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New Look of Trigonometric and Hyperbolic Inequalities

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Abstract

In this paper, we introduce a new approach for families to parameter-dependent inequalities. Depending on the values of the different parameters these inequalities are extending the classical ones.

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1 Introduction

Let us consider the following trigonometric functions

$$\frac{\sin x}{x}, \quad \frac{\tan x}{x}, \quad \cos x.$$

when x tends to 0 these functions tend to 1 . Therefore, for any real a, b, c one gets

$$a\frac{\sin x}{x} + b\frac{\tan x}{x} + c\cos x - a - b - c,$$

approaches 0 when x tends to 0 . More precisely,

Proposition 1.1 *For a, b, c real positive numbers such that $c \leq \frac{2b-a}{3}$ and for $x \in \left(0, \frac{\pi}{2}\right)$ the following inequality holds*

$$a\frac{\sin x}{x} + b\frac{\tan x}{x} + c\cos x \geq a + b + c.$$

Moreover, one have

$$lim_{x\to 0}\frac{a\frac{\sin x}{x}+b\frac{\tan x}{x}+c\cos x-a-b-c}{x^2}=\frac{a-2b+3c}{6}.$$

Proof Indeed, for $x\in\left(0,\frac{\pi}{2}\right)$ one gets

$$\frac{\sin x}{x}>1-\frac{x^2}{6},\quad \frac{\tan x}{x}>1+\frac{x^2}{3},\quad \cos x>1-\frac{x^2}{2}.$$

We then deduce

$$a\frac{\sin x}{x}+b\frac{\tan x}{x}+c\cos x>a+b+c-\frac{x^2}{6}(a-2b+3c)>a+b+c$$

since $c\le\frac{2b-a}{3}$ More generally, one proves the following for any real numbers p,q,r

Proposition 1.2 *For a,b,c,p,q,r real numbers and a,b,c positive such that $c\le\frac{|2bq-ap|}{3r}$ and for $x\in\left(0,\frac{\pi}{2}\right)$ the following inequality holds*

$$a\left(\frac{\sin x}{x}\right)^p+b\left(\frac{\tan x}{x}\right)^q+c(\cos x)^r\ge a+b+c.$$

Moreover, one have

$$lim_{x\to 0}\frac{a\left(\frac{\sin x}{x}\right)^p+b\left(\frac{\tan x}{x}\right)^q+c(\cos x)^r-a-b-c}{x^2}=\frac{(|2bq-ap|-3rc)}{6}.$$

Proposition 1.2 reduces to Proposition 1.1 for $p=q=r=1$

Proof Indeed, for $x\in\left(0,\frac{\pi}{2}\right)$ one gets

$$\left(\frac{\sin x}{x}\right)^p>\left(1-\frac{x^2}{6}\right)^p>\left(1-\frac{px^2}{6}\right),\quad \left(\frac{\tan x}{x}\right)^q>\left(1+\frac{x^2}{3}\right)^q>$$

$$\left(1+\frac{qx^2}{3}\right),\quad (\cos x)^r>\left(1-\frac{x^2}{2}\right)^r>\left(1-\frac{rx^2}{2}\right)$$

Then we derive for $p,q>0$

$$a\left(\frac{\sin x}{x}\right)^p+b\left(\frac{\tan x}{x}\right)^q+c(\cos x)^r\ge a\left(1-\frac{px^2}{6}\right)+b\left(1+\frac{qx^2}{3}\right)+c\left(1-\frac{rx^2}{2}\right)=$$

$$a+b+c+(2bq-ap-3rc)\frac{x^2}{6}>a+b+c$$

For $p, q < 0$ one has

$$a\left(\frac{\sin x}{x}\right)^p+b\left(\frac{\tan x}{x}\right)^q+c(\cos x)^r \geq a\left(1-\frac{px^2}{6}\right)+b\left(1+\frac{qx^2}{3}\right)+c\left(1-\frac{rx^2}{2}\right)=$$

$$a+b+c+(-2bq+ap-3rc)\frac{x^2}{6} > a+b+c$$

For the hyperbolic counter part, and by the same way we easily prove

Proposition 1.3 *For a, b, c, p, q, r real numbers and a, b, c, r positive such that $c \geq \frac{|2bq-ap|}{3r}$. and for $x \in (0, \infty)$ the following inequality holds*

$$a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + c(\cosh x)^r \geq a+b+c.$$

Moreover, one have

$$lim_{x\to 0}\frac{a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + c(\cosh x)^r - a - b - c}{x^2} = \frac{(3rc-|2bq-ap|)}{6}.$$

Proof Indeed, for $x \in (0, \infty)$ one gets

$$\left(\frac{\sinh x}{x}\right)^p > \left(1+\frac{x^2}{6}\right)^p > \left(1+\frac{px^2}{6}\right), \quad \left(\frac{\tanh x}{x}\right)^q > \left(1-\frac{x^2}{3}\right)^q >$$

$$\left(1-\frac{qx^2}{3}\right), \quad (\cosh x)^r > \left(1+\frac{x^2}{2}\right)^r > \left(1+\frac{rx^2}{2}\right)$$

Then we derive for $p, q > 0$

$$a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + c(\cosh x)^r \geq$$

$$a\left(1+\frac{px^2}{6}\right)+b\left(1-\frac{qx^2}{3}\right)+c\left(1+\frac{rx^2}{2}\right)=a+b+c+(-2bq+ap+3rc)\frac{x^2}{6} > a+b+c$$

For $p, q < 0$ one has

$$a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + c(\cosh x)^r \geq$$

$$a\left(1+\frac{px^2}{6}\right)+b\left(1-\frac{qx^2}{3}\right)+c\left(1+\frac{rx^2}{2}\right)=a+b+c+(2bq-ap+3rc)\frac{x^2}{6} > a+b+c$$

The interest of the preceding remarks is to give a more general framework allowing the insertion of classical inequalities both in trigonometric and hyperbolic cases. Propositions 1.2 and 1.3 allows us in particular to derive some known inequalities. Indeed, for $a = b = c = 1$ and $p = 2, q = 1, r = 0$ one finds again the Wilker inequality. For $p = -2, q = -1, r = 0$ one finds the second Wilker inequality. For $p = 3, q = 0, r = 1$ one finds the Cusa inequality (and the Lazarewic inequality for hyperbolic case).

The aim of the present paper is precisely to highlight for different real values of p, q, r, a, b, c intrinsic properties of new inequalities making it possible to generalize those well known of classical ones. Moreover, we will prove for certain values of the parameters the following sharpened inequalities

$$\beta x^d < a\left(\frac{\sin x}{x}\right)^p + b\left(\frac{\tan x}{x}\right)^q + c(\cos x)^r - a - b - c < \alpha x^d,$$

$$\beta x^d < a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + c(\cosh x)^r - a - b - c < \alpha x^d,$$

where α, β are real and d is even integer such that $2 \leq d \leq 10$.

The first interesting case to study occurs when

$$c = \frac{2bq - ap}{3r}$$

It corresponds to following functions which are depending of 5 parameters

$$a\left(\frac{\sin x}{x}\right)^p + b\left(\frac{\tan x}{x}\right)^q + \left(\frac{2bq - ap}{3r}\right)(\cos x)^r - a - b - \frac{2bq - ap}{3r}, \qquad (1)$$

that we propose to minimize and so to derive other interesting inequalities. In fact, for $a \neq 0$ that depends only of 4 parameters (in taking $a = 1$)

$$u_t(p, q, r, b, x) = \left(\frac{\sin x}{x}\right)^p + b\left(\frac{\tan x}{x}\right)^q + \left(\frac{2bq - p}{3r}\right)(\cos x)^r - 1 - b - \frac{2bq - p}{3r}, \qquad (2)$$

For the hyperbolic counterpart, one gets the function

$$u_h(p, q, r, b, x) = \left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + \left(\frac{2bq - p}{3r}\right)(\cosh x)^r - 1 - b - \frac{2bq - p}{3r}. \qquad (3)$$

The case $a = 0$ will be treated separately in part 5.

By Propositions 1.2 and 1.3 these functions above are positive. Moreover, we easily verify that

$$lim_{x \to 0} \frac{u_t(p, q, r, b, x)}{x^4} < \infty, \quad lim_{x \to 0} \frac{u_h(p, q, r, b, x)}{x^4} < \infty.$$

In [5] and [6] some particular cases have been developed.

Recall at first, for $p = n, q = 1, r = 0$ one proved ([5], Theorem 2.1, [6], Theorem 2.1)

Theorem A *Let us consider the functions*

$$f_t(n,x) = \left(\frac{\sin x}{x}\right)^n + \frac{n\tan x}{2x} - \frac{n+2}{2}.$$

Then for every integer $n \geq 2$ and for $x \in \left(0, \frac{\pi}{2}\right)$, $f_t(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold

$$\frac{n}{n-1} f_t(n-1,x) < f_t(n,x) < \frac{n(22+5n)}{(n-1)(5n+17)} f_t(n-1,x) < \ldots < \frac{74}{64} f_t(2,x).$$

Theorem A1 *Let us consider the function*

$$f_h(n,x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n+2}{2}$$

Then for every integer $n \geq 2$ and for $x \in (0,\infty)$, $f_h(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold:

$$\ldots > \frac{f_h(n,x)}{n(22+5n)} > \frac{f_h(n-1,x)}{(n-1)(5n+17)} > \frac{f_h(n-2,x)}{(n-2)(5n+12)} > \ldots > \frac{f_h(2,x)}{64}.$$

For $p = -n, q = -1, r = 0$ one proved ([5], Theorem 2.7, [6], Theorem 2.6)

Theorem B *Let us consider the function*

$$g_t(n,x) = \left(\frac{x}{\sin x}\right)^n + \frac{nx}{2\tan x} - \frac{n+2}{2}$$

Then for every integer $n \geq 3$ and for $x \in \left(0, \frac{\pi}{2}\right)$, $g_t(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold

$$\frac{2n}{n-1} g_t(n-1,x) > g_t(n,x) > \frac{n(5n-2)}{(n-1)(5n-7)} g_t(n-1,x) > \ldots > \frac{24}{13} g_t(3,x).$$

Theorem B1 Let us consider the function

$$g_h(n,x) = \left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n+2}{2}$$

Then for every integer $n \geq 3$ and for $x \in (0,\infty)$, $g_h(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold the following inequalities hold

$$\ldots < \frac{g(n,x)}{n(5n-2)} < \frac{g(n-1,x)}{(n-1)(5n-7)} < \frac{g(n-2,x)}{(n-2)(5n-12)} < \ldots < \frac{g(3,x)}{39}$$

For $p = n, q = 0, r = 3$ one proved ([5], Theorem 2.11, [6], Theorem 2.10)

Theorem C *let the function*

$$h_t(n,x) = \left(\frac{\sin x}{x}\right)^n - \frac{n\cos x}{3} + \frac{n-3}{3}.$$

Then for every integer $n \geq 3$ and for $x \in \left(0, \frac{\pi}{2}\right)$, $h_t(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold

$$\frac{n}{n-1}h_t(n-1,x) < h_t(n,x) < \frac{n(5n-7)}{(n-1)(5n-12)}h_t(n-1,x) < \ldots < \frac{13}{6}h_t(3,x).$$

Theorem C1 *Let us consider the function*

$$h_h(n,x) = \left(\frac{\sinh x}{x}\right)^n - \frac{n\cosh x}{3} + \frac{n-3}{3}.$$

Then for every integer $n \geq 3$ and for $x \in (0,\infty)$, $h_h(n,x)$ is increasing function with respect to n. Moreover, the following inequalities hold:

$$\frac{h_h(n,x)}{n(5n-7)} > \frac{h_h(n-1,x)}{(n-1)(5n-12)} > \frac{h_h(n-2,x)}{(n-2)(5n-17)} > \ldots \frac{h_h(3,x)}{24}.$$

Remarks Notice that also one can deduce the following limits

$$lim_{x\to 0}\frac{f_t(n,x)}{x^4} = \frac{f_h(n,x)}{x^4} = \frac{n^2}{72} + \frac{11n}{180},$$

$$lim_{x\to 0}\frac{g_t(n,x)}{x^4} = \frac{g_h(n,x)}{x^4} = \frac{n^2}{72} - \frac{n}{180},$$

$$lim_{x\to 0}\frac{h_t(n,x)}{x^4} = \frac{h_h(n,x)}{x^4} = \frac{n^2}{72} - \frac{7n}{360}.$$

One of the goals we propose to achieve is to is to extend the preceding results to more general cases using (1) and (2). The paper is organized as follows. In Parts 2, 3, 4 we provide improvements of the above Theorems $A, A1$, B, B1, C, C1. In Part 5, we provide generalizations, in particular of these of Cusa and Lazarevic inequalities. Open problems and perspectives will be discussed in Part 7 in order to discover other properties and consider future improvements. All the proofs are exposed in Part 6.

2 Generalized Wilker inequalities

That corresponds to

$$c = 0 \qquad or \quad b = \frac{ap}{2q}, \quad p, q > 0$$

Expression (1) becomes

$$a\left(\frac{\sin x}{x}\right)^p + \frac{ap}{2q}\left(\frac{\tan x}{x}\right)^q - a - \frac{ap}{2q},$$

or by simplifying by $a \neq 0$

$$\left(\frac{\sin x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tan x}{x}\right)^q - 1 - \frac{p}{2q}, \quad p, q > 0.$$

For the hyperbolic case, after simplifying by $a \neq 0$ expression (2) becomes

$$\left(\frac{\sinh x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tanh x}{x}\right)^q - 1 - \frac{p}{2q}.$$

We propose to prove the following which are an extension of Theorems A and A1

Theorem 2.1 *Let us consider the function*

$$f_t(p, q, x) = \left(\frac{\sin x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tan x}{x}\right)^q - 1 - \frac{p}{2q}.$$

Then, for $p \geq 1, q \geq 1$ *and for* $x \in \left(0, \frac{\pi}{2}\right)$ *one has* $f_t(p, q, x) > 0$. *Moreover, the following inequalities hold:*

$$(i) \qquad \frac{p}{p-1} f_t(p-1, q, x) < f_t(p, q, x),$$

$$(ii) \qquad q f_t(p, q, x) > (q-1) f_t(p, q-1, x).$$

Theorem 2.2 *Let us consider the function*

$$f_h(p, q, x) = \left(\frac{\sinh x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tanh x}{x}\right)^q - 1 - \frac{p}{2q}.$$

Then, for $p \geq 1, q \geq 1$ *and for* $x \in (0, \infty)$ *one has* $f_h(p, q, x) > 0$. *Moreover, the following inequalities hold:*

$$(i) \qquad \frac{p}{p-1} f_h(p-1, q, x) < f_h(p, q, x).$$

$$(ii) \qquad q f_h(p, q, x) > (q-1) f_t(p, q-1, x).$$

3 Generalized second Wilker inequalities

That corresponds to

$$c = 0 \quad or \quad b = \frac{ap}{2q}, \quad p, q < 0$$

Expression (1) becomes

$$a\left(\frac{\sin x}{x}\right)^p + \frac{ap}{2q}\left(\frac{\tan x}{x}\right)^q - a - \frac{ap}{2q}$$

or by simplifying by $a \neq 0$

$$\left(\frac{\sin x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tan x}{x}\right)^q - 1 - \frac{p}{2q}, \quad p, q < 0$$

For the hyperbolic case, after simplifying by $a \neq 0$ expression (2) becomes

$$\left(\frac{\sinh x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tanh x}{x}\right)^q - 1 - \frac{p}{2q}, \quad p, q < 0$$

We propose to prove the following which extend Theorems B and B1

Theorem 3.1 *Let us consider the function*

$$g_t(p, q, x) = \left(\frac{\sin x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tan x}{x}\right)^q - 1 - \frac{p}{2q}, \quad p, q < -1$$

Then, for $x \in \left(0, \frac{\pi}{2}\right)$, $g_t(p, q, x) > 0$. *Moreover, the following inequalities hold:*

$$(i) \qquad \frac{p}{p-1} g_t(p-1, q, x) < g_t(p, q, x),$$

$$(ii) \qquad q g_t(p, q, x) > (q-1) g_t(p, q-1, x).$$

Theorem 3.2 *Let us consider the function*

$$g_h(p, q, x) = \left(\frac{\sinh x}{x}\right)^p + \frac{p}{2q}\left(\frac{\tanh x}{x}\right)^q - 1 - \frac{p}{2q} \quad p, q < 0..$$

Then, for $x \in (0, \infty)$, $g_h(p, q, x) > 0$. *Moreover, the following inequalities hold:*

$$(i) \qquad \frac{p}{p-1} g_h(p-1, q, x) > g_h(p, q, x),$$

$$(ii) \qquad q g_h(p, q, x) > (q-1) g_h(p, q-1, x).$$

4 Generalized Adamovic and Lazarevic inequalities

That corresponds

$$c \le 0 \quad and \quad b = 0, \quad p, r > 0$$

Expression (1) becomes

$$a\left(\frac{\sin x}{x}\right)^p - \left(\frac{ap}{3r}\right)(\cos x)^r - a - \frac{-ap}{3r},$$

or by simplifying by $a \neq 0$

$$\left(\frac{\sin x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cos x)^r - 1 + \frac{p}{3r}.$$

For the hyperbolic case, after simplifying by $a \neq 0$ expression (2) becomes

$$\left(\frac{\sinh x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cosh x)^r - 1 + \frac{p}{3r}.$$

The case $p = 3, r = 1$ reduces to the classical Adamowic and Lazarewic inequalities.

We propose to prove the following which extend Theorems C and C1

Theorem 4.1 *Let us consider the function defined for* $x \in \left(0, \frac{\pi}{2}\right)$ *and* $p, r \ge 0$

$$h_t(p, r, x) = \left(\frac{\sin x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cos x)^r - 1 + \frac{p}{3r}$$

Then, for every integer $p \ge 3r - \frac{8}{5}$ *one has* $h_t(p, r, x) > 0$*. Moreover, the following inequalities hold*

$$(i) \qquad h_t(p, r, x) < h_t(p, r-1, x),$$

$$(ii) \qquad \frac{h_t(p, r, x)}{p} > \frac{h_t(p-1, r, x)}{p-1},$$

Theorem 4.2 *Let us consider the function defined for* $x \in (0, \infty)$ *and*

$$h_h(p, r, x) = \left(\frac{\sinh x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cosh x)^r - 1 + \frac{p}{3r}.$$

Then, for every integer $p \ge 3r - \frac{8}{5}$ *one has* $h_h(p, r, x) > 0$*. Moreover, the following inequalities hold*

$$(i) \qquad h_t(p, r, x) < h_t(p, r-1, x),$$

$$(ii) \qquad \frac{h_t(p, r, x)}{p} > \frac{h_t(p-1, r, x)}{p-1},$$

5 Other remarkable inequalities

Recall for p, q, r natural numbers and a, b positive such that $\frac{2bq-ap}{3r} > 0$ the following expressions

$$a\left(\frac{\sin x}{x}\right)^p + b\left(\frac{\tan x}{x}\right)^q + \left(\frac{2bq-ap}{3r}\right)(\cos x)^r - a - b - \frac{2bq-ap}{3r} \quad (1)$$

$$a\left(\frac{\sinh x}{x}\right)^p + b\left(\frac{\tanh x}{x}\right)^q + \left(\frac{2bq-ap}{3r}\right)(\cosh x)^r - a - b - \frac{2bq-ap}{3r} \quad (2)$$

5.1 Generalized Cusa inequalities

(i) Consider the following particular case

$$a = 0, \quad b = 1, \quad r = -\frac{2q}{3}$$

Then from expression (1) we find again to Adamovic inequality for $q \geq 0$

$$\left(\frac{\tan x}{x}\right)^q - (\cos x)^{\frac{-2q}{3}} = (\cos x)^{-q}\left[\left(\frac{\sin x}{x}\right)^q - (\cos x)^{\frac{q}{3}}\right] > 0$$

likewise from expression (2) we find again Lazarevic inequality

$$\left(\frac{\tanh x}{x}\right)^q - (\cosh x)^{\frac{-2q}{3}} = (\cosh x)^{-q}\left[\left(\frac{\sinh x}{x}\right)^q - (\cosh x)^{\frac{q}{3}}\right] > 0.$$

(ii) Consider also the particular case

$$a = 0, \qquad b = 1, \qquad q = 1, \qquad r = 1.$$

Then expression (1) becomes

$$\left(\frac{\tan x}{x}\right) - \frac{2}{3\cos x} - \frac{1}{3}.$$

That corresponds to Cusa inequality

$$\cos x\left(\left(\frac{\tan x}{x}\right) - \frac{2}{3\cos x} - \frac{1}{3}\right) = \frac{\sin x}{x} - \frac{2+\cos x}{3} < 0.$$

For the hyperbolic counterpart, one find

$$\cosh x\left(\left(\frac{\tanh x}{x}\right) - \frac{2}{3\cosh x} - \frac{1}{3}\right) = \frac{\sinh x}{x} - \frac{2+\cosh x}{3} < 0.$$

The following improve the last cases and generalize the Cusa and Lazarevic inequalities

Proposition 5.1 *For the case* $a = 0, b = 1, r = -1$ *in expression (1) and for* $x \in (0, \frac{\pi}{2})$ *the following inequalities hold*
(i) for $q = 1$ *one has*

$$\left(\frac{\tan x}{x}\right) < \frac{2}{3\cos x} + \frac{1}{3}.$$

(ii) for $q \geq 2$ *one has*

$$\frac{2q}{3\cos x} + 1 - \frac{2q}{3} < \left(\frac{\tan x}{x}\right)^q.$$

By the same way we easily prove

Proposition 5.2 *For the case* $a = 0, b = 1, r = -1$ *in expression (2) and for* $x \in (0, \infty)$ *the following inequalities hold*
(i) for $q = 1$ *one has*

$$\left(\frac{\tanh x}{x}\right) < \frac{2}{3\cosh x} + \frac{1}{3}.$$

(ii) for $q \geq 2$ *one has*

$$\frac{2q}{3\cosh x} + 1 - \frac{2q}{3} < \left(\frac{\tanh x}{x}\right)^q.$$

Theorem 5.3 *Let us consider the function*

$$u_t(q, x) = \left(\frac{\tan x}{x}\right)^q - \frac{2q}{3}\frac{1}{\cos x} - 1 + \frac{2q}{3}.$$

Then, for every integer $q > 2$ *and for* $x \in (0, \frac{\pi}{2})$, *the following inequalities hold*

$$\frac{u_t(q, x)}{q(10q - 11)} > \frac{u_t(q - 1, x)}{(q - 1)(10q - 21)}.$$

Theorem 5.4 *Let us consider the function*

$$u_h(q, x) = \left(\frac{\tanh x}{x}\right)^q - \frac{2q}{3}\frac{1}{\cosh x} - 1 + \frac{2q}{3}.$$

Then, for every integer $q > 2$ *and for* $x \in (0, \infty)$, *the following inequalities hold*

$$\frac{u_h(q, x)}{q(10q - 11)} < \frac{u_h(q - 1, x)}{(q - 1)(10q - 21)}.$$

5.2 Other interesting cases

A question that arises naturally: can we find other similar inequalities to that of Cusa for other q, r values.
Suppose now

$$c \geq 0, \qquad a = 0, \qquad and \qquad q, r > 0.$$

Expression (1) becomes

$$b\left(\frac{\tan x}{x}\right)^q + \frac{2bq}{3r}(\cos x)^r - b - \frac{2bq}{3r},$$

or by simplifying by b 6= 0

$$\left(\frac{\tan x}{x}\right)^q + \frac{2q}{3r}(\cos x)^r - 1 - \frac{2q}{3r}.$$

For the hyperbolic case, after simplifying by $b \neq 0$ expression (2) becomes

$$\left(\frac{\tanh x}{x}\right)^q + \frac{2q}{3r}(\cosh x)^r - 1 - \frac{2q}{3r}.$$

We propose now to prove the following

Theorem 5.5 *Let us consider the function*

$$u_t(q, r, x) = \left(\frac{\tan x}{x}\right)^q + \frac{2q}{3r}(\cos x)^r - 1 - \frac{2q}{3r}.$$

Then, for integers $q > 1, r > 1$ and for $x \in (0, \frac{\pi}{2})$, $u_t(q, r, x) > 0$. Moreover, the following inequalities hold

$$(i) \qquad r u_t(q, r, x) < (r-1) u_t(q, r-1, x),$$

$$(ii) \qquad \frac{u_t(q, r, x)}{q} > \frac{u_t(q-1, r, x)}{q-1}.$$

Theorem 5.6 *Let us consider the function*

$$u_h(q, r, x) = \left(\frac{\tanh x}{x}\right)^q + \frac{2q}{3r}(\cosh x)^r - 1 - \frac{2q}{3r}.$$

Then, for integers $q > 1, r > 1$ and for $x \in (0, \infty)$, $u_h(q, r, x) < 0$. Moreover, the following inequalities hold

$$(i) \qquad r u_h(q, r, x) < (r-1) u_h(q, r-1, x),$$

$$(ii) \qquad \frac{u_h(q, r, x)}{q} > \frac{u_h(q-1, r, x)}{q-1}.$$

6 Proofs

The following lemmas will be useful in the sequel

Lemma 6.1 *For $x \in (0, \frac{\pi}{2})$ and $p, q, r \geq 1$ or $p, q, r \leq -1$ or $p, q \geq 1$ the following inequalities holds*

$$(i)\quad 1-\frac{px^2}{6}+\left(-\frac{p}{180}+\frac{p^2}{72}\right)x^4+\left(-\frac{p}{2835}+\frac{p^2}{1080}-\frac{p^3}{1296}\right)x^6<\left(\frac{\sin x}{x}\right)^p<1-\frac{px^2}{6}+$$

$$\left(-\frac{p}{180}+\frac{p^2}{72}\right)x^4+\left(-\frac{p}{2835}+\frac{p^2}{1080}-\frac{p^3}{1296}\right)x^6+\left(-\frac{p}{37800}+\frac{101p^2}{1360800}-\frac{p^3}{12960}+\frac{p^4}{31104}\right)x^8,$$

$$(ii)\qquad 1+\frac{qx^2}{3}+\left(\frac{7}{90}\,q+\frac{q^2}{18}\right)x^4+\left(\frac{62}{2835}\,q+\frac{7}{270}\,q^2+\frac{1}{162}\,q^3\right)x^6<\left(\frac{\tan x}{x}\right)^q<$$

$$1+\frac{qx^2}{3}+\left(\frac{7q}{90}+\frac{q^2}{18}\right)x^4+\left(\frac{62q}{2835}+\frac{7q^2}{270}+\frac{q^3}{162}\right)x^6+\left(\frac{3509q^2}{340200}+\frac{127q}{18900}+\frac{7q^3}{1620}+\frac{q^4}{1944}\right)x^8,$$

$$(iii)\qquad 1-\frac{rx^2}{2}+\left(-\frac{r}{12}+\frac{r^2}{8}\right)x^4+\left(\frac{r^2}{24}-\frac{r}{45}-\frac{r^3}{48}\right)x^6<(\cos x)^r<$$

$$1-\frac{rx^2}{2}+\left(-\frac{r}{12}+\frac{r^2}{8}\right)x^4+\left(\frac{r^2}{24}-\frac{r}{45}-\frac{r^3}{48}\right)x^6+\left(\frac{7r^2}{480}-\frac{17r}{2520}-\frac{r^3}{96}+\frac{r^4}{384}\right)x^8,$$

$$(iv)\qquad \left(\frac{\log x}{x}\right)<-\frac{1}{6}x^2-\frac{1}{180}x^4-\frac{1}{2835}x^6-\frac{1}{37800}x^8,$$

$$(v)\quad \log\left(\frac{\sin x}{x}\right)>-\frac{1}{6}x^2-\frac{1}{180}x^4-\frac{1}{2835}x^6,\ \log\left(\frac{\tan x}{x}\right)>\frac{1}{3}x^2+\frac{7}{90}x^4+\frac{62}{2835}x^6.$$

Lemma 6.2 *For $x \in (0, \infty)$, and $p, q \geq 1$ or $p, q \leq -1$ the following inequalities holds*

$$(i)\quad 1+\frac{px^2}{6}+\left(\frac{1}{120}\,p+\frac{1}{72}\,p\,(p-1)\right)x^4<\left(\frac{\sinh x}{x}\right)^p<$$

$$1+\frac{px^2}{6}+\left(\frac{1}{120}\,p+\frac{1}{72}\,p\,(p-1)\right)x^4+\left(\frac{1}{5040}\,p+\frac{1}{720}\,p\,(p-1)+\frac{1}{1296}\,p\,(p-1)\,(p-2)\right)x^6,$$

$$(ii)\quad 1-\frac{qx^2}{3}+\left(\frac{2q}{15}+\frac{q(q-1)}{18}\right)x^4+\left(-\frac{17q}{315}-\frac{2q(q-1)}{45}-\frac{q(q-1)(q-2)}{162}\right)x^6<\left(\frac{\tanh x}{x}\right)^q,$$

$$(iii)\qquad \left(\frac{\log x}{\sinh x}\right)>-\frac{1}{6}x^2+\frac{1}{180}x^4-\frac{1}{2835}x^6+\frac{1}{37800}x^8,$$

$$(iv)\quad \log\left(\frac{\sinh x}{x}\right)<\frac{1}{6}x^2-\frac{1}{180}x^4+\frac{1}{2835}x^6,\ \log\left(\frac{\tanh x}{x}\right)<-\frac{1}{3}x^2+\frac{7}{90}x^4-\frac{62}{2835}x^6.$$

6.1 Proof of Theorem 2.1

By Lemma 6.1 one gets

$$\left(\frac{\sin x}{x}\right)^p > 1 - \frac{px^2}{6}, \qquad \left(\frac{\tan x}{x}\right)^q > 1 + \frac{qx^2}{3} + \left(\frac{7q}{90} + \frac{q^2}{18}\right)x^4,$$

we then deduce

$$f_t(p,q,x) > -\frac{px^2}{6} + \frac{p\,(1 + 1/3\,qx^2 + (2/15\,q + 1/18\,q\,(q-1))\,x^4)}{2q} - \frac{p}{2q} = \frac{1}{180}\,px^4\,(7+5\,q) > 0.$$

Consider the function $\phi(p) = \frac{f_t(p,q,x)}{p}$ and compute its derivative with respect to p

$$\phi'(p) = \left(\left(\frac{\sin(x)}{x}\right)^p \ln\left(\frac{\sin(x)}{x}\right) + 1/2\left(\frac{\tan(x)}{x}\right)^q q^{-1} - 1/2\,q^{-1}\right)p^{-1} -$$
$$\left(\left(\frac{\sin(x)}{x}\right)^p + 1/2\,p\left(\frac{\tan(x)}{x}\right)^q q^{-1} - 1 - 1/2\,\frac{p}{q}\right)p^{-2} =$$
$$p\left(\frac{\sin(x)}{x}\right)^p \ln\left(\frac{\sin(x)}{x}\right) - \left(\frac{\sin(x)}{x}\right)^p + 1 > 0,$$

since its second derivative

$$\phi''(p) = p\left(\frac{\sin(x)}{x}\right)^p \left(\ln\left(\frac{\sin(x)}{x}\right)\right)^2 > 0.$$

That means $\phi(p) > \phi(p-1)$ or equivalently

$$(p-1)f_t(p,q,x) - pf_t(p-1,q,x) > 0.$$

Inequality (i) of Theorem 2.1 is then proved. Now to prove (ii) let us consider the derivative of $qf_t(p,q,x)$ with respect to q

$$\left(\frac{\sin(x)}{x}\right)^p + 1/2\,p\left(\frac{\tan(x)}{x}\right)^q q^{-1} - 1 - 1/2\,\frac{p}{q} +$$
$$q\left(1/2\,p\left(\frac{\tan(x)}{x}\right)^q \ln\left(\frac{\tan(x)}{x}\right) q^{-1} - 1/2\,p\left(\frac{\tan(x)}{x}\right)^q q^{-2} + 1/2\,\frac{p}{q^2}\right) =$$
$$\left(\frac{\sin(x)}{x}\right)^p - 1 + 1/2\,p\left(\frac{\tan(x)}{x}\right)^q \ln\left(\frac{\tan(x)}{x}\right) >$$
$$\left(\frac{\sin(x)}{x}\right)^p - 1 + 1/2\,p\left(\frac{\tan(x)}{x}\right)^q \left(\frac{\tan(x)}{x} - 1 - 1/2\left(\frac{\tan(x)}{x} - 1\right)^2\right) >$$
$$\left(1/30\,p + \frac{1}{72}\,p^2 + 1/18\,pq\right)x^4 > 0.$$

6.2 Proof of Theorem 2.2

By Lemma 6.2 one gets

$$1+1/6\,px^2<\left(\frac{\sinh(x)}{x}\right)^p,\quad 1/2\,\frac{p}{q}-1/6\,px^2<1/2\,p\left(\frac{\tanh(x)}{x}\right)^q q^{-1},$$

then

$$f_h(p,q,x)>0.$$

Consider the function $\phi(p)=\frac{f_h(p,q,x)}{p}$ and compute its derivative with respect to p

$$\phi'(p)=-\left(\frac{\sinh(x)}{x}\right)^p p^{-2}+\left(\frac{\sinh(x)}{x}\right)^p \ln\left(\frac{\sinh(x)}{x}\right)p^{-1}+p^{-2}=$$

$$p\left(\frac{\sinh(x)}{x}\right)^p\ln\left(\frac{\sinh(x)}{x}\right)-\left(\frac{\sinh(x)}{x}\right)^p+1>0,$$

since its second derivative

$$\phi''(p)=p\left(\frac{\sinh(x)}{x}\right)^p\left(\ln\left(\frac{\sinh(x)}{x}\right)\right)^2>0.$$

That means $\phi(p)>\phi(p-1)$ or equivalently

$$(p-1)f_h(p,q,x)-pf_h(p-1,q,x)>0.$$

Inequality (i) is then proved.To prove (ii) let us consider the derivative of $qf_h(p,q,x)$ with respect to q

$$\left(\frac{\sinh(x)}{x}\right)^p+1/2\,p\left(\frac{\tanh(x)}{x}\right)^q\ln\left(\frac{\tanh(x)}{x}\right)-1>$$

$$\left(\frac{\sinh(x)}{x}\right)^p-1+1/2\,p\left(\frac{\tanh(x)}{x}\right)^q\left(\frac{\tanh(x)}{x}-1-1/2\left(\frac{\tanh(x)}{x}-1\right)^2\right)>$$

$$\left(1/30\,p+\frac{1}{72}p^2+1/18\,pq\right)x^4>0.$$

6.3 Proof of Theorem 3.1

Let us write in the sequel

$$s=-p,\quad t=-q$$

Let us recall

$$\left(\frac{x}{\sin x}\right)^s+\frac{s}{2t}\left(\frac{x}{\tan x}\right)^t-\frac{s}{2t}-1.$$

By Lemma 6.1 this function is upper than

$$> \frac{sx^2}{6} + \left(\frac{7s}{360} + \frac{s(s-1)}{72}\right)x^4 + \frac{s\left(1-\frac{tx^2}{3}\right)}{2t} - \frac{s}{2t} = \frac{sx^4}{360}(2+5s) > 0.$$

Consider the function $\phi(p) = \frac{g_t(p,q,x)}{p}$ and compute its derivative with respect to p

$$\phi'(p) = \left(\left(\frac{x}{\sin(x)}\right)^p \ln\left(\frac{x}{\sin(x)}\right) + 1/2\left(\frac{x}{\tan(x)}\right)^q q^{-1} - 1/2\,q^{-1}\right)p^{-1} -$$

$$\left(\left(\frac{x}{\sin(x)}\right)^p + 1/2\,p\left(\frac{x}{\tan(x)}\right)^q q^{-1} - 1 - 1/2\,\frac{p}{q}\right)p^{-2}.$$

The second derivative is

$$\phi''(p) = \left(\frac{x}{\sin(x)}\right)^p \left((\sin(x))^{-1} - \frac{x\cos(x)}{(\sin(x))^2}\right)\sin(x)\ln\left(\frac{x}{\sin(x)}\right)x^{-1} > 0.$$

Since $\phi'(0) = 0$ then $\phi'(p) > 0$ and $\phi(p)$ is increasing. That means $(p-1)g_t(p,q,x) - pg_t(p-1,q,x) > 0$.
To prove (ii) let us consider difference

$$qg_t(p,q,x) - (q-1)g_t(p,q-1,x) = q\left(\left(\frac{x}{\sin(x)}\right)^p + 1/2\,p\left(\frac{x}{\tan(x)}\right)^q q^{-1} - 1 - 1/2\,\frac{p}{q}\right) -$$

$$(q-1)\left(\left(\frac{x}{\sin(x)}\right)^p + 1/2\,p\left(\frac{x}{\tan(x)}\right)^{q-1}(q-1)^{-1} - 1 - 1/2\,\frac{p}{q-1}\right) =$$

$$-1/2\,p\left(\frac{x}{\tan(x)}\right)^q \tan(x)\,x^{-1} + 1/2\,p\left(\frac{x}{\tan(x)}\right)^q + \left(\frac{x}{\sin(x)}\right)^p - 1 >$$

$$\left(-\frac{11}{180}\,p + 1/18\,qp + \frac{1}{72}\,p^2\right)x^4 > 0$$

by Lemma 6.1 since $p > 1, q > 1$.

6.4 Proof of Theorem 3.2

Let us write in the sequel

$$s = -p, \quad t = -q$$

Let us recall

$$g_h(s,t,x) = \left(\frac{x}{\sinh x}\right)^s + \frac{s}{2t}\left(\frac{x}{\tanh x}\right)^t - \frac{s}{2t} - 1$$

By Lemma 6.2 this function is upper than

$$> -\frac{sx^2}{6} + \left(\frac{7s}{360} + \frac{s(s-1)}{72}\right)x^4 + \frac{s\left(1 - \frac{tx^2}{3}\right)}{2t} - \frac{s}{2t} = \frac{sx^4}{180}(-7+5t) > 0$$

for $t \geq 2$.
Consider the function $\phi(p) = \frac{g_h(p,q,x)}{p}$ and compute its derivative with respect to p

$$\phi'(p) = \left(\frac{x}{\sinh(x)}\right)^p \left(\ln\left(\frac{x}{\sinh(x)}\right)\right)^2 p^{-1} - 2\left(\frac{x}{\sinh(x)}\right)^p \ln\left(\frac{x}{\sinh(x)}\right) p^{-2} +$$

$$2\left(\frac{x}{\sinh(x)}\right)^p p^{-3} - 2\,p^{-3} =$$

$$\left(\frac{x}{\sinh(x)}\right)^p \ln\left(\frac{x}{\sinh(x)}\right)\left(\ln\left(\frac{x}{\sinh(x)}\right)p - 2\right)p^{-2} + 2\left(\frac{x}{\sinh(x)}\right)^p p^{-3} - 2\,p^{-3} < 0$$

because

$$\left(\ln\left(\frac{x}{\sinh(x)}\right)p - 2\right) < 0,\ 2\left(\frac{x}{\sinh(x)}\right)^p p^{-3} - 2\,p^{-3} < 0.$$

To prove (ii) let us consider the derivative of $\phi(q) = qg_t(p,q,x)$ with respect to q

$$\phi'(q) = q\left(\frac{x}{\sinh(x)}\right)^p \ln\left(\frac{x}{\sinh(x)}\right) + 1/2\left(\frac{x}{\tanh(x)}\right)^q - 1/2$$

as well as

$$\phi''(q) = \left(\frac{x}{\sinh(x)}\right)^p \ln\left(\frac{x}{\sinh(x)}\right) + 1/2\left(\frac{x}{\tanh(x)}\right)^q \ln\left(\frac{x}{\tanh(x)}\right) >$$

$$\left(-\frac{1}{30} + 1/36\,p + 1/18\,q\right)x^4 > 0,$$

by Lemma 7.4 since $p > 1, q > 1$.

6.5 Proof of Theorem 4.1

Let us recall

$$h_t(p,r,x) = \left(\frac{\sin x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cos x)^r - 1 + \frac{p}{3r}$$

By Lemma 6.1 this function is upper than

$$-\frac{px^2}{6} + \left(\frac{p}{120} + \frac{p(p-1)}{72}\right) x^4 - \frac{p\left(1 - \frac{r}{2}x^2 + (\frac{r}{24} + \frac{r(r-1)}{8})x^4\right)}{3r} + \frac{p}{3r} =$$

$$\frac{1}{360} px^4(8 + 5p - 15r) > 0$$

since by hypothesis $8 + 5p - 15r > 0$.

Turn now to the inequality of Theorem 4.1. Let us compute the derivative of $h_t(p, r, x)$ with respect to r and we use Lemma 7.3

$$-1/3\,\frac{p\,(\cos(x))^r \ln(\cos(x))}{r} + 1/3\,\frac{p\,(\cos(x))^r}{r^2} - 1/3\,\frac{p}{r^2} <$$

$$-1/3\,\frac{(-1/12\,p + 1/4\,pq)\,x^4}{q} + 1/3\,\frac{p}{q^2} + 1/3\,\frac{p\,(1/24\,q + 1/8\,q\,(q-1))\,x^4}{q^2} - 1/3\,\frac{p}{q^2} =$$

$$-1/3\,\frac{(-1/12\,p + 1/4\,pq)\,x^4}{q} + 1/3\,\frac{p\,(1/24\,q + 1/8\,q\,(q-1))\,x^4}{q^2} = -1/24\,px^4 < 0.$$

Thus $h_t(p, r, x) < h_t(p, r - 1, x)$.
Let us compute the derivative of $\frac{h_t(p,r,x)}{p}$ with respect to p and we use Lemma 7.3

$$-\left(\frac{\sin(x)}{x}\right)^p p^{-2} + \left(\frac{\sin(x)}{x}\right)^p \ln\left(\frac{\sin(x)}{x}\right) p^{-1} + p^{-2} >$$

$$\left(-\left(\frac{1}{120}\,p + \frac{1}{72}\,p\,(p-1)\right) p^{-2} - \frac{1}{180}\,p^{-1} + \frac{1}{36}\right) x^4 = (\frac{1}{72} x^4 > 0.$$

6.6 Proof of Theorem 4.2

Let us consider

$$h_h(p, r, x) = \left(\frac{\sinh x}{x}\right)^p - \left(\frac{p}{3r}\right)(\cosh x)^r - 1 + \frac{p}{3r}$$

By Lemma 6.2 this function is upper than

$$\frac{px^2}{6} - \left(\frac{p}{120} + \frac{p(p-1)}{72}\right) x^4 + \frac{p\left(1 - \frac{rx^2}{2} + \left(\frac{r}{24} + \frac{r(r-1)}{18}\right) x^4\right)}{3r} + \frac{p}{3r} = \frac{px^4}{360}(8 + 5p - 15r) > 0,$$

since $8 + 5p - 15r > 0$.
Turn now to the inequality of Theorem 4.2. Let us compute the derivative of $h_h(p, r, x)$ with respect to r and we use Lemma 7.4

$$-1/3\,\frac{p\,(\cosh(x))^r \ln(\cosh(x))}{r} + 1/3\,\frac{p\,(\cosh(x))^r}{r^2} - 1/3\,\frac{p}{r^2} <$$

$$-1/3\,\frac{(-1/12\,p+1/4\,pq)\,x^4}{q}+1/3\,\frac{p}{q^2}+1/3\,\frac{p\,(1/24\,q+1/8\,q\,(q-1))\,x^4}{q^2}-1/3\,\frac{p}{q^2}=$$
$$-1/3\,\frac{(-1/12\,p+1/4\,pq)\,x^4}{q}+1/3\,\frac{p\,(1/24\,q+1/8\,q\,(q-1))\,x^4}{q^2}=-1/24\,px^4<0.$$

Thus $h_h(p,r,x)<h_h(p,r-1,x)$.
Let us compute the derivative of $\frac{h_h(p,r,x)}{p}$ with respect to p and we use Lemma 7.4

$$-\left(\frac{\sinh(x)}{x}\right)^p p^{-2}+\left(\frac{\sinh(x)}{x}\right)^p\ln\left(\frac{\sinh(x)}{x}\right)p^{-1}+p^{-2}>$$
$$\left(-\left(\frac{1}{120}\,p+\frac{1}{72}\,p\,(p-1)\right)p^{-2}-\frac{1}{180}\,p^{-1}+\frac{1}{36}\right)x^4=(\frac{1}{72}x^4>0.$$

6.7 Proof of Proposition 5.1

As we have seen (i) is directly deduced from the Cusa inequality. On the other hand, the following inequality

$$\frac{1+2\cos x}{3}<\left(\frac{\sin x}{x}\right)^2$$

implies

$$\frac{4\cos x}{3}-\frac{(\cos x)^2}{3}<\frac{1+2\cos x}{3}<\left(\frac{\sin x}{x}\right)^2$$

We then deduce (ii) for $q=2$ since

$$\frac{1}{(\cos x)^2}\left[\frac{4\cos x}{3}-\frac{(\cos x)^2}{3}\right]<\frac{1}{(\cos x)^2}\left(\frac{\sin x}{x}\right)^2$$

Consider now the case $q\geq 2$. The derivative with respect to q of

$$\alpha(x,q)=\frac{3}{2q}\left(\frac{\tan x}{x}\right)^q-\frac{1}{\cos x}+1-\frac{3}{2q}$$

is

$$-\frac{3}{2q^2}\left(\frac{\sin x}{x\cos x}\right)^q+\frac{3}{2q}\left(\frac{\sin x}{x\cos x}\right)^q\ln\left(\frac{\sin x}{x\cos x}\right)+\frac{3}{2q^2}=$$
$$\left(\frac{\sin x}{x\cos x}\right)^q\left(-\frac{3}{2q^2}+\frac{3}{2q}\ln\left(\frac{\sin x}{x\cos x}\right)\right)+\frac{3}{2q^2}>$$
$$\left(1+\frac{qx^2}{3}+\left(\frac{2q}{15}+\frac{q(q-1)}{18}\right)x^4\right)\left(-\frac{3}{2q^2}+\frac{x^2}{2q}+\frac{7}{60}\frac{x^4}{q}\right)+\frac{3}{2q^2}=$$
$$\frac{1}{5400}x^4\left(450+420x^2+49x^4+150qx^2+35x^4q\right)>0$$

That means $\alpha(x,q)$ is increasing with respect to q. Then

$$\alpha(x,q)>\alpha(x,2)=\left(\frac{\tan x}{x}\right)^2-\frac{4}{3\cos x}-1+\frac{4}{3}>0.$$

6.8 Proof of Proposition 5.2

As we have seen (i) is directly deduced from the Lazarevic inequality. On the other hand, the following inequality

$$\frac{1+2\cosh x}{3} < \left(\frac{\sinh x}{x}\right)^2$$

implies

$$\frac{4\cosh x}{3} - \frac{(\cosh x)^2}{3} < \frac{1+2\cosh x}{3} < \left(\frac{\sinh x}{x}\right)^2.$$

We then deduce (ii) for $q = 2$ since

$$\frac{1}{(\cosh x)^2}\left[\frac{4\cosh x}{3} - \frac{(\cosh x)^2}{3}\right] < \frac{1}{(\cosh x)^2}\left(\frac{\sinh x}{x}\right)^2$$

Consider now the case $q \geq 2$. The derivative with respect to q of

$$\beta(x,q) = \frac{3}{2q}\left(\frac{\tanh x}{x}\right)^q - \frac{1}{\cosh x} + 1 - \frac{3}{2q}$$

is

$$-\frac{3}{2q^2}\left(\frac{\sinh x}{x\cosh x}\right)^q + \frac{3}{2q}\left(\frac{\sinh x}{x\cosh x}\right)^q \ln\left(\frac{\sinh x}{x\cosh x}\right) + \frac{3}{2q^2} =$$

$$\left(\frac{\sinh x}{x\cosh x}\right)^q\left(-\frac{3}{2q^2} + \frac{3}{2q}\ln\left(\frac{\sinh x}{x\cosh x}\right)\right) + \frac{3}{2q^2} >$$

$$\left(1 - \frac{qx^2}{3} + \left(-\frac{2q}{15} + \frac{q(q-1)}{18}\right)x^4\right)\left(-\frac{3}{2q^2} - \frac{x^2}{2q} + \frac{7}{60}\frac{x^4}{q}\right) + \frac{3}{2q^2} =$$

$$\frac{1}{5400}x^4\left(450 + 420x^2 + 49x^4 + 150qx^2 + 35x^4q\right) > 0$$

That means $\alpha(x,q)$ is increasing with respect to q. Then

$$\beta(x,q) > \beta(x,2) = \left(\frac{\tanh x}{x}\right)^2 - \frac{4}{3\cosh x} - 1 + \frac{4}{3} > 0$$

6.9 Proof of Theorem 5.3

Let us prove that the function

$$\frac{u_t(q,x)}{q(10q-11)} = \frac{\left(\frac{\tan x}{x}\right)^q - \frac{2q}{3\cos x} - 1 + \frac{2q}{3}}{q(10q-11)}$$

is increasing with respect to q. Let us compute its derivative with respect to q :

$$\left(\left(\frac{\sin x}{\cos(x)x}\right)^q \ln\left(\frac{\sin x}{x\cos x}\right) - \frac{2}{3\cos x} + \frac{2}{3}\right) q^{-1}(-11+10q)^{-1} -$$

$$\left(\left(\frac{\sin x}{x\cos x}\right)^q - \frac{2q}{3\cos x} - 1 + \frac{2}{3q}\right)(-11+10q)^{-1}q^{-2} -$$

$$10\left(\left(\frac{\sin x}{\cos xx}\right)^q - \frac{2q}{3\cos x} - 1 + \frac{2}{3}q\right) q^{-1}(-11+10q)^{-2} =$$

$$\left(\frac{\sin x}{x\cos x}\right)^q \left(q(-11+10q)\ln\left(\frac{\sin x}{x\cos x}\right) + 11 - 20q\right) q^{-2}(-11+10q)^{-2} +$$

$$\frac{20}{3}\frac{1}{\cos(x)(-11+10q)^2} - \frac{203q^2-60q+33}{q^2(-11+10q)^2}$$

By Lemma 6.1 the last expression is upper than

$$> \frac{\left(1+\frac{qx^2}{3}+\left(\frac{2q}{15}+\frac{q(q-1)}{18}\right)x^4\right)\left(q(-11+10q)\left(\frac{x^2}{3}+\frac{7x^4}{90}+\frac{62x^6}{2835}\right)+11-20q\right)}{q^2(-11+10q)^2} +$$

$$\frac{20}{3}\frac{\left(1+\frac{x^2}{2}+\frac{5}{24}x^4\right)}{(-11+10q)^2} - \frac{20q^2-60q+33}{3q^2(-11+10q)^2} =$$

$$\frac{1}{510300}\frac{\left(2205q^2x^2+620x^4q^2+9450q^2+6807qx^2+26460q+868x^4q+11160\right)x^6}{q(-11+10q)}.$$

The last expression is positive for $q \geq 2$.
Thus we deduce this derivative is positive and the function is increasing with respect to q. We then deduce that

$$\frac{u_t(q,x)}{q(10q-11)} > \frac{u_t(q-1,x)}{(q-1)(10q-21)}$$

6.10 Proof of Theorem 5.4

Let us prove that the function

$$\frac{u_h(q,x)}{q(10q-11)} = \frac{\left(\frac{\tanh x}{x}\right)^q - \frac{2q}{3\cosh x} - 1 + \frac{2q}{3}}{q(10q-11)}$$

is decreasing with respect to q. Let us compute its derivative with respect to q :

$$\frac{\left(\left(\frac{\sinh x}{x\cosh x}\right)^q \ln\left(\frac{\sinh x}{x\cosh}\right) - \frac{2}{3\cosh x} + \frac{2}{3}\right)}{q(-11+10q)} - \frac{\left(\left(\frac{\sinh x}{x\cosh x}\right)^q - \frac{2q}{3\cosh x} - 1 + \frac{2q}{3}\right)}{q^2(-11+10q)} -$$

$$10\frac{\left(\left(\frac{\sinh x}{x\cosh x}\right)^q-\frac{2q}{3\cosh x}-1+\frac{2q}{3}\right)}{q(-11+10q)^2}=$$

$$\left(\frac{\sinh x}{x\cosh x}\right)^q\frac{\left(q(-11+10q)\ln\left(\frac{\sinh(x)}{x\cosh x}\right)+11-20q\right)}{q^2(-11+10q)^2}+$$

$$\frac{20}{3}\frac{1}{\cosh x(-11+10q)^2}-\frac{20q^2-60q+33}{3q^2(-11+10q)^2}$$

By Lemma 6.2 the last expression is less than

$$<\frac{\left(1-\frac{qx^2}{3}+\left(\frac{2q}{15}+\frac{q(q-1)}{18}\right)x^4\right)\left(q(-11+10q)\left(-\frac{x^2}{3}+\frac{7x^4}{90}-\frac{62x^6}{2835}\right)+11-20q\right)}{q^2(-11+10q)^2}$$

6.11 Proof of Theorem 5.5

It is easy to see that fo $x\in(0,\frac{\pi}{2})$

$$\left(\frac{\tan(x)}{x}\right)^q-1>0,\quad \frac{2q}{3p}(1-\frac{q\,(\cos(x))^p}{p})>0,$$

that implies

$$u_t(q,r,x)=\left(\frac{\tan(x)}{x}\right)^q-2/3\,\frac{q\,(\cos(x))^p}{p}-1+2/3\,\frac{q}{p}>0.$$

To prove (i) let us derive

$$ru_t(q,r,x)=r\left(\frac{\tan(x)}{x}\right)^q-2/3\,q\,(\cos(x))^r-r+2/3\,q$$

with respect to r

$$\left(\frac{\tan(x)}{x}\right)^q-2/3\,q\,(\cos(x))^p\ln(\cos(x))-1<0.$$

This means $ru_t(q,r,x)$ is decreasing with r.
Let us consider now

$$\frac{u_t(q,r,x)}{q}=\left(\frac{\tan(x)}{x}\right)^q q^{-1}-2/3\,\frac{(\cos(x))^p}{p}-q^{-1}+2/3\,p^{-1},$$

its derivative with respect to q is

$$-\left(\frac{\tan(x)}{x}\right)^q q^{-2}+\left(\frac{\tan(x)}{x}\right)^q\ln\left(\frac{\tan(x)}{x}\right)q^{-1}+q^{-2}>$$

$$\left(-\left(2/15\,q+1/18\,q\,(q-1)\right)q^{-2}+\frac{7}{90}\,q^{-1}+\frac{1}{9}\right)x^4=\frac{1}{18}x^4>0,$$

by Lemma 6.1.

6.12 Proof of Theorem 5.6

It is easy to see that fo $x \in (0, \infty)$

$$\left(\frac{\tanh(x)}{x}\right)^q - 1 < 0, \quad \frac{2q}{3p}(1 - \frac{q\,(\cosh(x))^p}{p}) < 0,$$

that implies

$$u_h(q,r,x) = \left(\frac{\tanh(x)}{x}\right)^q - 2/3\,\frac{q\,(\cosh(x))^p}{p} - 1 + 2/3\,\frac{q}{p} < 0.$$

To prove (i) let us derive

$$ru_h(q,r,x) = r\left(\frac{\tanh(x)}{x}\right)^q - 2/3\,q\,(\cosh(x))^r - r + 2/3\,q$$

with respect to r

$$\left(\frac{\tanh(x)}{x}\right)^q - 2/3\,q\,(\cosh(x))^p \ln(\cosh(x)) - 1 < 0.$$

This means $ru_h(q,r,x)$ is decreasing with r.
Let us consider now

$$\frac{u_h(q,r,x)}{q} = \left(\frac{\tanh(x)}{x}\right)^q q^{-1} - 2/3\,\frac{(\cosh(x))^p}{p} - q^{-1} + 2/3\,p^{-1},$$

its derivative with respect to q is

$$-\left(\frac{\tanh(x)}{x}\right)^q q^{-2} + \left(\frac{\tanh(x)}{x}\right)^q \ln\left(\frac{\tanh(x)}{x}\right) q^{-1} + q^{-2} >$$

$$\left(-\left(2/15\,q + 1/18\,q\,(q-1)\right) q^{-2} + \frac{7}{90}\,q^{-1} + \frac{1}{9}\right) x^4 = \frac{1}{18}x^4 > 0,$$

by Lemma 6.2.

7 Open Problem

It is interesting that this study can be developed, allowing new horizons to be opened regarding this type of inequalities. Theorems 5.5 and 5.6 really give us hope. To that end, it would be skillful to consider inequalities in a more global approach, including (1) and (2) so that the theorems 2.1, 2.2, 3.1, 3.2, 4.1, 4.2 appear as particular cases. On the other hand, it would also be interesting to evaluate the functions considered in parts 2, 3, 4 and 5. In this sense, we can frame the following differences (in the two cases trigonometric and hyperbolic):

$$\frac{f(p,q,x)}{p(12+5p+10q)}-\frac{x^4}{360},\qquad \frac{g(p,q,x)}{p(12+5p+10q)}-\frac{x^4}{360},$$

$$\frac{h(p,q,x)}{p(12+5p+10q)}-\frac{x^4}{360},\qquad \frac{h(q,r,x)}{q(4+15r+10q)}-\frac{x^4}{180}.$$

For example can we have the following inequalities (or their converse) for certain values of p, q ?

$$-\frac{-480-588q-140q^2-42p+35p^2}{(45360)(12+5p+10q)}<\frac{f(p,q,x)}{p(12+5p+10q)}-\frac{x^4}{360}<0.$$

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