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Solutions to two open problems in integral theory

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Abstract

Open problems play a central role in advancing mathematical research. In recent years, several such problems have been formulated in integral theory. This paper addresses two of these open problems and provides complete, detailed proofs of their solutions.

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1 Introduction

The number of known integral formulas is vast and continues to grow each year. In addition to those compiled in [1, 7], several recent contributions can be found in [8, 9, 10, 11, 4, 6, 2, 3, 5].

In addition to established results, open problems are sometimes formulated, inspiring new developments and theoretical advances. In this article, we address two such open problems– one posed in [5] and another in [3].

The first problem, formulated in [5], asserts that

$$\int_0^{\pi} \tan x \, \arctan(\cot x) \, dx = \pi \ln 2,$$

without providing a proof. We close this gap by presenting a rigorous derivation.

The second problem, proposed in [3], concerns the evaluation of the integral

$$\int_{-\infty}^{\infty} \frac{\tanh |x|}{1+x^2} \, dx.$$

Here again, we resolve this open question by employing series techniques.

The remainder of the paper is organized as follows. Section 2 presents the solution to the first open problem. Section 3 is devoted to the resolution of the second open problem. Finally, Section 4 offers concluding remarks and perspectives for future research.

2 Solution of the first open problem

The problem is precisely formulated in the following proposition, followed by a detailed proof.

Proposition 2.1. *We have*

$$\int_0^{\pi} \tan x \, \arctan(\cot x) \, dx = \pi \ln 2,$$

Proof. Let us set

$$I := \int_0^{\pi} \tan x \, \arctan(\cot x) \, dx.$$

For any $x \in (0, \pi)$, we have

$$\cot x = \tan\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \frac{\pi}{2} - x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

so

$$\arctan(\cot x) = \arctan\left(\tan\left(\frac{\pi}{2} - x\right)\right) = \frac{\pi}{2} - x.$$

Hence, we can write

$$I = \int_0^{\pi} \tan x \left(\frac{\pi}{2} - x\right) dx.$$

Since the integrand $\tan x \left(\frac{\pi}{2} - x\right)$ is continuous on $(0, \pi)$, it is integrable on $(0, \pi)$. Performing an integrate by parts with

$$u = \frac{\pi}{2} - x, \quad dv = \tan x \, dx,$$

so that $du = -dx$ and $v = -\ln|\cos x|$, we obtain

$$I = \left[-\left(\frac{\pi}{2} - x\right)\ln|\cos x|\right]_0^{\pi} - \int_0^{\pi} \ln|\cos x|\, dx.$$

The boundary term vanishes since $\ln|\cos 0| = \ln|\cos \pi| = 0$. Hence, we have

$$I = -\int_0^{\pi} \ln|\cos x|\, dx = -2\int_0^{\pi/2} \ln(\cos x)\, dx.$$

Decomposing the integral by the Chasles integral relation and using the change of variables $y = \pi - x$, we get

$$\begin{aligned} I &= -\int_0^{\pi/2} \ln(\cos x)\, dx - \int_{\pi/2}^{\pi} \ln(-\cos x)\, dx \\ &= -\int_0^{\pi/2} \ln(\cos x)\, dx - \int_0^{\pi/2} \ln(\cos y)\, dy \\ &= -2\int_0^{\pi/2} \ln(\cos x)\, dx. \end{aligned}$$

It is well-known that

$$\int_0^{\pi/2} \ln(\cos x)\, dx = -\frac{\pi}{2}\ln 2.$$

See, for instance, [7, Entry 4.224 (6), Page 531].

Therefore, we derive

$$I = -2\left(-\frac{\pi}{2}\ln 2\right) = \pi \ln 2.$$

This completes the proof. □

3 Solution of the second open problem

The problem is precisely formulated in the following proposition, followed by a detailed proof.

Proposition 3.1. *We have*

$$\int_{-\infty}^{\infty} \frac{\tanh|x|}{1+x^2}\, dx = 8\sum_{k=1}^{\infty} \frac{\ln\left(\pi^2(2k-1)^2\right) - \ln 4}{\pi^2(2k-1)^2 - 4}.$$

Proof. Using [1, Entry 1.42 (2), Page 44], for any $x \in \mathbb{R}$, we have

$$\tanh\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2} .$$

Now set $\frac{\pi x}{2} = y$, equivalently $x = \frac{2y}{\pi}$. Substituting into the previous series expansion gives

$$\tanh y = \frac{8y}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + \frac{4y^2}{\pi^2}}.$$

Replacing the letter y by x, we obtain

$$\frac{\tanh x}{1+x^2} = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{x}{\left((2k-1)^2 + \frac{4x^2}{\pi^2}\right)(1+x^2)}$$

and this is an absolute convergence.

Integrating both sides over $[0, \infty)$ yields

$$\int_0^{\infty} \frac{\tanh x}{1+x^2}\, dx = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \int_0^{\infty} \frac{x}{\left((2k-1)^2 + \frac{4x^2}{\pi^2}\right)(1+x^2)}\, dx.$$

Let us set

$$I_k := \int_0^{\infty} \frac{x}{\left((2k-1)^2 + \frac{4x^2}{\pi^2}\right)(1+x^2)}\, dx.$$

We now evaluate I_k by the change of variables $u = x^2$, so $du = 2x\, dx$ and $x\, dx = \frac{1}{2}\, du$. Denoting

$$a = (2k-1)^2, \qquad b = \frac{4}{\pi^2},$$

we obtain

$$I_k = \frac{1}{2} \int_0^{\infty} \frac{1}{(bu+a)(u+1)}\, du.$$

Performing a partial-fraction decomposition, we get

$$\frac{1}{(bu+a)(u+1)} = \frac{1}{a-b} \left[\frac{1}{u+1} - \frac{b}{bu+a}\right].$$

Therefore, we have

$$\begin{aligned}
I_k &= \frac{1}{2}\left[\frac{1}{a-b} \int_0^{\infty} \frac{du}{u+1} - \frac{b}{a-b} \int_0^{\infty} \frac{du}{bu+a}\right] \\
&= \frac{1}{2(a-b)} \lim_{m\to\infty} \left[\ln(u+1) - \ln(bu+a)\right]_0^m \\
&= \frac{1}{2(a-b)} \lim_{m\to\infty} \left[\ln\left(\frac{m+1}{bm+a}\right) + \ln a\right] \\
&= \frac{1}{2} \frac{\ln\left(\frac{a}{b}\right)}{a-b} = \frac{\pi^2}{2} \frac{\ln\left(\pi^2(2k-1)^2\right) - \ln 4}{\pi^2(2k-1)^2 - 4}.
\end{aligned}$$

Finally, substituting back into the series, we obtain

$$\int_0^\infty \frac{\tanh x}{1+x^2}\,dx = \frac{8}{\pi^2}\sum_{k=1}^\infty I_k = \frac{8}{\pi^2}\sum_{k=1}^\infty \frac{\pi^2}{2}\,\frac{\ln\left(\pi^2(2k-1)^2\right)-\ln 4}{\pi^2(2k-1)^2-4}.$$

This simplifies to the compact form

$$\int_0^\infty \frac{\tanh x}{1+x^2}\,dx = 4\sum_{k=1}^\infty \frac{\ln\left(\pi^2(2k-1)^2\right)-\ln 4}{\pi^2(2k-1)^2-4}.$$

Finally, by symmetry, we get

$$\int_{-\infty}^\infty \frac{\tanh |x|}{1+x^2}\,dx = 2\int_0^\infty \frac{\tanh x}{1+x^2}\,dx = 8\sum_{k=1}^\infty \frac{\ln\left(\pi^2(2k-1)^2\right)-\ln 4}{\pi^2(2k-1)^2-4}.$$

This completes the proof. □

A numerical study also reveals that

$$\int_{-\infty}^\infty \frac{\tanh |x|}{1+x^2}\,dx \approx 2.09651.$$

The open problem is thus solved, both theoretically and practically.

4 Conclusion

This paper contributes to the advancement of integral theory by providing complete solutions to two open problems, one formulated in [5] and another in [3]. The proofs are presented in full detail and may serve as inspiration for addressing additional open problems, as well as for formulating new ones.

References

[1] M. Abramowitz and I.A. Stegun (eds.), Handbook of Mathematical Functions, Dover Publications, 7th Edition (2007).

[2] I. Ayoob, *On the evaluation of certain unsolved definite integrals*, European Journal of Pure and Applied Mathematics, 18, 3, (2025), 1–12.

[3] D.F.M. Barreto and C. Chesneau, *Study of a particular class of integrals*, International Journal of Open Problems in Computer Science and Mathematics, 18, 4, (2025), 118-127.

[4] C. Chesneau, *On a new one-parameter arctangent-power integral*, International Journal of Open Problems in Computer Science and Mathematics, 17, 4, (2024), 1–8.

[5] C. Chesneau, *On some trigonometric and inverse trigonometric integral dormulas*, Annals of Communications in Mathematics, 8, 3, 2025, 386-392.

[6] M. W. Coffey, *Integral and series representations of the digamma and polygamma functions*, 32, 4, (2012), 317–337.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, (2007).

[8] R. Reynolds and A. Stauffer, *A definite integral involving the logarithmic function in terms of the Lerch function*, Mathematics, 7 (2019), 1–5.

[9] R. Reynolds and A. Stauffer, *Definite integral of arctangent and polylogarithmic functions expressed as a series*, Mathematics, 7 (2019), 1–7.

[10] R. Reynolds and A. Stauffer, *Derivation of logarithmic and logarithmic hyperbolic tangent integrals expressed in terms of special functions*, Mathematics, 8 (2020), 1–6.

[11] R. Reynolds and A. Stauffer, *A quadruple definite integral expressed in terms of the Lerch function*, Symmetry, 13 (2021), 1–8.