\gamma : I \to \mathbb{R}^3_1$ be a regular curve with Frenet frame $\{T, N, B\}$. An involute of γ is a curve that orthogonally intersects all the tangents of the base curve at the corresponding points. Then the parametrization of the involute of γ according to the Frenet frame is

$$\overline{\gamma}\left(t\right) = inv_F\left(\gamma\left(t\right)\right) = \gamma\left(t\right) + \left(c - t\right) T\left(t\right),$$

where c is a real constant and $T = \dot{\gamma}$ [15].

The definition of lightcone frame for a regular curve can be given as follows

$$\left\{L^+_\theta = \left(1, \cos\theta, \sin\theta\right), \ L^-_\theta = \left(1, -\cos\theta, -\sin\theta\right), \ M_\theta = \left(0, \sin\theta, -\cos\theta\right)\right\},$$

where $\theta : \theta\left(t\right)$ is a smooth function (Figure 1). For more details, see[2].

Also, the inner product and outer product provide the following properties.

$$\begin{aligned} \left\langle L^+_\theta, \ L^+_\theta \right\rangle &= \left\langle L^-_\theta, \ L^-_\theta \right\rangle = \left\langle L^+_\theta, \ M_\theta \right\rangle = \left\langle L^-_\theta, \ M_\theta \right\rangle = 0, \\ \left\langle L^+_\theta, \ L^-_\theta \right\rangle &= -2, \quad \left\langle M_\theta, M_\theta \right\rangle = 1. \\ L^+_\theta \wedge L^+_\theta &= L^-_\theta \wedge L^-_\theta = 0, \quad L^+_\theta \wedge L^-_\theta = 2M_\theta, \\ L^+_\theta \wedge M_\theta &= L^+_\theta, \quad L^-_\theta \wedge M_\theta = -L^-_\theta. \end{aligned}$$

Let $\gamma : \gamma\left(t\right)$ be a regular curve. There exists a smooth function

$$\left(\alpha, \beta, \theta\right) : I \to \mathbb{R}^3 / \left\{\left(0, 0, \theta\right)\right\}$$

such that

$$\dot{\gamma} = \alpha L^+_\theta + \ \beta L^-_\theta. \tag{3}$$

If above condition is satisfied for γ, then this curve is called a regular curve with *the lightcone semipolar coordinates* (l.s.c) $\left(\alpha, \beta, \theta\right)$ [2].

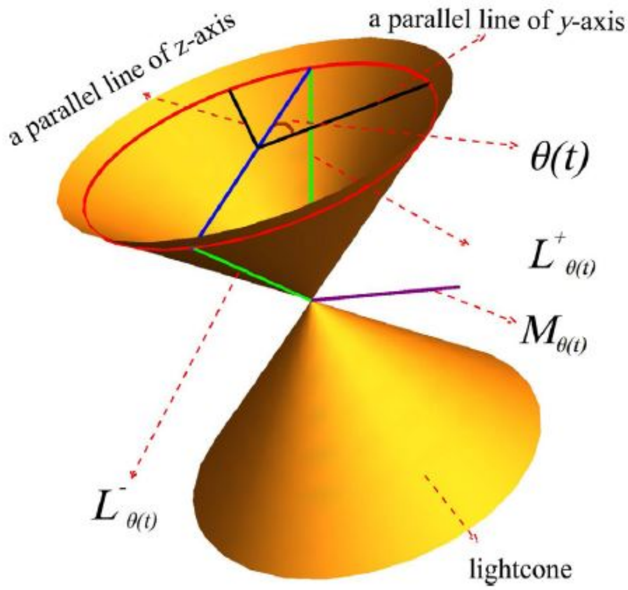

Figure 1: The lightcone frame

On the other hand, the following result was established in [2].

Theorem 2.1 *Let* $(\alpha, \beta, \theta) : I \to \mathbb{R}^3 / \{(0,0,\theta)\}$ *be a smooth function. There exists a regular curve* $\gamma : I \to \mathbb{R}^3_1$

$$\gamma = ((\alpha + \beta)\,dt, (\alpha - \beta)\cos\theta dt, (\alpha - \beta)\sin\theta dt)$$

with the (l.s.c) (α, β, θ).

3 Main results

In this section, we will define the involute of a regular curve using the lightcone frame and get the Frenet apparatus of this type of curve.

Definition 3.1 *Let* $\gamma : I \to \mathbb{R}^3_1$ *be a regular curve with (l.s.c)* (α, β, θ), *then we define the involute of* γ *according to lightcone frame by*

$$inv_L(\gamma) = \widetilde{\gamma} = \gamma + \frac{(c-t)\,\alpha}{2\,|\alpha\beta|^{\frac{1}{2}}} L^+_\theta + \frac{(c-t)\,\beta}{2\,|\alpha\beta|^{\frac{1}{2}}} L^-_\theta.$$

Since $\left\| \dot{\gamma} \right\| = 0$ when γ is a lightlike, it is probably not always possible to define an involute curve of a regular curve. We suppose γ is a unit speed curve i.e., $\left\| \dot{\gamma} \right\| = 2\,|\alpha\beta|^{\frac{1}{2}} = 1$, then we have $\alpha\beta = \pm\frac{1}{4}$. In this case, γ is a non lightlike curve and the above definition can be given by

$$inv_L(\gamma) = \widetilde{\gamma} = \gamma + (c-t)\,\alpha L^+_\theta + \ (c-t)\,\beta L^-_\theta. \tag{4}$$

Theorem 3.2 *Let $\widetilde{\gamma}$ be an involute of a regular curve γ in $\mathbb{R}^3_1$. The distance between the points $\gamma(t)$ and $\widetilde{\gamma}(t)$ is given by*

$$d(\gamma,\widetilde{\gamma}) = 2|c-t|\,|\alpha\beta|^{\frac{1}{2}}.$$

From the equation (4) we can write,

$$\widetilde{\gamma}-\gamma = (c-t)\,\alpha L^+_\theta + \ (c-t)\,\beta L^-_\theta,$$

$$\langle\widetilde{\gamma}-\gamma,\widetilde{\gamma}-\gamma\rangle = -4(c-t)^2\,\alpha\beta.$$

Using the definition of the norm, we easily find

$$d(\gamma,\widetilde{\gamma}) = 2|c-t|\,|\alpha\beta|^{\frac{1}{2}}.$$

Theorem 3.3 *Let $\widetilde{\gamma}$ be an involute of a regular curve γ in $\mathbb{R}^3_1$. The relations between the Frenet frame and the lightcone frame of $\widetilde{\gamma}$ are as follows.*

$$\begin{cases} \widetilde{T} = \frac{(c-t)}{N_1}\left(\dot{\alpha}L^+_\theta + \dot{\beta}L^-_\theta + \dot{\theta}(\beta-\alpha)M_\theta\right), \\ \widetilde{N} = \varepsilon_2\frac{(c-t)}{N_1N_2}\begin{pmatrix} \dot{\theta}\left((\beta-\alpha)-\dot{\alpha}\delta_3\right)L^+_\theta \\ \left(\dot{\beta}\delta_3-\dot{\theta}(\beta-\alpha)\delta_2\right)L^-_\theta \\ 2\left(\dot{\beta}\delta_1-\dot{\alpha}\delta_2\right)M_\theta \end{pmatrix}, \\ \widetilde{B} = -\varepsilon_1\varepsilon_2\frac{1}{N_2}\left(\delta_1L^+_\theta+\delta_2L^-_\theta+\delta_3M_\theta\right). \end{cases}$$

We can calculate the Frenet vector fields of $\widetilde{\gamma}$ according to lightcone frame as follows. From the definition of the involute curve $\widetilde{\gamma}$ we can write

$$\widetilde{\gamma} = \gamma + (c-t)\,\alpha L^+_\theta + (c-t)\,\beta L^-_\theta.$$

Let us calculate following differentiations respect to t

$$\dot{\widetilde{\gamma}} = k_1L^+_\theta + k_2L^-_\theta + k_3M_\theta, \tag{5}$$

where

$$\begin{aligned} k_1 &= (c-t)\,\dot{\alpha}, \\ k_2 &= (c-t)\,\dot{\beta}, \\ k_3 &= (c-t)\,\dot{\theta}(\beta-\alpha). \end{aligned}$$

$$\ddot{\widetilde{\gamma}} = \lambda_1L^+_\theta + \lambda_2L^-_\theta + \lambda_3M_\theta, \tag{6}$$

where

$$\begin{aligned}\lambda_1 &= -\dot{\alpha} + (c-t)\left(\ddot{\alpha} + \frac{1}{2}\dot{\theta}^2(\beta-\alpha)\right),\\ \lambda_2 &= \dot{\beta} + (c-t)\left(\ddot{\beta} + \frac{1}{2}\dot{\theta}^2(\beta-\alpha)\right),\\ \lambda_3 &= \left(\dot{\theta} + \ddot{\theta}\right)(\beta-\alpha) + 2\dot{\theta}\left(\dot{\beta} - \dot{\alpha}\right).\end{aligned}$$

$$\dddot{\widetilde{\gamma}} = \mu_1 L_\theta^+ + \mu_2 L_\theta^- + \mu_3 M_\theta, \tag{7}$$

where

$$\begin{aligned}\mu_1 &= \dot{\theta}^2\left(\beta - \alpha + \dot{\beta} - \dot{\alpha}\right) + \frac{1}{2}\dot{\theta}\ddot{\theta}(\beta-\alpha)\\ &\quad + (c-t)\left(\dddot{\alpha} + \dot{\theta}\ddot{\theta}(\beta-\alpha) + \frac{1}{2}\dot{\theta}\left(\dot{\beta}-\dot{\alpha}\right)\right),\\ \mu_2 &= \dot{\theta}^2\left(\beta - \alpha + \dot{\beta} - \dot{\alpha}\right) + \frac{1}{2}\dot{\theta}\ddot{\theta}(\beta-\alpha)\\ &\quad + (c-t)\left(\dddot{\beta} + \dot{\theta}\ddot{\theta}(\beta-\alpha) + \frac{1}{2}\dot{\theta}\left(\dot{\beta}-\dot{\alpha}\right)\right),\\ \mu_3 &= \dot{\theta}\left(2\dot{\beta} - \dot{\alpha} + (c-t+2)\left(\ddot{\beta} - \ddot{\alpha}\right)\right)\\ &\quad + \ddot{\theta}\left(3\dot{\beta} - 2\dot{\alpha} + \beta - \alpha\right) + \ddot{\theta}(\beta-\alpha).\end{aligned}$$

Direct comptation indicates

$$\begin{aligned}\left\langle \widetilde{\gamma}, \widetilde{\gamma}\right\rangle &= (c-t)^2\left(\dot{\theta}^2(\beta-\alpha)^2 - 4\dot{\alpha}\dot{\beta}\right)\\ &\Longrightarrow\ N_1 = \left\|\widetilde{\gamma}\right\| = |c-t|\left|\dot{\theta}^2(\beta-\alpha)^2 - 4\dot{\alpha}\dot{\beta}\right|^{\frac{1}{2}}.\end{aligned} \tag{8}$$

Here we note that

$$\begin{cases}\dot{\theta}^2(\beta-\alpha)^2 > 4\dot{\alpha}\dot{\beta}: \ \overline{\gamma}\ \textit{spacelike},\\ \dot{\theta}^2(\beta-\alpha)^2 > 4\dot{\alpha}\dot{\beta}: \ \overline{\gamma}\ \textit{timelike},\\ \dot{\theta}^2(\beta-\alpha)^2 = 4\dot{\alpha}\dot{\beta}: \ \overline{\gamma}\ \textit{lightlike}.\end{cases}$$

$$\widetilde{\gamma} \wedge \ddot{\widetilde{\gamma}} = \delta_1 L_\theta^+ + \delta_2 L_\theta^- + \delta_3 M_\theta,$$

where

$$\begin{aligned}\delta_1 &= k_1\lambda_3 - k_3\lambda,\\ \delta_2 &= k_3\lambda_2 - k_2\lambda_3,\\ \delta_3 &= 2\left(k_1\lambda_2 - k_2\lambda_1\right).\end{aligned}$$

$$\begin{aligned}&\Longrightarrow \quad \left\|\widetilde{\gamma}\wedge\ddot{\widetilde{\gamma}}\right\|^2 = -4\delta_1\delta_2 + 4\delta_3^2\\ &\Longrightarrow \quad N_2 = \left\|\widetilde{\gamma}\wedge\ddot{\widetilde{\gamma}}\right\| = 2\sqrt{|\delta_3^2 - \delta_1\delta_2|}, \qquad (9)\end{aligned}$$

$$\begin{aligned}\det\left(\widetilde{\gamma},\ddot{\widetilde{\gamma}},\dddot{\widetilde{\gamma}}\right) &= -\left\langle\widetilde{\gamma}\wedge\ddot{\widetilde{\gamma}},\dddot{\widetilde{\gamma}}\right\rangle\\ &= 2\left(\delta_1\mu_2 + \delta_2\mu_1\right) - \delta_3\mu_3. \qquad (10)\end{aligned}$$

Using the above equalities and equation (2) we obtain

$$\begin{aligned}\widetilde{T} &= \frac{\widetilde{\gamma}}{\left\|\widetilde{\gamma}\right\|}\\ &= \frac{(c-t)}{N_1}\left(\dot{\alpha}L_\theta^+ + \dot{\beta}L_\theta^- + \dot{\theta}\left(\beta-\alpha\right)M_\theta\right),\end{aligned}$$

$$\begin{aligned}\widetilde{B} &= -\varepsilon_1\varepsilon_2\frac{\widetilde{\gamma}\wedge\ddot{\widetilde{\gamma}}}{\left\|\widetilde{\gamma}\wedge\ddot{\widetilde{\gamma}}\right\|}\\ &= -\varepsilon_1\varepsilon_2\frac{1}{N_2}\left(\delta_1L_\theta^+ + \delta_2L_\theta^- + \delta_3M_\theta\right),\end{aligned}$$

$$\begin{aligned}\widetilde{N} &= \varepsilon_2\left(\overline{B}\wedge\overline{T}\right)\\ &= \varepsilon_2\frac{(c-t)}{N_1N_2}\begin{pmatrix}\dot{\theta}\left(\left(\beta-\alpha\right)-\dot{\alpha}\delta_3\right)L_\theta^+\\ \left(\dot{\beta}\delta_3-\dot{\theta}\left(\beta-\alpha\right)\delta_2\right)L_\theta^-\\ 2\left(\dot{\beta}\delta_1-\dot{\alpha}\delta_2\right)M_\theta\end{pmatrix}.\end{aligned}$$

Now, let us find the $\widetilde{\kappa}$ and $\widetilde{\tau}$ of the involute curve $\widetilde{\gamma}$ in terms of its (l.s.c) (α,β,θ).

Theorem 3.4 *Let $\widetilde{\gamma}$ be an involute of a regular curve γ in $\mathbb{R}^3_1$. The curvature $\widetilde{\kappa}$ and torsion $\widetilde{\tau}$ of $\widetilde{\gamma}$ is given by*

$$\widetilde{\kappa} = \frac{N_2}{N_1^3}, \quad \widetilde{\tau} = \frac{2\left(\delta_1\mu_2 + \delta_2\mu_1\right) - \delta_3\mu_3}{N_2^2}.$$

Using the equations (8), (9) and (10) into equation (2), the desired results are obtained.

Theorem 3.5 *Let γ be a regular curve with (l.s.c) (α, β, θ). If $\widetilde{\gamma}$ is the involute of γ then $\alpha\beta$=constant.*

Let $\widetilde{\gamma}$ be the involute of γ. Since the curve $\widetilde{\gamma}$ orthogonally intersects all the tangents of γ at the corresponding points we can write

$$\left\langle T, \widetilde{T} \right\rangle = 0.$$

Using (3) and Theorem 3.3 we have

$$\left\langle \alpha L_\theta^+ + \ \beta L_\theta^-, \dot{\alpha} L_\theta^+ + \dot{\beta} L_\theta^- + \dot{\theta}\left(\beta - \alpha\right) M_\theta \right\rangle = 0,$$

it follows that

$$\alpha\dot{\beta} + \beta\dot{\alpha} = 0.$$

From here the desired result can be easily obtained.

Example 3.6 *Let $\eta : I \to \mathbb{R}^3_1$ be a regular curve defined by*

$$\eta\left(t\right) = \left(\frac{17}{18}t, \frac{15}{18}\sin t, -\frac{15}{8}\cos t\right).$$

From Theorem 2.1 we can write

$$\alpha\left(t\right) = 2, \quad \beta\left(t\right) = \frac{1}{8}, \quad \theta\left(t\right) = t.$$

Using equation (4), the expression for inv $\left(\eta\left(t\right)\right) = \widetilde{\eta}\left(t\right)$ is as follows (see Figs. 2 and 3)

$$\widetilde{\eta}\left(t\right) = \left(0, \frac{15}{8}\sin t - \frac{15}{8}t\cos t, -\frac{15}{8}\cot t - \frac{15}{8}t\sin t\right).$$

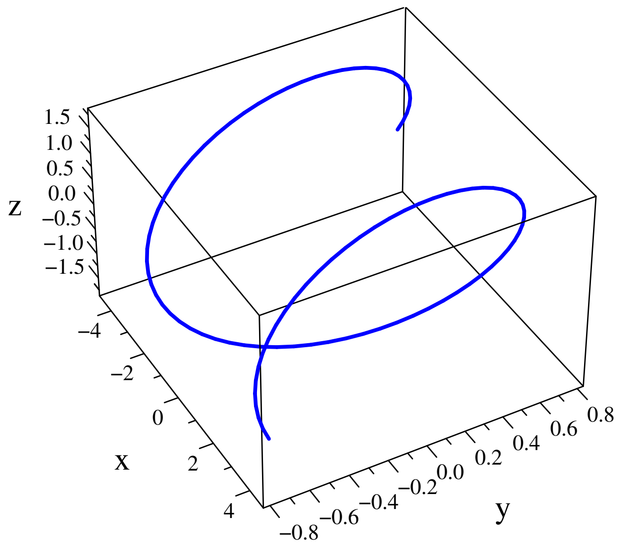

Figure 2: The regular curve η for $-5 \leq t \leq 5$

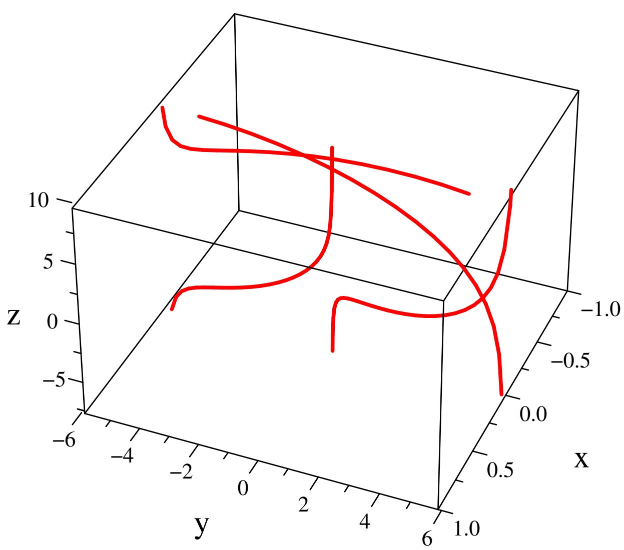

Figure 3: The involute curve $\widetilde{\eta}(t)$ for $-5 \leq t \leq 5$

4 Open Problem

The concept of the involute curve defined with respect to the lightcone frame opens several potential directions for future research. One possible extension is to generalize the present definition to higher-dimensional Minkowski spaces or the Lorentzian manifolds. Another interesting problem is to investigate the geometric properties of the surfaces generated by such involute curves. Furthermore, the study of the lightcone involute can also be carried out for other special curves such as Bertrand curves, Mannheim curves, or circular helices. Finally, it would be valuable to study possible applications of the proposed construction in relativistic kinematics or the geometry of spacetime trajectories.

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