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Achromatic Numbers of n-Complete Starfish and $SF(n, 1)$ Graphs

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Abstract

The achromatic number of a graph is the largest number of colors that can be assigned to each vertex of the graph such that adjacent vertices are assigned different colors and any two different colors are assigned to some pair of adjacent vertices. In this paper, we find exact values of the achromatic numbers of n-complete starfish graphs, and bounds of the achromatic numbers of $SF(n, 1)$ graphs when n is a natural number and $n \geq 3$.

Keywords: *Chromatic numbers, Achromatic numbers, n-complete starfish graphs, $SF(n, 1)$ graphs.*
2010 Mathematics Subject Classification: 05C15.

1 Introduction

The study of graph theory originated with the publication of Euler, L. entitled "Solutio problematis and geometriamsitus pertinentis" or known as "The seven bridges of Königsberg problem" in 1736. Later in the year 1852, Guthrie, F. made a conjecture entitled "The four-color problem". This problem has been studied extensively, leading to the creation of basic concepts and definitions on graph theory. So it can be said that the starting point for the study of graph theory was this four-color problem. The definition of a complete n-coloring of

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a graph was first introduced by Harary, F. et al. [8] in 1967. Three years later, Harary, F. and Hedetniemi, S. [7] were able to find the achromatic number of the join of two graphs. They obtained that the achromatic number of the join of two graphs is the sum of their achromatic numbers. In 1998, Cairnie, N. and Edwards, K. [2] determined the achromatic number of bounded degree trees. In the next three years, MacGillivray, G. and Rodriguez, A. [9] were able to find the achromatic number of the union of paths. Furthermore, in 2013 Edwards, K. [5] gave simple necessary and sufficient conditions for a graph of maximum degree 2 to have a complete coloring with k colors, provided the graph is large enough, and use this to give the achromatic number for such a graph. In 2018, Aparna, K. M. et al. [1] determined the achromatic numbers of some special graphs such as Central graph of the Tadpole graph, Central graph of a Spider graph and Central graph of a Double Triangular Snake graph. Next year, Nithyadevi, N. and Vijayalakshmi, D. [10] obtained the achromatic numbers for Central graph of Ladder graph, Central graph of Dutch-Windmill graph, Central graph of Fan graph and Central graph of Flower graph.

Graph coloring has many practical applications like computer science, telecommunications, operation research, designs of experiments, etc. Since then, many mathematicians and graph theorists have studied on the achromatic numbers of some graphs. So, this is the reason that why we are interesting to find the achromatic numbers of some simple graphs, including n-complete starfish graphs, and $SF(n, 1)$ graphs when n is a natural number and $n \geq 3$.

2 Preliminaries

In this section, we recall some definitions and some results follow the book entitled "Introduction to Graph Theory" of Chartrand, G. and Zhang, P. [4] that will be used throughout this paper. Since our work is the study on finite simple labeled undirected graphs, that means our graphs have no loops and multiple edges, the definition of graphs is defined as the following.

A (simple, labeled, undireted) *graph* G consists of a finite nonempty set $V(G)$ of elements called *vertices* and a set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. If $e = \{u, v\}$ (for simply uv) is an edge of G, then u and v are called *adjacent* vertices. We also say u and v are joined by the edge e. The vertices u and v are referred to as *neighbors* of each other. In this case the vertex u and the edge e (as well as v and e) are said to be *incident* with each other. If distinct edges incident with a common vertex, then the edges are also called *adjacent* edges. A vertex of a graph is said to be *pendant* if it has only one neighbor. An edge of a graph is also said to be *pendant* if one of its vertices is a pendant vertex. The number of vertices and the number of edges in G is often called the *order* and the *size* of G, respectively. A graph with exactly one vertex is called a *trivial* graph, implying that a graph with order at least 2 is called a *nontrivial* graph. The number of edges incident on a vertex v in a graph G is called the *degree* of the vertex v and denoted by $\deg(v)$. The vertex of degree 1 in a graph G is called an *end* vertex of the graph G.

Next, we recall some graphs which will be referred to throughout this paper.

If the vertices of a graph G of order n, where $n \geq 3$, can be labeled as $v_1, v_2, \ldots, v_n$, so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ and v_nv_1, then G is called a *cycle* of order n and denoted by C_n. So the size of C_n is n.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n and so the size of K_n is $\binom{n}{2}$.

3 Main results

First of all, we would like to recall the definitions of a coloring and a complete coloring of a graph.

A (proper) *coloring* of a graph G is an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently. Here, we use natural numbers to represent n different colors and will say that a surjective map $f : V(G) \longrightarrow \{1, 2, \ldots, n\}$ is an *n-coloring function* of a graph G if $f(u) \neq f(v)$ for every $u, v \in V(G)$ and $uv \in E(G)$. And if there exists an n-coloring function of a graph G, we say that G has an *n-coloring*.

A *complete n-coloring* of a graph G is an n-coloring of G such that for every pair of colors there is at least one edge in G whose end vertices are colored with this pair of colors. As a consequence, if a graph G has a complete n-coloring then the size of the graph G must be at least $\binom{n}{2}$. So, if the size of a graph G is less that $\binom{n}{2}$ for some natural number n, then G does not have a complete n-coloring.

Most of mathematicians familiar with the chromatic number of a graph. The chromatic number of a graph is defined as the smallest number n such that the graph has an n-coloring. The chromatic number of a graph G is denoted by $\chi(G)$. But for the achromatic number of a graph, it is defined differently. The achromatic number of a graph is defined as the largest number n in which the graph has a complete n-coloring. The achromatic number of a graph G is denoted by $\psi(G)$. It follows that $\psi(G) \geq \chi(G)$ for every graph G and if G is a graph of order n then $\psi(G) \leq n$. Moreover, we can easily see that $\chi(K_n) = n = \psi(K_n)$.

3.1 Achromatic numbers of n-complete starfish graphs

Definition 1. *Let n be any natural number such that $n \geq 3$. An n-*complete starfish* graph, denoted by $K_n \odot K_1$, is a graph constructed by joining every* *vertex of a complete graph K_n to a corresponding pendant vertex with a pendant edge.*

The following figure shows the 5-complete starfish graph.

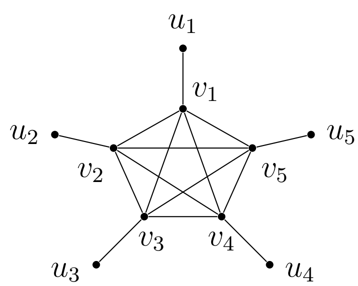

Figure 1: The 5-complete starfish graph $K_5 \odot K_1$

By Definition 1, we have that the order and the size of an n-complete starfish graph $K_n \odot K_1$ are $2n$ and $\binom{n}{2} + n = \frac{n(n+1)}{2}$, respectively.

To get the achromatic numbers of n-complete starfish graphs when n is a natural number and $n \geq 3$, we need the following lemmas.

Lemma 1. *Let n be any natural number such that $n \geq 3$. The n-complete starfish graph $K_n \odot K_1$ has a complete $(n+1)$-coloring.*

Proof. Since $\psi(K_n) = n$, K_n has a complete n-coloring. Let $f : V(K_n) \longrightarrow \{1, 2, 3, \ldots, n\}$ by $f(u_i) = i$ where $u_i \in V(K_n)$ for every $i = 1, 2, 3, \ldots, n$ be a complete n-coloring of K_n. Since the graph $K_n \odot K_1$ is constructed by joining every vertex of a complete graph K_n to a corresponding pendant vertex with a pendant edge, every pendant vertex is adjacent to a corresponding vertex of K_n and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to u_i for every $i = 1, 2, 3, \ldots, n$. We define $g : V(K_n \odot K_1) \longrightarrow \{1, 2, 3, \ldots, n+1\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n + 1$ for every $i = 1, 2, 3, \ldots, n$. It is obvious that g is a complete $(n+1)$-coloring function. Hence, the n-complete starfish graph $K_n \odot K_1$ has a complete $(n+1)$-coloring. □

Lemma 2. *Let n be any natural number such that $n \geq 3$. The n-complete starfish graph $K_n \odot K_1$ does not have a complete $(n+2)$-coloring.*

Proof. Since the size of $K_n \odot K_1$ is $\frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} < \frac{(n+2)(n+1)}{2} = \binom{n+2}{2}$, the n-complete starfish graph $K_n \odot K_1$ does not have a complete $(n+2)$-coloring. □

A consequence of Lemma 1 and Lemma 2 is the following theorem

Theorem 1. *Let n be any natural number such that $n \geq 3$. Then $\psi(K_n \odot K_1) = n + 1$.*

3.2 Bounds of achromatic numbers of $SF(n,1)$ graphs

Definition 2. *Let n be a natural number in which $n \geq 3$. A $SF(n,1)$ graph is a graph constructed by joining every vertex of a cycle graph C_n to a corresponding pendant vertex with a pendant edge.*

The following figure shows the $SF(4,1)$ graph.

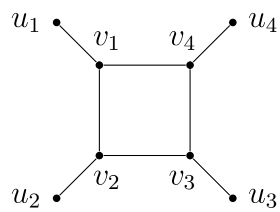

Figure 2: The $SF(4,1)$ graph

By Definition 2, we have that the order and the size of a $SF(n,1)$ graph are equal, i.e. equal to $2n$.

In 2009, Fuller, S. [6] was able to find the achromatic number of the cycle graphs $C_{\binom{n}{2}}$ for any odd number $n \geq 3$ and the achromatic number of the cycle graphs $C_{\binom{n}{2}+\frac{n}{2}}$ for any even number $n \geq 4$ as the following theorem.

Theorem 2. *Let n be any odd number such that $n \geq 3$. Then $\psi(C_{\binom{n}{2}}) = n$.*

Theorem 3. *Let n be any even number such that $n \geq 4$. Then $\psi(C_{\binom{n}{2}+\frac{n}{2}}) = n$.*

Later in 2018, Chainurak, P. [3] was able to find the achromatic number of cycle graphs as the following lemma.

Lemma 3. *Let n be any odd number such that $n \geq 5$. Then $\psi(C_{\binom{n}{2}+1}) = n-1$.*

Lemma 4. *Let n be any odd number such that $n \geq 3$ and $m \in \{2,3,4,\ldots,n-1\}$. Then $\psi(C_{\binom{n}{2}+m}) = n$.*

Lemma 5. *Let n be any even number such that $n \geq 4$ and $m \in \{0,1,2,\ldots,\frac{n}{2}-1\}$. Then $\psi(C_{\binom{n}{2}+m}) = n-1$.*

Lemma 6. *Let n be any even number such that $n \geq 4$ and $m \in \{1,2,\ldots,\frac{n}{2}-1\}$. Then $\psi(C_{\binom{n}{2}+\frac{n}{2}+m}) = n$.*

Furthermore, we also need some results of natural numbers which can be proved by the proof of Mathematical Induction, as the following lemmas.

Lemma 7. *Let n be any natural number such that $n \geq 2$. Then $\frac{18n-3}{8} > 2n$.*

Proof. Consider case $n = 2$, we have $\frac{18(2)-3}{8} = \frac{33}{8} > 4 = 2(2)$. Assume that for any natural number $k \geq 2$, $\frac{18k-3}{8} > 2k$. Consider $\frac{18(k+1)-3}{8} = \frac{18k-3}{8} + \frac{18}{8} > 2k+2 = 2(k+1)$. By the proof of mathematical induction, we have $\frac{18n-3}{8} > 2n$ for every natural number $n \geq 2$. □

By using the same method of proof as in Lemma 7, we obtain the following lemma.

Lemma 8. *Let n be any natural number such that $n \geq 4$. Then $\frac{9(2n+1)}{8} > 2(n+1)$.*

Lemma 9. *Let n be any natural number such that $n \geq 3$. Then $\frac{3(6n+1)}{8} > 2n + 1$.*

Lemma 10. *Let n be any natural number such that $n \geq 5$. Then $\frac{3(3n-1)(n-1)}{8} > n(n-1)$.*

Lemma 11. *Let n be any natural number such that $n \geq 7$. Then $\frac{3(3n-1)(n-1)}{8} > n(n-1) + 2$.*

Lemma 12. *Let n be any natural number such that $n \geq 1$. Then $\frac{18}{4}n + \frac{15}{4} > 2(n+1)$.*

Lemma 13. *Let n be any natural number such that $n \geq 7$ and $m \in \{2, 3, 4, \ldots, n-1\}$. Then $\frac{(3n+1)(3n-1)}{8} > n(n-1) + 2m$.*

Proof. Let $m = max\{2, 3, 4, \ldots, n-1\}$. Consider case $n = 7$, we have

$$\frac{(3(7)+1)(3(7)-1)}{8} = 55 > 54 = 7(6) + 2(6).$$

Assume that for any natural number $k \geq 7$, $\frac{(3k+1)(3k-1)}{8} > k(k-1) + 2(k-1)$. Consider $\frac{(3(k+1)+1)(3(k+1)-1)}{8} = \frac{((3k+1)+3)((3k-1)+3)}{8} = \frac{(3k+1)(3k-1)}{8} + \frac{9(2k+1)}{8}$. By the assumption of induction and Lemma 8, we conclude that

$$\begin{aligned}
\frac{(3(k+1)+1)(3(k+1)-1)}{8} &> k(k-1) + 2(k-1) + 2(k+1) \\
&= ((k+2)(k-1) + 2(k+1) \\
&= ((k+2)+1)((k-1)+1) \\
&= ((k+1)+2)((k+1)-1) \\
&= (k+1)(k+1) - (k+1) + 2(k+1) - 2 \\
&= (k+1)((k+1)-1) + 2((k+1)-1).
\end{aligned}$$

By the proof of mathematical induction, we have $\frac{(3n+1)(3n-1)}{8} > n(n-1) + 2m$ for every natural number $n \geq 7$ and $m \in \{2, 3, 4, \ldots, n-1\}$. □

By using the same method of proof as in Lemma 13, we obtain the following lemma.

Lemma 14. *Let n be any natural number such that $n \geq 4$ and $m \in \{0,1,2,\ldots,\frac{n}{2}-1\}$. Then $\frac{3n(3n-2)}{8} > n(n-1)+2m$.*

Lemma 15. *Let n be any natural number such that $n \geq 4$ and $m \in \{1,2,3,\ldots,\frac{n}{2}-1\}$. Then $\frac{3n(3n+2)}{4} > n^2+2m$.*

Theorem 4. ([11]) *Let x be any real number and n be any integer. Then $\lfloor x+n \rfloor = \lfloor x \rfloor + n$.*

Up to now, none has been able to find the achromatic numbers of $SF(n,1)$ graphs for any natural number n such that $n \geq 4$. We use these previous theorems and lemmas as the tool to determine bounds of the achromatic numbers of the graphs.

To determine bounds of achromatic numbers of $SF(n,1)$ graphs for any natural number n such that $n \geq 4$, we consider six cases:

1. Bounds of $\psi(SF(\binom{n}{2},1))$ for any odd number n such that $n \geq 5$.
2. Bounds of $\psi(SF(\binom{n}{2}+1,1))$ for any odd number n such that $n \geq 7$.
3. Bounds of $\psi(SF(\binom{n}{2}+m,1))$ for any odd number n such that $n \geq 7$ and $m \in \{2,3,4,\ldots,n-1\}$.
4. Bounds of $\psi(SF(\binom{n}{2}+\frac{n}{2},1))$ for any even number n such that $n \geq 4$.
5. Bounds of $\psi(SF(\binom{n}{2}+m,1))$ for any even number n such that $n \geq 4$ and $m \in \{0,1,2,\ldots,\frac{n}{2}-1\}$.
6. Bounds of $\psi(SF(\binom{n}{2}+\frac{n}{2}+m,1))$ for any even number n such that $n \geq 4$ and $m \in \{1,2,3,\ldots,\frac{n}{2}-1\}$.

Lemma 16. *Let n be any odd number such that $n \geq 3$. Then the graph $SF(\binom{n}{2},1)$ has a complete $(n+1)$-coloring.*

Proof. By Theorem 2, $\psi(C_{\binom{n}{2}}) = n$. So there is a complete n-coloring, say $f : V(C_{\binom{n}{2}}) \longrightarrow \{1,2,3,\ldots,n\}$. Since the graph $SF(\binom{n}{2},1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}})$ for every $i = 1,2,3,\ldots,\binom{n}{2}$. We define $g : V(SF(\binom{n}{2},1)) \longrightarrow \{1,2,3,\ldots,n+1\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n+1$ for every $u_i, v_i \in V(SF(\binom{n}{2},1))$ and $i = 1,2,3,\ldots,\binom{n}{2}$. It is obvious that g is a complete $(n+1)$-coloring. Hence the graph $SF(\binom{n}{2},1)$ has a complete $(n+1)$-coloring. □

Lemma 17. *Let n be any odd number such that $n \geq 5$. Then the graph $SF(\binom{n}{2}, 1)$ does not have a complete $(n + \frac{\binom{n}{2}}{n})$-coloring.*

Proof. Consider $|E(SF(\binom{n}{2}, 1))| = 2\binom{n}{2} = n(n-1)$ and $\binom{n+\frac{\binom{n}{2}}{n}}{2} = \frac{3(3n-1)(n-1)}{8}$. Then by Lemma 10, $|E(SF(\binom{n}{2}, 1))| < \binom{n+\frac{\binom{n}{2}}{n}}{2}$. Therefore, the graph $SF(\binom{n}{2}, 1)$ does not have a complete $(n + \frac{\binom{n}{2}}{n})$-coloring. □

A consequence of Lemma 16 and Lemma 17 is the following theorem.

Theorem 5. *Let n be any odd number such that $n \geq 5$. Then*

$$n + 1 \leq \psi(SF(\binom{n}{2}, 1)) \leq n + \frac{\binom{n}{2}}{n} - 1.$$

Lemma 18. *Let n be any odd number such that $n \geq 3$. Then the graph $SF(\binom{n}{2} + 1, 1)$ has a complete n-coloring.*

Proof. By Lemma 3, $\psi(C_{\binom{n}{2}+1}) = n-1$. So there is a complete $(n-1)$-coloring, say $f : V(C_{\binom{n}{2}+1}) \longrightarrow \{1, 2, 3, \ldots, n-1\}$. Since the graph $SF(\binom{n}{2} + 1, 1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}+1}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}+1}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}+1})$ for every $i = 1, 2, 3, \ldots, \binom{n}{2}+1$. We define $g : V(SF(\binom{n}{2} + 1, 1)) \longrightarrow \{1, 2, 3, \ldots, n\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n$ for every $u_i, v_i \in V(SF(\binom{n}{2} + 1, 1))$ and $i = 1, 2, 3, \ldots, \binom{n}{2} + 1$. It is obvious that g is a complete n-coloring. Hence the graph $SF(\binom{n}{2} + 1, 1)$ has a complete n-coloring. □

Lemma 19. *Let n be any odd number such that $n \geq 7$. Then the graph $SF(\binom{n}{2} + 1, 1)$ does not have a complete $(n + \lfloor\frac{\binom{n}{2}+1}{n}\rfloor)$-coloring.*

Proof. Consider $\lfloor\frac{\binom{n}{2}+1}{n}\rfloor = \lfloor\frac{n(n-1)+2}{2n}\rfloor = \lfloor\frac{n-1}{2} + \frac{1}{n}\rfloor$ such that $\frac{n-1}{2}$ is an integer. By Theorem 4, $\lfloor\frac{\binom{n}{2}+1}{n}\rfloor = \frac{n-1}{2} + \lfloor\frac{1}{n}\rfloor = \frac{n-1}{2}$. Hence $\binom{n+\lfloor\frac{\binom{n}{2}+1}{n}\rfloor}{2} = \binom{n+\frac{n-1}{2}}{2} = \binom{\frac{3n-1}{2}}{2} = \frac{3(3n-1)(n-1)}{8}$. By Lemma 11, $\binom{n+\lfloor\frac{\binom{n}{2}+1}{n}\rfloor}{2} > n(n-1) + 2$. Since $|E(SF(\binom{n}{2} + 1, 1)))| = 2(\binom{n}{2} + 1) = n(n-1) + 2$, $|E(SF(\binom{n}{2} + 1, 1)))| < \binom{n+\lfloor\frac{\binom{n}{2}+1}{n}\rfloor}{2}$. Therefore, the graph $SF(\binom{n}{2} + 1, 1)))$ does not have a complete $(n + \lfloor\frac{\binom{n}{2}+1}{n}\rfloor)$-coloring. □

A consequence of Lemma 18 and Lemma 19 is the following theorem.

Theorem 6. *Let n be any odd number such that $n \geq 7$. Then*

$$n \leq \psi(SF(\binom{n}{2}+1,1))) \leq n + \lfloor \frac{\binom{n}{2}+1}{n} \rfloor - 1.$$

Lemma 20. *Let n be any odd number such that $n \geq 3$ and $m \in \{2,3,4,\ldots, n-1\}$. Then the graph $SF(\binom{n}{2}+m,1)$ has a complete $(n+1)$-coloring.*

Proof. By Lemma 4, $\psi(C_{\binom{n}{2}+m}) = n$. So there is a complete n-coloring, say $f : V(C_{\binom{n}{2}+m}) \longrightarrow \{1,2,3,\ldots,n\}$. Since the graph $SF(\binom{n}{2}+m,1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}+m}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}+m}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}+m})$ for every $i = 1,2,3,\ldots,\binom{n}{2}+m$. We define $g : V(SF(\binom{n}{2}+m,1)) \longrightarrow \{1,2,3,\ldots,n+1\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n+1$ for every $u_i, v_i \in V(SF(\binom{n}{2}+m,1))$ and $i = 1,2,3,\ldots,\binom{n}{2}+m$. It is obvious that g is a complete $(n+1)$-coloring. Hence the graph $SF(\binom{n}{2}+m,1)$ has a complete $(n+1)$-coloring. □

Lemma 21. *Let n be any odd number such that $n \geq 7$ and $m \in \{2,3,4,\ldots, n-1\}$. Then the graph $SF(\binom{n}{2}+m,1)$ does not have a complete $(n+\lfloor \frac{\binom{n}{2}+m}{n} \rfloor + 1)$-coloring.*

Proof. Let $m = \max\{2,3,4,\ldots,n-1\}$. Consider $\lfloor \frac{\binom{n}{2}+m}{n} \rfloor = \lfloor \frac{n(n-1)+2m}{2n} \rfloor = \lfloor \frac{n-1}{2} + \frac{m}{n} \rfloor$ such that $\frac{n-1}{2}$ is an integer. By Theorem 4, $\lfloor \frac{\binom{n}{2}+m}{n} \rfloor = \frac{n-1}{2} + \lfloor \frac{m}{n} \rfloor = \frac{n-1}{2}$. Hence $\binom{n+\lfloor \frac{\binom{n}{2}+m}{n} \rfloor+1}{2} = \binom{n+\frac{n-1}{2}+1}{2} = \binom{\frac{3n+1}{2}}{2} = \frac{(3n+1)(3n-1)}{8}$. By Lemma 13, $\binom{n+\lfloor \frac{\binom{n}{2}+m}{n} \rfloor+1}{2} > n(n-1)+2m$. Since $|E(SF(\binom{n}{2}+m,1)))| = 2(\binom{n}{2}+m) = n(n-1)+2m$, we have $|E(SF(\binom{n}{2}+m,1)))| < \binom{n+\lfloor \frac{\binom{n}{2}+m}{n} \rfloor}{2}$. Therefore, the graph $SF(\binom{n}{2}+m,1)))$ does not have a complete $(n+\lfloor \frac{\binom{n}{2}+m}{n} \rfloor+1)$-coloring. □

A consequence of Lemma 20 and Lemma 21 is the following theorem.

Theorem 7. *Let n be any odd number such that $n \geq 7$ and $m \in \{2,3,4,\ldots, n-1\}$. Then $n+1 \leq \psi(SF(\binom{n}{2}+m,1))) \leq n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor$.*

Lemma 22. *Let n be any even number such that $n \geq 4$. Then the graph $SF(\binom{n}{2}+\frac{n}{2},1)$ has a complete $(n+1)$-coloring.*

Proof. By Theorem 3, $\psi(C_{\binom{n}{2}+\frac{n}{2}}) = n$. So there is a complete n-coloring, say $f : V(C_{\binom{n}{2}+\frac{n}{2}}) \longrightarrow \{1, 2, 3, \ldots, n\}$. Since the graph $SF(\binom{n}{2} + \frac{n}{2}, 1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}+\frac{n}{2}}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}+\frac{n}{2}}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}+\frac{n}{2}})$ for every $i = 1, 2, 3, \ldots, \binom{n}{2}+\frac{n}{2}$. We define $g : V(SF(\binom{n}{2}+\frac{n}{2}, 1)) \longrightarrow \{1, 2, 3, \ldots, n+1\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n+1$ for every $u_i, v_i \in V(SF(\binom{n}{2} + \frac{n}{2}, 1))$ and $i = 1, 2, 3, \ldots, \binom{n}{2} + \frac{n}{2}$. It is obvious that g is a complete $(n + 1)$-coloring. Hence the graph $SF(\binom{n}{2} + \frac{n}{2}, 1)$ has a complete $(n + 1)$-coloring. □

Lemma 23. *Let n be any even number such that $n \geq 4$. Then the graph $SF(\binom{n}{2} + \frac{n}{2}, 1)$ does not have a complete $(n + \frac{\binom{n}{2}+\frac{n}{2}}{n} + 1)$-coloring.*

Proof. Consider $|E(SF(\binom{n}{2} + \frac{n}{2}, 1))| = 2(\binom{n}{2} + \frac{n}{2}) = n^2$ and $\binom{n+\frac{\binom{n}{2}+\frac{n}{2}}{\frac{n}{2}}+1}{2} = \binom{2n+1}{2} = 2n^2 + n$. Since $n^2 < 2n^2 + n$ for every natural number n, $|E(SF(\binom{n}{2} + \frac{n}{2}, 1))| < \binom{n+\frac{\binom{n}{2}+\frac{n}{2}}{\frac{n}{2}}+1}{2}$. Therefore, the graph $SF(\binom{n}{2} + \frac{n}{2}, 1)$ does not have a complete $(n + \frac{\binom{n}{2}+\frac{n}{2}}{n} + 1)$-coloring. □

A consequence of Lemma 22 and Lemma 23 is the following theorem.

Theorem 8. *Let n be any even number such that $n \geq 4$. Then*

$$n + 1 \leq \psi(SF(\binom{n}{2} + \frac{n}{2}, 1))) \leq n + \frac{\binom{n}{2} + \frac{n}{2}}{n}.$$

Lemma 24. *Let n be any even number such that $n \geq 4$ and $m \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}$. Then the graph $SF(\binom{n}{2} + m, 1)$ has a complete n-coloring.*

Proof. By Lemma 5, $\psi(C_{\binom{n}{2}+m}) = n-1$. So there is a complete $(n-1)$-coloring, say $f : V(C_{\binom{n}{2}+m}) \longrightarrow \{1, 2, 3, \ldots, n - 1\}$. Since the graph $SF(\binom{n}{2} + m, 1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}+m}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}+m}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}+m})$ for every $i = 1, 2, 3, \ldots, \binom{n}{2} + m$. We define $g : V(SF(\binom{n}{2} + m, 1)) \longrightarrow \{1, 2, 3, \ldots, n\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n$ for every $u_i, v_i \in V(SF(\binom{n}{2} + m, 1))$ and $i = 1, 2, 3, \ldots, \binom{n}{2} + m$. It is obvious that g is a complete n-coloring. Hence the graph $SF(\binom{n}{2} + m, 1)$ has a complete n-coloring. □

Lemma 25. *Let n be any even number such that $n \geq 4$ and $m \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}$. Then the graph $SF(\binom{n}{2} + m, 1)$ does not have a complete $(n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor + 1)$-coloring.*

Proof. Let $m = max\{0, 1, 2, \ldots, \frac{n}{2} - 1\}$. Consider $\lfloor \frac{\binom{n}{2}+m}{n} \rfloor = \lfloor \frac{n-1}{2} + \frac{m}{n} \rfloor = \lfloor \frac{n-2}{2} + \frac{1}{2} + \frac{m}{n} \rfloor = \lfloor \frac{n-2}{2} + \frac{n+2m}{2n} \rfloor$ such that $\frac{n-2}{2}$ is an integer. By Theorem 4, $\lfloor \frac{\binom{n}{2}+m}{n} \rfloor = \frac{n-2}{2} + \lfloor \frac{n+2m}{2n} \rfloor = \frac{n-2}{2}$. Hence $\binom{n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor + 1}{2} = \binom{n + \frac{n-2}{2} + 1}{2} = \binom{\frac{3n}{2}}{2} = \frac{3n(3n-2)}{8}$. Then by Lemma 14, $\frac{3n(3n-2)}{8} > n(n-1) + 2m$.
Since $|E(SF(\binom{n}{2} + m, 1))| = 2(\binom{n}{2} + m) = 2(\frac{n(n-1)}{2} + m) = n(n-1) + 2m$, we have $|E(SF(\binom{n}{2} + m, 1)| < (n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor + 1)$. Therefore, the graph $SF(\binom{n}{2} + m, 1))$ does not have a complete $(n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor + 1)$-coloring. □

A consequence of Lemma 24 and Lemma 25 is the following theorem.

Theorem 9. *Let n be any even number such that $n \geq 4$ and $m \in \{0, 1, 2, \ldots, \frac{n}{2} - 1\}$. Then $n \leq \psi(SF(\binom{n}{2} + m, 1)) \leq n + \lfloor \frac{\binom{n}{2}+m}{n} \rfloor$.*

Lemma 26. *Let n be an even number such that $n \geq 4$ and $m \in \{1, 2, 3, \ldots, \frac{n}{2} - 1\}$. Then the graph $SF(\binom{n}{2} + \frac{n}{2} + m, 1)$ has a complete $(n + 1)$-coloring.*

Proof. By Lemma 6, $\psi(C_{\binom{n}{2}+\frac{n}{2}+m}) = n$. So there is a complete n-coloring, say $f : V(C_{\binom{n}{2}+\frac{n}{2}+m}) \longrightarrow \{1, 2, 3, \ldots, n\}$. Since the graph $SF(\binom{n}{2} + \frac{n}{2} + m, 1)$ is constructed by joining every vertex of the cycle graph $C_{\binom{n}{2}+\frac{n}{2}+m}$ to a corresponding pendant vertex with a corresponding pendant edge, every pendant vertex is adjacent to a corresponding vertex of $C_{\binom{n}{2}+\frac{n}{2}+m}$ and all pendant vertices are not adjacent to each other. Suppose that v_i is a corresponding pendant vertex which is adjacent to $u_i \in V(C_{\binom{n}{2}+\frac{n}{2}+m})$ for every $i = 1, 2, 3, \ldots, \binom{n}{2} + \frac{n}{2} + m$. We define $g : V(SF(\binom{n}{2} + \frac{n}{2} + m, 1)) \longrightarrow \{1, 2, 3, \ldots, n + 1\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = n + 1$ for every $u_i, v_i \in V(SF(\binom{n}{2} + \frac{n}{2} + m, 1))$ and $i = 1, 2, 3, \ldots, \binom{n}{2} + \frac{n}{2} + m$. It is obvious that g is a complete $(n+1)$-coloring. Hence the graph $SF(\binom{n}{2} + \frac{n}{2} + m, 1)$ has a complete $(n + 1)$-coloring. □

Lemma 27. *Let n be any even number such that $n \geq 4$ and $m \in \{1, 2, 3, \ldots, \frac{n}{2} - 1\}$. Then the graph $SF(\binom{n}{2} + \frac{n}{2} + m, 1)$ does not have a complete $(n + \lfloor \frac{\binom{n}{2}+\frac{n}{2}+m}{n} \rfloor + 1)$-coloring.*

Proof. Let $m = max\{1, 2, 3, \ldots, \frac{n}{2} - 1\}$. Consider $\lfloor \frac{\binom{n}{2}+\frac{n}{2}+m}{n} \rfloor = \lfloor \frac{\frac{n^2}{2}+m}{n} \rfloor = \lfloor \frac{n}{2} + \frac{m}{n} \rfloor$ where $\frac{n}{2}$ is an integer. By Theorem 4, $\lfloor \frac{\binom{n}{2}+\frac{n}{2}+m}{n} \rfloor = \frac{n}{2} + \lfloor \frac{m}{n} \rfloor = \frac{n}{2}$. Hence $\binom{n + \lfloor \frac{\binom{n}{2}+\frac{n}{2}+m}{n} \rfloor + 1}{2} = \binom{n + \frac{n}{2} + 1}{2} = \binom{\frac{3n+2}{2}}{2} = \frac{(3n+2)(3n)}{4}$. Then by Lemma 15,

$\binom{n+\lfloor\frac{\binom{n}{2}+\frac{n}{2}+m}{\frac{n}{2}}\rfloor+1}{2} = \binom{n+\frac{n}{2}+1}{2} = \frac{(3n+2)(3n)}{4}$. Since $|E(SF(\binom{n}{2}+\frac{n}{2}+m,1))| = 2(\binom{n}{2}+\frac{n}{2}+m) = n^2+m$, we have

$$|E(SF(\binom{n}{2}+\frac{n}{2}+m,1)| < (n+\lfloor\frac{\binom{n}{2}+\frac{n}{2}+m}{n}\rfloor+1).$$

Therefore, the graph $SF(\binom{n}{2}+\frac{n}{2}+m,1)))$ does not have a complete $(n+\lfloor\frac{\binom{n}{2}+\frac{n}{2}+m}{n}\rfloor+1)$-coloring. □

A consequence of Lemma 26 and Lemma 27 is the following theorem.

Theorem 10. *Let n be any even number such that $n \geq 4$ and $m \in \{1,2,3,\ldots,\frac{n}{2}-1\}$. Then $n+1 \leq \psi(SF(\binom{n}{2}+\frac{n}{2}+m,1))) \leq n+\lfloor\frac{\binom{n}{2}+\frac{n}{2}+m}{n}\rfloor$.*

4 Conclusion and Open Problem

In this paper, we found the exact value of the achromatic number of a complete starfish graph n and the bounds of the achromatic number of the graph $SF(n,1)$ when n is a natural number and $n \geq 3$. For the next study, we can extend our results by further research to find sharper bounds of the achromatic number of the graph $SF(n,1)$.

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