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Study of a particular class of integrals

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Abstract

This paper investigates a specific class of integrals that produce interesting results, including evaluations involving fundamental mathematical constants. Rigorous proofs are provided, and several directions for future research are proposed.

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1 Introduction

Classical integral formulas have long played a central role in analysis and its applications. The most notable examples are collected in [7], although these are presented without proofs. In recent years, several authors have revisited and extended these results. Notable developments and related contributions can be found in [8, 9, 10, 11, 2, 5, 1], where various extensions, refinements, and applications of such integrals have been systematically explored.

Inspired by this line of research, the present study explores a new class of integrals of the form

$$\int_{-\infty}^{+\infty} \frac{f(x)}{1+x^2}\, dx,$$

where, conceptually, $f(x)$ denotes an elementary function. Despite its potential to yield closed-form evaluations involving classical mathematical constants and special functions, this class has not been the subject of systematic investigation in the literature. Our aim is to identify and rigorously prove new identities within this framework, while also drawing attention to links with existing results and suggesting ways in which it can be further generalized.

The rest of the paper can be summarized as follows: Section 2 contains the main results. An open problem is given in Section 3. Section 4 provides a conclusion.

2 Results

The results relate to the class of integrals involving odd, power, trigonometric, logarithmic and exponential functions. Each of these is the subject of a subsection below.

2.1 Odd functions

The proposition below is obtained for the odd functions, yielding an elementary integral result.

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function. Then, provided the integral exists,*

$$\int_{-\infty}^{+\infty} \frac{f(x)}{1+x^2}\, dx = 0.$$

Proof. Since f is odd, for any $x \in \mathbb{R}$, we have $f(-x) = -f(x)$. Since the kernel $(1+x^2)^{-1}$ is even, the integrand is odd and the integral over a symmetric interval vanishes whenever it converges. (A formal justification requires checking absolute or conditional convergence; standard dominated convergence or symmetric limits can be applied.) □

2.2 Power functions

The proposition below considers the square root function, a particular power function.

Proposition 2.2. *We have*

$$\int_{-\infty}^{+\infty} \frac{\sqrt{|x|}}{1+x^2}\,dx = \pi\sqrt{2}.$$

Proof. For this integral, note that we have an even function, so it can be rewritten as

$$2\int_{0}^{+\infty} \frac{\sqrt{x}}{1+x^2}\,dx.$$

Using the substitution $u = \sqrt{x}$, so that $2u\,du = dx$, we obtain

$$4\int_{0}^{+\infty} \frac{u^2}{1+u^4}\,du = 2\int_{-\infty}^{+\infty} \frac{u^2}{1+u^4}\,du.$$

Thus, using the residue theorem applied to improper integrals, we have

$$2\int_{-\infty}^{+\infty} \frac{u^2}{1+u^4}\,du = 4\pi i \operatorname{Res}\left(\frac{z^2}{1+z^4}, e^{i\pi/4}\right) + 4\pi i \operatorname{Res}\left(\frac{z^2}{1+z^4}, -e^{-i\pi/4}\right),$$

since we consider the poles in the upper half-plane. Therefore

$$\int_{-\infty}^{+\infty} \frac{\sqrt{|x|}}{1+x^2}\,dx = 4\pi i\left(\frac{e^{-i\pi/4}}{4} - \frac{e^{i\pi/4}}{4}\right) = \pi\sqrt{2}.$$

□

This result can be generalized to other power functions, but we need to adapt the gamma/beta function computations accordingly.

2.3 Trigonometric functions

The proposition below examines the application of the cosine function.

Proposition 2.3. *We have*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}.$$

Proof. A standard approach involves analyzing the following known Fourier transform:

$$\int_{-\infty}^{+\infty} \frac{e^{itx}}{1+x^2}\,dx = \pi e^{-|t|},$$

and take the real part at $t = 1$ to obtain the result.

It is also possible to use differentiation under the integral sign. Consider the function

$$f(t) = \int_{-\infty}^{+\infty} \frac{\cos(tx)}{1+x^2}\,dx.$$

Note that $f(1)$ is the result we are looking for. Moreover, since the function in the integral is even, we have

$$f(t) = 2\int_0^{+\infty} \frac{\cos(tx)}{1+x^2}\,dx = 2\int_0^{+\infty}\int_0^{+\infty} (\sin s)\cos(tx)e^{-sx}\,ds\,dx.$$

Using Fubini's theorem, we have

$$f(t) = 2\int_0^{+\infty}\int_0^{+\infty} (\sin s)\cos(tx)e^{-sx}\,dx\,ds = 2\int_0^{+\infty} \frac{x}{x^2+t^2}\sin s\,ds.$$

Performing the change of variable $s = tx$, then $ds = t\,dx$, we obtain

$$f(t) = 2\int_0^{+\infty} \frac{x\sin(tx)}{1+x^2}\,dx, \quad t \neq 0.$$

On the other hand, note that

$$f'(t) = -2\int_0^{+\infty} \frac{x\sin(tx)}{1+x^2}\,dx = -f(t).$$

Furthermore, from the first form of $f(t)$, note that

$$f(0) = 2\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \pi.$$

Thus, we have the initial value problem

$$\begin{cases} f'(t) + f(t) = 0 \\ f(0) = \pi \end{cases}$$

Therefore, $f(t) = \pi e^{-t}$ for any $t \geq 0$ and thus

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = f(1) = \frac{\pi}{e}.$$

□

2.4 Logarithmic functions

The proposition below investigates the case of the logarithmic function.

Proposition 2.4. *We have*

$$\int_{-\infty}^{+\infty} \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi\ln(4).$$

Proof. One approach uses contour integration or differentiating a parameterized integral such as

$$f(t) = \int_{-\infty}^{+\infty} \frac{\ln(1+tx^2)}{1+x^2}\,dx$$

and evaluating at $t = 1$, together with known integrals for the arctangent and logarithm. Differentiating the function, we obtain

$$f'(t) = \int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)(1+tx^2)}\,dx.$$

Using the residue theorem for improper integrals, we have

$$f'(t) = 2\pi i \left(\operatorname{Res}\left(\frac{x^2}{(1+x^2)(1+tx^2)}, i \right) + \operatorname{Res}\left(\frac{x^2}{(1+x^2)(1+tx^2)}, i/\sqrt{t} \right) \right),$$

where we consider $t \geq 0$. Thus

$$f'(t) = 2\pi i \left(-\frac{1}{2i(1-t)} + \frac{1}{2i\sqrt{t}(t-1)} \right).$$

Integrating, we obtain

$$f(t) = \pi \ln|t-1| - \pi \ln|\sqrt{t}-1| + \pi \ln|t+1| + k,$$

where k denotes a certain constant to determine. Thus, applying boundary conditions, we have $k = 0$. On the other hand, when evaluating at $t = 1$, we must take into account that the function does not exist at $t = 1$, since the obtained pole turns out to be of order two, so we can only calculate its limit. Therefore, we have

$$\lim_{t\to 1} \pi \left(\ln\left| \frac{t-1}{\sqrt{t}-1} \right| + \ln|t+1| \right) = 2\pi \ln 2 = \pi \ln 4.$$

This is obtained using L'Hôpital's rule on the first logarithm, completing the proof. □

Another result of the same type is given in the proposition below.

Proposition 2.5. *We have*

$$\int_{-\infty}^{+\infty} \frac{\ln(1+|x|)}{1+x^2}\,dx = 2C + \frac{\pi}{2}\ln 2,$$

where C denotes Catalan's constant.

Proof. A sketch of the proof is as follows: first, split the integral at zero. Then, either apply a suitable series expansion or express $\ln(1+|x|)$ as an integral involving rational functions. This allows one to relate the resulting expression to known series representations that yield Catalan's constant.

Initially, we can rewrite the integral to be in the interval $(0,1)$. First, note that the function is even and can be separated as follows:

$$2\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx + 2\int_1^{+\infty} \frac{\ln(1+x)}{1+x^2}\,dx.$$

Performing the change of variable $x = 1/y$ for the second integral and combining the integrals, we have

$$2\int_0^1 \frac{2\ln(1+x) - \ln x}{1+x^2}\,dx.$$

Using the geometric series on the interval $(0,1)$ and taking into account the uniform convergence, we obtain

$$4\sum_{n=0}^{+\infty}(-1)^n\int_0^1 \ln(1+x)x^{2n}\,dx - \sum_{n=0}^{+\infty}(-1)^n\int_0^1 (\ln x)x^{2n}\,dx.$$

Performing integration by parts on both integrals and using the fact that polynomials decay faster to zero than a logarithm, we then have

$$4\ln 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1} - 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{x^{2n+1}}{1+x}\,dx + 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}\,dx.$$

Naturally, from the Fourier expansion of $f(t) = t$ on the interval $(-\pi,\pi)$, we obtain

$$t = \sum_{n=1}^{+\infty}\frac{2(-1)^{n+1}}{n}\sin(nt).$$

Thus, evaluating at $t = \pi/2$ and using the odd terms in the series, since they are the ones that do not vanish, we have

$$\frac{\pi}{4} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{(2n-1)} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)}.$$

Furthermore, using the fact that

$$C = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2},$$

we continue the result as

$$\pi \ln 2 - 4 \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{x^{2n+1}}{1+x}\, dx + 2C.$$

On the other hand, note that

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \int_0^{\pi/4} \ln(1+\tan t)\, dt = \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)\, dt.$$

Then, using angle subtraction formula for tangent, we have

$$\int_0^{\pi/4} \ln(1+\tan t)\, dt = \int_0^{\pi/4} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)\, dt.$$

Therefore, we obtain

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \int_0^{\pi/4} \ln(1+\tan t)\, dt = \frac{\pi}{8} \ln 2.$$

In this way, we complete the proof, since

$$2\int_0^1 \frac{2\ln(1+x) - \ln x}{1+x^2}\, dx = \frac{\pi}{2} \ln 2 + 2C.$$

□

Note that additionally, we obtained the following interesting result

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{x^{2n+1}}{1+x}\, dx = \frac{\pi}{8} \ln 2.$$

2.5 Exponential functions

The proposition below considers the exponential function.

Proposition 2.6. *We have*

$$\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1+x^2}\, dx = e\pi \operatorname{erfc}(1).$$

Proof. We may use convolution representation or express as integral involving the complementary error function. For instance, write

$$\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1+x^2}\, dx = \int_0^{+\infty} e^{-t} \left(\int_{-\infty}^{+\infty} e^{-x^2} e^{-tx^2}\, dx\right)\, dt$$

and evaluate inner Gaussian integrals, then exchange integrals and integrate in t, which leads to expressions in terms of erfc(1). In this case, we will use the substitution $x\sqrt{1+t} = u$, so $dx = du/\sqrt{1+t}$. On the other hand, the function to integrate is also even. Substituting, we obtain

$$2\int_0^{+\infty} e^{-t} \left(\int_0^{+\infty} \frac{e^{-u^2}}{\sqrt{1+t}}\, du \right) dt.$$

Solving the Gaussian integral, we get

$$\int_0^{+\infty} e^{-t} \frac{\sqrt{\pi}}{\sqrt{1+t}}\; dt.$$

Now, performing the substitution $1 + t = v^2$, then $dt = 2v\, dv$, we obtain

$$\pi \left(\frac{2}{\sqrt{\pi}} \int_1^{+\infty} e^{-v^2}\, dv \right) e.$$

Or using the definition $\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^{+\infty} e^{-t^2}\, dt$, we have

$$\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1+x^2}\, dx = e\pi \operatorname{erfc}(1).$$

□

3 Open problems

Open problems include some unconsidered integrals of our class, especially those defined with trigonometric and hypergeometric functions. One may think of

$$\int_{-\infty}^{+\infty} \frac{\sin(|x|)}{1+x^2}\, dx$$

and

$$\int_{-\infty}^{+\infty} \frac{\tanh(|x|)}{1+x^2}\, dx.$$

For the first, we conjecture that

$$\int_{-\infty}^{+\infty} \frac{\sin(|x|)}{1+x^2}\, dx = e^{-1}E_i(1) - eE_i(1),$$

where E_i denotes the standard exponential integral function. However, a complete and rigorous proof of this identity remains to be established.

Regarding the second integral, we can verify that it converges, but a full proof of its exact value is still an open problem.

4 Conclusion and perspectives

In this paper, we have compiled and established several integral identities of the form

$$\int_{-\infty}^{+\infty} \frac{f(x)}{1+x^2}\, dx,$$

including evaluations that involve classical mathematical constants and special functions. Future work may explore multi-parameter generalizations and investigate potential connections with representations involving the digamma and polygamma functions.

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