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On Spectra of Symmetrized Two-Sided Multiplication Operators Implemented by Orthogonal Isometries

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Abstract

Characterization of Symmetrized Two-Sided Multiplication Operators have been done over years in-terms of their properties which include numerical ranges and norms among others. However, characterizing the spectrum and norms have not been exhausted still remains interesting. There exists an open question requiring the determination of the norms of elementary operators in a general Banach space setting. Since there is a strong relation ship between the norm and spectral radius, it is in the interest of this study to characterize spectral properties of Symmetrized Two-Sided Multiplication Operators as an avenue that can help in solving the general problem.

Keywords: *Spectrum, Elementary operator, Isometry, Orthogonality.*
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1 Introduction

Several studies on characterization of symmetrized two-sided multiplication Operators (STSMO) and elementary operators (EO) in general have been done over years in-terms of their properties which include numerical ranges [2] and norms [5], spectrum (Sp), spectral radius (Sr) among others. However, characterizing the spectrum and norms have not been exhausted and still remains

interesting [4]. Many authors ([1], [7] and the references therein) considered elementary operators acting on Hilbert-Schmidt class $C_2(H)$ and conditions for operators to be 2-symmetric and 3-symmetric [6] have been established. In the investigations it was established that if $M_{T,S}$ is an elementary operator and $M^2_{T,S} = -M^2_{T,S^*}$ then the operator $M_{T,S}$ is 2-symmetric if there exists a scalar β whereby $T^2 + T^{*2} = 2\beta T^*T$ and $SS^* = \beta(S^2 + S^{*2})$ whereby $\frac{1}{2} \leq\mid \beta \mid\leq 1$.
The work of [3] further characterized the 3-symmetric elementary operators and established that for two sided multiplication operators acting on $C_2(H)$ with $T, S \in B(H)$ whereby $\{T^3, T^{*2}T\}$ and $\{S^3, S^{*2}S\}$ are commutants, STSMO are linear independent [8]. The research was further extended by [9] to binomial operators and it was established that for any two sided multiplication operator, $M_{T,S}$, it is binomial if we have a scalar α such that $TT^{*2}T = \alpha T^*T^2T^*$ and also $\alpha SS^2B^* = \alpha SS^{*2}S$ with $\mid \alpha \mid= 1$.
In our research, it was interesting to establish the spectral properties of β whereby $T^2 + T^{*2} = 2\beta T^*T$ and $SS^* = \beta(S^2 + S^{*2})$, further we also extended our investigation to the spectral properties of STSMOs. In [10] the work researched on the norm equality and established sufficient conditions for the equality to hold for uniformly convex spaces. Additionally, an equivalance relationship was established. In our research we investigated various spectra properties of the symmetric norm ideals in $B(X)$. We extended our in the norms to other technical approaches and established the link between various spectra properties.
The work of [12] considered lower bound for completely bounded norms of $M_{T,S}$. In the investigation, it was established that $\| M_{T,S} \|_{kl}\geq\| T \|\| S \|$. Furthermore, using tensor products, and given that injective norm is the minimal tensor cross norm then, for $T, S \in B(H)$ and if $M_{T,S} = T \otimes S + S \otimes T$, it follows that $\| M_{T,S} \|_{\beta}\geq 2(\sqrt{2}-1) \| T \|\| S \|$ as seen in [13]. Further research established that for operators of rank 1, $M_{T,S}$ does not attain its norm [14]. Moreover, link between between spatial numerical range and numerical radius was also established in the research. The research was further extended by [15] to real mappings defined by $R_{T,S}(x) = T^*xS + S^*x^*T$. Examples were utilized to establish the relationship $M_{T,S}(x) = TxS + SxT = exe + uxu$ holds.
Of interest in our research was to investigate various spectral properties of various inequalities such as $\| M_{T,S} \|_{kl}\geq\| T \|\| S \|$ that were satisfied in the norms. We extended and spectral properties of $M_{T,S}(x) = TxS + SxT = exe + uxu$ and other operators established. For [16], their investigation established that equality holds for spectral norms and the investigation was extended in cones and it was proved that if operators admit the two-sided multiplication property, then they are normal and norm attainable.
In [17] the authors characterized positive operators particularly self adjoint operators. In the research, conditions for a polynomial's eigenvalue of a real polynomial to have negative real part were established. Furthermore, the research

established that if an operator is positive then its eigenvalues are non-negative and extended to polynomials and it was established that characteristic polynomial of such self adjoint operators is real. In other words the polynomial has real coefficients. A link between self adjoint operators and orthogonality was also established in [20]. In the investigation it was established that T is self adjoint if there exist orthogonal isometries and a sequence of real values which are normal as supported by [19].

The study of [18] also considered the relationship between identity operators and orthogonal isometric operators. In the investigation, it was established that if I is an identity operator and if β is faithful, then $IP_j = P_i j = 1, 2, ..., n$ and the reduction is irreducible. In operator spaces, it was established in [21] that for normal cones C and $C^{'}$, they coincide if they are positive and β is in the spectrum of S. The investigation was further extended to resolvents which are real positivity was established.

Additionally, it was proved that the leading coefficient in the principal part of the resolvent at $\omega = \beta$ is a positive operator. Moreover, it was proved in [26] that if C is properly contained in the cone $C^{'}$ then there exists one characteristic vector s for β in C. The research was extended to power STSMO and it was established that for the the interior point of C, the point spectrum is a singleton set containing β.

The authors of [25] researched on positive operators that can be presented in the form $S(X) = \Sigma_{j=i}^{\xi} \alpha_j X \beta_j$ where α_j and β_j are square matrices. In the research it was proved that if $\xi(S)$ is a block matrix in $M_j(M_j(\mathcal{C}))$, then the operator $S : (M_j(\mathcal{C})) \to (M_j(\mathcal{C}))$ is i-positive if $I_j \otimes P)\xi(S)(I_j \otimes P)$ is positive for all rank i. The investigation was extended to operators of the form $Sy = \sum_{j=1}^{\xi} \mu_j \alpha_j y \alpha_j^*$. Again, it also proved that S is i-positive under certain conditions. Moreover, it was shown that if S has length of at most $(k^2 - 1)$ and S is $(K - 1)$ positive, then S is completely positive.

The authors of [43] researched on positive operators and more particularly compactly strong operators. They heavily utilized the upper and lower spectral radius in there work. In the research they proved that lower spectral radius also serves as the upper spectral radius. They also established the relationship between positive operators, irreducible ideal and the spectral radius, quasi-interior C

Recent studies also did the comparison between ideal irreducible operators semi-strong positivity of operators [27]. In the investigation, it was established that ideal irreducibility implies semi-strongly positivity if X is a Banach lattice.

To sum up the introduction, we realized that little has been given attention with regard to spectra of STSMO. This forms the basis of this research.

2 Literature review

We discuss literature on spectral analysis of symmetrized two sided multiplication operators in this work. We consider various studies, their relevance and critical contributions to this study.
The study of [28] considered complex operators that are symmetric. They explored a wide range of this class of operators including the Jordan canonical model. They extensively explored the C-symmetric property where they established that S is C-symmetric if if it is a commutant. They established that the point spectra of S and S^* corresponds and the same assertion also holds for the approximate spectra of the two operators as supported by [29]. They further established that if S is invertible then S also satisfies the C-symmetric property. Further investigation also established that for bounded linear operators if S is symmetric and an orthonormal basis exists to which S is also symmetric. Another work in [30] further proved that if $S = S^t$ is symmetric, then for unitary operator U and a normal and symmetric N, $S = UNU^t$. In this case N was diagonalized to a real orthogonal isometry.

Theorem 2.1 *([31]) If $M = M_{S^*S}$ is the spectral measure of S^*S then $CSM(\delta) = M(\delta)CS$ for every Borel subset δ of $[0, \infty]$.*

Another study by [31] considered the spectral measure of S^*S and established that if $M = M_{S^*S}$ is the spectral measure of S^*S then $CSM(\delta) = M(\delta)CS$ for every Borel subset δ of $[0, \infty]$. The research was extended to rank order one operators and established that if $S = x \otimes y$ then $CS = S^*C$ if S is a constant multiple of $x \otimes Cy$. The study extended to consider the compression of S_u to H_u. In the investigation it was established that (H_u, S_u, C) admit the C-symmetry property. The study also considered the relationship between closed graph densely defined symmetric operators and the isometric involution operators C and established that the symmetric property $SC = CS$ also exist between this two classes of operators, hence S can be parameterized by all isometric operators Q such that $Q^*C = CQ$ as elaboated in [32].

Theorem 2.2 *For jointly C-symmetric operators $S_\beta g = \alpha_\beta g$, if g is orthogonal to the conjugate kernel Q_β and $S^*_\beta g = g/\alpha_\beta$ then g is orthogonal to the reproducing kernel K_β.*

Some recent study of [22] considered antilinear involution and established that if S is two sided then $RU = U^*R$. Additionally, for jointly C-symmetric operators $S_\beta g = \alpha_\beta g$, if g is orthogonal to the conjugate kernel Q_β and $S^*_\beta g = g/\alpha_\beta$ then g is orthogonal to the reproducing kernel K_β. They finally established the link between invariant subspaces, reducing subspaces and orthogonal isometries. In the research of [24] it was established that M is C-invariant if $M^\perp$ is C-invariant, and moreover, the compression $T = PSP$ of S to M also admits

$CT = T^*C$ property.
In another investigation, [23] investigated bounded linear operators in H. They considered the operator $S = CSC^*$ and proved that $S = CJ \mid S \mid$. The study of [35] emphasized this and proved that $JU = U^*J$ where U and J are both symmetric. The following result provide a methods that can be used to construct symmetric operators.

Theorem 2.3 *([33]) For cones, all STSMO can construct symmetric operators.*

Now [34] established that the product of orthogonal isomery with $\mid S \mid$ is unitary. The research further proved that S^*S and SS^* are equivalent and they admit the C-symmetric property. The investigation was extended to compact operators and it was established that eigenvectors of $\mid S \mid$ are the nonzero eigenvalues of $\mid S \mid$. Further research proved that $Sx_m = \delta_m Cx_m$.
In our research it was interesting to establish that C-symmetric operators are also two sided multiplication operators. As such we investigated the spectral properties and established that the spectral measure that belongs to the Borel subset. We extended the investigation to STSMO and also investigated the spectral properties of such operators. We also considered the spectral properties of C-symmetric operator with its adjoint. We extended our research to scaled C-symmetric operators given that scaled C-symmetric operators are also C-symmetric operators.
The work of [36] researched on the spectra of different classes of operators in H. In the research it was established that the spectrum of continuous operators is in the operator norm as seen in the next theorem.

Theorem 2.4 *([33]) The spectrum of a continuous operator is in the operator norm.*

The investigation was extended and it was established that the spectrum (Sp) of such operators is also compact and also non empty. Additionally, it was established that the Sp of is nonvoid in $\mathcal{R}$. The work also considered l^2 space and it proved that two-sided shift operators has an empty spectrum. It is worth to note that compact operators are not two sided multiplication operators but they can be view as two sided multiplication operators if they are compact and when they are multiplied by other operators from both sides.
Some authors [37] in researched on C-symmetric operators in H. The investigation determined the Jordan ideal of S_C coincides with their dual space. In the research, it was established that $C^*(S) \subset S_C; W^*(S) \subset S_C; \mid S \mid \in S_C$: $C^*S = J^*(S)$. It was further proved that if $A \in S_C$, then $\parallel J_A \parallel = \parallel A \parallel$ and $\delta(J_A) = \delta_r(J_A) = \delta_l(J_A) = \delta_\pi(J_A) = \frac{1}{2}\{\delta(A) + \delta(A)\}$. Again, it was established that if $T \in S_C$, then the following also hold, that is, $\delta_\pi(T) = \delta_\delta(T) = \delta(T); \delta(J_T) \subset \frac{1}{2}\{\delta(T) + \delta(T)\}$, furthermore it was established in [38]

that $\delta_\pi(J_A) = \delta_\delta(J_{A,1})$. Using the dual relationship, it was established that if $\delta_\pi(J_{A,0}) = \delta_\delta(J_{A,1})$ and $\delta_\delta(J_{A,0}) = \delta_\pi(J_{A,1})$.
In our research we also considered the ideal of STSMO. We also established various links between various types of spectra. As a result we represented compact operators by multiplying it with a function then proceeded to investigate the spectral properties of the resultant operator. Furthermore, it is also well established that finite rank compact operators are two sided. Therefore, we investigated the spectrum to determine if it consists of eigenvalues plus the zero. We extended our investigation to shift operators and established that the spectrum of shift operators is the closed unit disk in the $\mathcal{C}$ plane as illustrated in [39].
For [40], they considered elementary operators acting on Hilbert-Schmidt class C_2H. The worked on conditions for operators to be 2-symmetric and 3-symmetric. In the investigation, it was established that if $M_{T,S}$ is an elementary operator and $M^2_{T,S} = -M^2_{T,S^*}$, then the operator $M_{T,S}$ is 2-symmetric if there exists a scalar $\beta : T^2+T^{*2} = 2\beta T^*T$ and $SS^* = \beta(S^2+S^{*2}) : \frac{1}{2} \leq\mid \beta \mid\leq 1$.
Again, [41] characterized the 3-symmetric elementary operators and established that for two sided multiplication operators acting on C_2H with $T, S \in B(H) : \{T^3, T^{*2}T\}$ and $\{S^3, S^{*2}S\}$ being linear independent then $M_{T,S}$ is 3-symmetric if there exist constants α and β.

Theorem 2.5 *We have for cones,*

(i). $3T^{*3}T = \alpha T^{*3} - \beta T^3, S^3 = \overline{\alpha} S^2S^* + \beta SS^{*2}$

(ii). $T^{*3} = \alpha T^3, T^*A^2 = \beta T^{*2}T, S^3 = \alpha S^{*3}, SS^{*2} = \beta S^2S^*$ *and* $\mid \alpha \mid=\mid \beta \mid= 1$

The research was further extended by [42] to binomial operators and it was established that for any two sided multiplication operator, $M_{T,S}$ is binomial if we have a scalar α such that $TT^{*2}T = \alpha T^*T^2T^*$ and also $\alpha SS^2B^* = \alpha SS^{*2}S$ with $\mid \alpha \mid= 1$.
In our research it was interesting to establish the spectral properties of β whereby $T^2 + T^{*2} = 2\beta T^*T$ and $SS^* = \beta(S^2 + S^{*2})$, further we also extended our investigation to the spectral properties of operators that satisfies the following condition provided that $M_{T,S}$ is a STSMO.

Theorem 2.6 *We have for symmetric cones,*

(i). $3T^{*3}T = \alpha T^{*3} - \beta T^3, S^3 = \overline{\alpha} S^2S^* + \beta SS^{*2}$

(ii). $T^{*3} = \alpha T^3, T^*A^2 = \beta T^{*2}T, S^3 = \alpha S^{*3}, SS^{*2} = \beta S^2S^*$ *and* $\mid \alpha \mid=\mid \beta \mid= 1$

We finally closed our research by considering the spectral properties of two sided multiplication binomial operators. In our research we considered both spectrum, approximation spectrum, spectral radius any other spectral properties for operators under review as suggested by [43]. We also investigated the

spectral properties of the norms of the following equation that are $3T^{*3}T = \alpha T^{*3} - \beta T^3, S^3 = \overline{\alpha} S^2 S^* + \beta S S^{*2}$, $T^3 = 3\alpha T^* T^2 + \beta T^{*2} T, S^2 S^* = \alpha S^3 + \beta S^{*3}$ and $\mid \alpha \mid = \mid \beta \mid$, $T^{*3} = \alpha T^3, T^* A^2 = \beta T^{*2} T, S^3 = \alpha S^{*3}, S S^{*2} = \beta S^2 S^*$ and $\mid \alpha \mid = \mid \beta \mid = 1$.
The work of [3] concentrated on the norm equality and established sufficient conditions for the equality to hold. In our research we investigated various spectra properties of the symmetric norm ideals in $B(X)$. We extended our in the norms to other technical approaches and established the link between various spectra properties. Also [7] researched on lower bound for completely bounded norms of $M_{T,S}$. In the investigation, it was established that $\| M_{T,S} \|_{kl} \geq \| T \| \| S \|$.
Further using tensor product, and given that injective norm is the minimal tensor cross norm then if $T, S \in B(H)$ and if $M_{T,S} = T \otimes S + S \otimes T$, it follows from [11] that $\| M_{T,S} \|_{\beta} \geq 2(\sqrt{2} - 1) \| T \| \| S \|$. Further research established that for operators of rank 1, $M_{T,S}$ does not attain its norm. A link between between spatial numerical range and numerical radius was also established in the research. The research was further extended to real mappings. Examples were utilized to establish the relationship $M_{T,S}(x) = TxS + SxT = exe + uxu$ holds [20].
Of interest in our research was to investigate various spectral properties of various inequalities such as $\| M_{T,S,K} \| \geq \| T \| \| S \|$ that were satisfied in the norms. We extended this work and spectral properties of $M_{T,S}(x) = TxS + SxT = exe + uxu$ and other operators have been established [24]. We also investigated the spectral properties of operators that admit the following conditions: $\| M_{T,S,K} \mid B(H) \| = \| T \| \| S \|$ and $\| M_{T,S} \mid B(H) \| = \| M^*_{T^*,S^*} \mid B(H) \|$ for self adjoint operators.
In [23], they characterized positive operators particularly self adjoint operators. In the research, conditions for an a polynomial's eigenvalue of a real polynomial to have negative real part were established.

Theorem 2.7 *([24]) We have that*

(i). STSMO have symmetric bases

(ii). STSMO can be symmetrized under norm

Further more the research established that eigenvalues are non-negative for STSMO and it was extended to polynomials and it was established that characteristic polynomial of such self adjoint operators is real. In other words the polynomial has real coefficients. A link between self adjoint operators and orthogonality was also established.
Recently, [27] also considered the relationship between identity operators and orthogonal isometric operators. In the investigation, it was established that if I is an identity operator and if β is faithful, then $IP_j = P_i j = 1, 2, ..., n$ and

the reduction is irreducible.
Additionally, it was proved in [3] that the leading coefficient in the principal part of the resolvent at $\omega = \beta$ is a positive operator. So, it was proved that if C is properly contained in the cone then there exists one characteristic vector s for β in C. The research was extended to power operators and it was established in [42] that for the the interior point of C, the point spectrum is a singleton set containing β.

Theorem 2.8 *([36]) The point spectrum of STSMO is a singleton set.*

Another work of [38] researched on positive operators that can be presented in the form $S(X) = \Sigma_{j=i}^{\xi} \alpha_j X \beta_j$ where α_j and β_j are square matrices. In the research it was proved that if $\xi(S)$ is a block matrix in $M_j(M_j(\mathcal{C}))$, then the operator $S : (M_j(\mathcal{C})) \to (M_j(\mathcal{C}))$ is i-positive if $I_j \otimes P)\xi(S)(I_j \otimes P)$ is positive for all rank i. The investigation was extended to operators of the form $Sy = \sum_{j=1}^{\xi} \mu_j \alpha_j y \alpha_j^*$. In the research, it also proved in [30] that S is i-positive under certain conditions. Additionally it was shown that for S has length of at most $(k^2 - 1)$ and S is $(K - 1)$ positive, then S is completely positive.
The next results provided stronger conditions for S to be completely positive.

Theorem 2.9 *([15]) Suppose an elementary operator S has length at most $(i + 1)^2 - 1$ and S is i-positive then S is completely positive.*

The research was further extended to infinite dimensional spaces and it was proved that $Sy = \sum_{j=1}^{\xi} \mu_j \alpha_j y \alpha_j^*$ is also positive and of minimal length l and S is positive with $i \geq 1$. Wang and Wu [43] researched on positive operators and more particularly compactly strong operators. They heavily utilized the upper and lower spectral radius in there work. In the research they proved that lower spectral radius also serves as the upper spectral radius. In the work, they used the following result to establish the relationship between positive operators, irreducible ideal and the spectral radius, quasi-interior C.

Theorem 2.10 *([40]) For a positive, irreducible ideal, compact operator in a cone with dimension greater than* 1*, we have $r(S) > 0$.*

Additionally, the comparison between ideal irreducible operators semi-strong positivity of operators was done. In the investigation it was established that ideal irreducibility implies semi-strongly positivity.

Theorem 2.11 *([30]) Spectral radii for normal STSMO is equal to its non-symmetrized counterpart.*

The research was extended to the resolvent $(\beta I - S)^{-1}$ of S and it was established that for positive quasi-nilpotent operators, $r_*(S) = r^*(S) = 0$. In our research since cones and Banach spaces can be decomposed in two or more subspace then we investigated spectral properties of positive operators under direct sum. We extended our investigation to power operators and invertible positive operators. We also considered scaled operator in our investigation.

At this point we focus on the Sp in general and also Sp of STSMOs. The spectral analysis of operators provides fundamental contributions into both pure and applied mathematics [37]. It has been used in areas like non-self-adjoint spectral theory, computational methods, and applications in physics and engineering [6].

In cones, an operator's Sp is forms the set of its eigenvalues [45]. The spectral decomposition theorem states that normal operators can be represented using their eigenvalues and eigenvectors. Unlike the compact cones, operators in general cones may have a continuous Sp, making spectral analysis more complex [33]. There are several types of spectra namely: Point Sp, continuous Sp, residual Sp among others [12].

With this broader perspective on Sp of operators, we now consider spectrum of STSMOs implemented by orthogonal isometries which is the main property under consideration in this study. There are several important types of EOs, each with distinct spectral properties. When the underlying Banach space is a HS, the spectral properties of STSMO depend on the adjoint structure.

It is known that Sp is always a conditioned any space to a larger space [16]. Studies involving comparison of spaces and subspaces normally has certain restrictions. Sp being a set also falls under such restrictions. As stated in [7], Sp has this restriction. Given any subspace of a larger space the Sp can fall under this space when the larger space is invariant. The restriction criterion ensures that Sp preserves all its characteristics under invariance. However, this applies for operators on cones. It is therefore interesting to find out if this criterion applies to Sp of STSMO.

Another property of Sp that has been studied is its link to the set of infima that relates to the invertible operators when the invertible operator acts as a unitary operator [43]. It has been shown that the Sp is always conditioned for any space to a larger subspace. Characterization of sets and their infima is considered in [2] for Sp that has been studied and its link to the set of infima that relates to the invertible mapping when the map acts as a unitary one. This result shows that Sr belong to this set of infima. The open question that seeks answers states that: Do STSMOs have Sp sets with infima that contains its Sr?

The character of convex hull is also important for Sp as seen below. The convex hull Sp is always the intersection of the closure of the set that has the operator that operates as the unitary one. As stated in [31], the convex hull Sp is always the intersection of the closure of the set that has the unique operator. The unitary operator ensures that this set is unique. It is important to establish whether this set is also unique in the case of STSMOs.

Recently, a characterization involving Drazin Sp of the tensor products of operators has been given. Research shows that the Drazin Sp is equal to the Sp of the TP of maps. The equality given in [23] applies for general operators in BA. The result shows that The Drazin Sp is equal to the Sp of the TP of maps. This equality is not known for the Sp of STSMOs. It is therefore useful to determine whether this equality holds in the case of EO by comparing the Drazin Sp ofSTSMOs and the Sp of tensor product of STSMOs.
The other aspect of STSMOs worth considering is the Sp. We give a detailed literature that has covered this aspect. We begin with the restriction criterion.

Theorem 2.12 *([31]). The Sp is always conditioned for any space to a cone.*

Studies involving comparison of spaces and subspaces normally has certain restrictions [34]. Sp being a set also falls under such restrictions. As stated in Theorem 2.12, Sp has this restriction. Given any subspace of a larger space the Sp can fall under this space when the larger space is invariant (see[47] for details). The restriction criterion ensures that Sp preserves all its characteristics under invariance. However, this applies for operators on cones only [1]. It is therefore interesting to find out if this criterion applies to Sp of STSMOs.
Another property of Sp that has been studied is its link to the set of infima that relates to the invertible operators when the invertible operator acts as a unitary operator [14].

Theorem 2.13 *([17]). The Sr is in the set of infima of invertible maps*

Characterization of sets and their infima is considered in Proposition 2.13 for Sp that has been studied and its link to the set of infima that relates to the invertible operators when the invertible map operates like unitaries. This result shows that Sr belong to this set of infima. The open question that seeks answers states that: Do STSMOs have Sp sets with infima that contains its Sr?
The character of convex hull is also important for Sp as seen below.

Theorem 2.14 *([13]). The convex hull Sp is always the intersection of the closure of the set that has the map that operates like unitaries.*

As stated in Theorem 2.14, the convex hull Sp is always the intersection of the closure of the set that has the map that operates like unitaries. This shows that it is the smallest set since its an intersection of other sets. The unitary operator ensures that this set is unique. It is important to establish whether this set is also unique in the case of STSMOs as recommended in [46]. Recently, a characterization involving Drazin Sp of the tensor products of operators has been given.

Theorem 2.15 *([4]) The Drazin Sp is equal to the Sp of the TP of maps.*

The equality given in Theorem 2.15 applies for general operators in cones. The result shows that The Drazin Sp is equal to the Sp of the TP of maps. This equality is not known [25] for the Sp of STSMOs. It is therefore useful to determine whether this equality holds in the case of STSMOs by comparing the Drazin Sp of STSMOs and the Sp of tensor product of STSMOs.
In our study we were investigating on essential spectrum characterization of STSMO. This was suggested by [45] on elementary operator and Aluthge transform investigated on various structural properties of elementary operators. In our study we narrowed down to STSMO and found its polar decomposition which helped us to have its spectra in terms of the symbols defining it. The author in [46] studied approximate and defect spectra on analytic elementary operators on $B(H)$ which was parallel result of Lumer-Rosenblums Spectral theorem.
Therefore in our study we studied the structure of two-side multiplication operators which preserve point-spectrum as well as the structure of surjective and non-surjective elementary operators which preserve other subsections of spectra.From this the open question is whether this is true for entire spectrum. Moreover we investigated on the entire spectrum of symmetrized two side multiplication operators induced by orthogonal isometries as suggested by [47].
From spectra theory there is a strong link relating the operator distances and its Sp [44]. For some operators their norm is equal to the spectral radius. In our study we worked on spectra of STSMO where we used the already calculated norms to know the boundary of the spectra.

3 Main results

We carry out our investigation under different technical approaches. We begin by providing the necessary condition that guarantee $(A_1 \oplus ... \oplus A_n, m)-$ spectral analysis. In this section, we investigate various spectral properties of symmetrized two sided multiplication operators. We note that all the STSMO are induced by orthogonal isometries unless stated otherwise.

Proposition 3.1 *Let C and J be conjugate STSMOs on $B(H)$ and suppose $U = CJ$ and $CJ = JC$, then $\delta(U) = \delta(CJ) \subseteq \delta(C)\delta(J)$.*

Proof. Suppose C and J are conjugates, then by Jacobson's Lemma, nonzero points of the spectrum of CJ and JC coincide. Further, since CJ is invertible then it follows that CJ is also invertible. Therefore, $JC = J(PJP^{-1}) = (JP)JP^{-1}$. Since CJ is invertible, then 0 is not in the residual spectrum of C^* hence J is invertible. Finally, $CJ = (CP)P$ and so $\delta(CJ) = \delta(JC) = \delta(PCP)$.

In the next result, we provided the link between point spectrum and the numerical radius.

Lemma 3.2 *Let C and J be conjugate STSMOs on $B(H)$ then $\delta_p(CJ) \subseteq W(CJ)$.*

Proof. Suppose $\overline{\beta} \in \delta_p(CJ)$ then we have that $\overline{\beta}x = CJx$. Therefore, $\overline{\beta} = \overline{\beta}\langle x, x\rangle = \langle \overline{\beta}x, x\rangle = \langle \overline{\lambda}Cx, x\rangle = \langle CJx, x\rangle$. This implies that $\overline{\beta} \in \delta_p(CJ)$. Thus $\delta_p(CJ) \subseteq W(CJ)$.

Again we show that the above result also holds under direct sum.

Theorem 3.3 *Let C_i and J_i be conjugate STSMOs on $B(H)$ then $\delta_p(C_1J_1 \oplus ... \oplus C_nJ_n) \subseteq W(C_1J_1 \oplus ... \oplus C_nJ_n)$.*

Proof. Suppose $(\overline{\beta}, ..., \overline{\beta}) \in \delta(C_1J_1 \oplus ... \oplus C_nJ_n)$, then we have $x_1 + ... + x_n \in H_1 \oplus ... \oplus H_n$ and $(\overline{\beta}, ..., \overline{\beta}) = \delta(C_1J_1) \cup ... \cup \delta(C_nJ_n) = \langle C_1J_1 \oplus ... \oplus C_nJ_nx_1 + ... + x_n, x_1 + ... + x_n\rangle$. Therefore, $(\overline{\beta}, ..., \overline{\beta}) \in W(C_1J_1) \cup ... \cup \delta(C_nJ_n)$.

Next we extend our investigation to summation technique for operators as a consequence.

Corollary 3.4 *Let C_i and J_i be conjugate STSMOs on $B(H)$ and suppose that $U_i = C_iJ_i$ and $U_1 \oplus ... \oplus U_n = C_1J_1 \oplus ... \oplus C_nJ_n$, then $\delta(U_1 \oplus ... \oplus U_n) = \delta(C_1J_1 \oplus ... \oplus C_nJ_n)$.*

Proof. Suppose $C_1 \oplus ... \oplus C_n$ and $J_1 \oplus ... \oplus J_n$ are conjugates, then again by Jacobson's Lemma, the nonzero points of the spectrum of $(C_1J_1 \oplus ... \oplus C_nJ_n)$ and $J_1C_1 \oplus ... \oplus J_nC_n$ are equal. Since C_i and J_i are invertible, then if follows that $J_1C_1 \oplus ... \oplus J_nC_n$ is invertible. Therefore, $\delta(U_1 \oplus ... \oplus U_n) = \delta(C_1J_1 \oplus ... \oplus C_nJ_n)$.

We now examine $(C_1J_1) \cup ... \cup \delta(C_nJ_n)$ and its adjoint.

Proposition 3.5 *Suppose T_i are STSMOs, then $(C_1 \oplus ... \oplus C_n)(T_1 \oplus ... \oplus T_n)$ commutes with the spectral measure of $(T_1 \oplus ... \oplus T_n)^*(T_1 \oplus ... \oplus T_n)$.*

Proof. Since $(C_1 \oplus ... \oplus C_n)(T_1 \oplus ... \oplus T_n)$ commutes with its adjoint, then its square is also a commutant. Therefore, $(C_1 \oplus ... \oplus C_n)(T_1 \oplus ... \oplus T_n)$ also commutes with $p(T_1 \oplus ... \oplus T_n)^*(T_1 \oplus ... \oplus T_n)$ where p is a polynomial.

At this point, we examine the spectral properties of elementary operators within the norms.

Lemma 3.6 *Let X_i be uniformly convex Banach spaces and $T_i \in B(X_i)$ then $\| (I_1 \oplus ... \oplus I_n) - (T_1 \oplus ... \oplus T_n) \| = 1 + \| (T_1 \oplus ... \oplus T_n) \|$ if $\| (T_1 \oplus ... \oplus T_n) \| \in \delta(T_1 \oplus ... \oplus T_n)$.*

Proof. Suppose that $T_i : i = 1, ..., n$ are elementary operators, then they are all bounded linear operators on the uniformly convex Banach space. So, $\| (I_1 \oplus ... \oplus I_n) - (T_1 \oplus ... \oplus T_n) \| = 1 + \| (T_1 \oplus ... \oplus T_n) \|$ iff $\| (T_1 \oplus ... \oplus T_n) \| \in \delta_{ap}(T_1 \oplus ... \oplus T_n)$. Now, by containment for all elementary operators T_i, $\delta_{ap}(T_1 \oplus ... \oplus T_n) \subseteq \delta(T_1 \oplus ... \oplus T_n)$. Therefore, $\| (T_1 \oplus ... \oplus T_n) \| \in \delta(T_1 \oplus ... \oplus T_n)$ iff $\| (T_1 \oplus ... \oplus T_n) \|$ is in the numerical range of $(T_1 \oplus ... \oplus T_n)$.

We extend our investigation to the scaled elementary operators.

Lemma 3.7 *Let T_i, S_i on $B(H_i)$, $i = 1, ..., n$ be STSMOs and that $1 < p < \infty$. Suppose $\| (I_1 \oplus ... \oplus I_n) - M_{p,(T_1 \oplus ... \oplus T_n),(S_1 \oplus ... \oplus S_n)} \| = 1 + \| (T_1 \oplus ... \oplus T_n) \| \| (S_1 \oplus ... \oplus S_n) \|$, then for all $\lambda \in \mathcal{C}$ with $| \lambda | = 1, \lambda \| (T_1 \oplus ... \oplus T_n) \| \in \delta(T_1 \oplus ... \oplus T_n)$ and $\overline{\lambda} \| (S_1 \oplus ... \oplus S_n) \| \in \delta(S_1 \oplus ... \oplus S_n)$.*

Proof. Suppose $1 < p < \infty$, then $C_p(H_1 \oplus ... \oplus H_n)$ is uniformly convex. Now since T_i and S_i are STSMOs, then we have $\| (I_1 \oplus ... \oplus I_n) - (T_1 \oplus ... \oplus T_n)(S_1 \oplus ... \oplus S_n) \| = 1 + \| (T_1 \oplus ... \oplus T_n)(S_1 \oplus ... \oplus S_n) \| = 1 + \| (T_1 \oplus ... \oplus T_n) \| \| (S_1 \oplus ... \oplus S_n) \|$. Thus $\| (T_1 \oplus ... \oplus T_n) \| \| (S_1 \oplus ... \oplus S_n) \| \in \delta_{ap}((T_1 \oplus ... \oplus T_n)(S_1 \oplus ... \oplus S_n)$. This implies that $\| (T_1 \oplus ... \oplus T_n) \| \| (S_1 \oplus ... \oplus S_n) \| \in \delta(M_{p,(T_1 \oplus ... \oplus T_n),(S_1 \oplus ... \oplus S_n)}$. We note that $\delta(M_{p,(T_1 \oplus ... \oplus T_n),(S_1 \oplus ... \oplus S_n)}) = \delta(T_1 \oplus ... \oplus T_n)\delta(S_1 \oplus ... \oplus S_n)$ Therefore, for $\lambda \mathcal{C}$ such that $| \lambda | = 1$, we have $\lambda \| (T_1 \oplus ... \oplus T_n) \| \in \delta(T_1 \oplus ... \oplus T_n)$ and $\overline{\lambda} \| (S_1 \oplus ... \oplus S_n) \| \in \delta(S_1 \oplus ... \oplus S_n)$.

In the next result, we show that under certain conditions the spectrum of scaled and non-scaled operators coincide.

Theorem 3.8 *Let $M_{T,S}$ be a STSMO. Let $T, S \in B(H)$ and $| \lambda | = 1$ be a scalar such that $TT^{*2}T = \lambda T^* T^2 T^*$, then $\| TT^{*2}T \|$ and $\| T^* T^2 T^* \|$ coincide.*

Proof. Suppose T is a STSMO, then we have $\| (I - TT^{*2}T) \| = 1 - \| TT^{*2}T \|$. Since $\| TT^{*2}T \| \in \delta_{ap}(TT^{*2}T)$, then it follows that for any $\lambda \in \mathcal{C}$ and for $| \lambda | = 1$, we have that $\| TT^{*2}T \| = | \lambda | \| T^* T^2 T^* \|$. This implies that $\| TT^{*2}T \| = \| T^* T^2 T^* \|$. Therefore, from $\| (I - TT^{*2}T) \| = 1 - \| TT^{*2}T \| = 1 - \| T^* T^2 T^* \|$. This implies $\| T^* T^2 T^* \| \in \delta_{ap}(T^* T^2 T^*)$. Hence, $\| T^* T^2 T^* \| \in \delta(T^* T^2 T^*)$.

In the next result, we provide conditions under which STSMOs have real spectrum.

Theorem 3.9 *Let T be a STSMO. Let $T = T^*$ and also let $x \perp y$, then the spectrum of T is real.*

Proof. Let T be self adjoint and $x, y \in H$ be nonzero vectors, then $\lambda \langle x, x \rangle = \langle x, Tx \rangle = \langle T^* x, x \rangle = \overline{\lambda} \langle x, x \rangle$. Therefore, if $\lambda \in \mathcal{R}$, then $\lambda = \overline{\lambda}$. Now suppose that $Ty = \beta y$, then we have that $\lambda \langle x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle = \langle x, Ty \rangle =$

$\beta\langle x, y\rangle$. Now, $\lambda\langle x, y\rangle = \beta\langle x, y\rangle$ implies that $(\lambda - \beta)\langle x, y\rangle) = 0$. Since $\lambda \neq \beta$, then it follows that $\langle x, y\rangle = 0$ which implies that $x \perp y$.

Next we examine essential spectral properties of quasi-normal operators after establishing a condition that ensures that T is a STSMO.

Corollary 3.10 *Let T on $B(H)$ be quasi-normal and suppose that $TTT = TT^2$, then $\delta_e(TTT) = \delta_e(\lambda TTT) = \lambda\delta_e(TTT)$.*

Proof. If T is quasi-normal, then TTT is quasi-normal and so $(TTT)^2$ is also quasi-normal. Therefore, both TTT and $(TTT)^*$ are $p-$hyponormal. Thus λTTT and TTT are also quasi-normal. Since it is well established that T_1 and T_2 are quasi-similar then they are quasi-normal. So, $\delta_e(T_1) = \delta_e(T_2)$. Hence, $\delta_e(TTT) = \delta_e(\lambda TTT) = \lambda\delta_e(TTT)$.

Extension of our investigation to commutants of STSMOs is carried out and we investigate the approximate point spectrum of such operators.

Proposition 3.11 *Let T and S be commutants, suppose $M_{TS}(X) = TXS + SXT$, then $\delta(TS) = \delta(ST) = \delta_\pi(TS)$.*

Proof. Let $\lambda = 1$ and $A = TS, A^* = ST^*$. Then $A^*A = ST^*TS = STT^*S$. Now if $Q, R \in B(H)$ and $QR = RT$ where say R is normal, then $QR^* = R^*T$ then we have $T^*S = ST^*$. Thus $A^*A = TSST^*$. But $TSST^* = AA^*$ This implies that $A^*A = AA^*$. Therefore A is normal. Consequently $\delta(TS) = \delta(ST) = \delta_\pi(TS) = \lambda\delta_\pi(TS)$.

We now investigate the spectral properties when they are STSMOs are induced by $2-$isometries.

Lemma 3.12 *Let T_i be a symmetrized two side multiplication $2-$ isometries. Then $1 \in \delta(T_1 \oplus ... \oplus T_n)^*(T_1 \oplus ... \oplus T_n)$.*

Proof. We proof this by contradiction. Let 1 is not in $\delta(T_1\oplus...\oplus T_n)^*(T_1\oplus...\oplus T_n)$, then it follows that $A_1\oplus...\oplus A_n = (T_1\oplus...\oplus T_n)^*(T_1\oplus...\oplus T_n) - (I_1\oplus...\oplus I_n)$ is invertible. Now by definition of $2-$ isometry, $\delta(T_1 \oplus ... \oplus T_n)^*(A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n) = \delta(A_1 \oplus ... \oplus A_n)$. Alternatively, $(A_1 \oplus ... \oplus A_n)^{\frac{1}{2}}(T_1 \oplus ... \oplus T_n)(A_1 \oplus ... \oplus A_n)^{-\frac{1}{2}})^*(A_1 \oplus ... \oplus A_n)^{\frac{1}{2}}(T_1 \oplus ... \oplus T_n)(A_1 \oplus ... \oplus A_n)^{\frac{1}{2}} = I_1\oplus...\oplus I_n$. So, $T_1\oplus...\oplus T_n$ is is semi-isometric and hence isometric. Therefore, $\delta(I_1 \oplus ... \oplus I_n) = \delta(I_1) \cup ... \cup \delta(I_1) = \{1\} \cup ... \cup \{1\} = \{1\}$. This implies that $1 \in \delta(T_1 \oplus ... \oplus T_n)^*(T_1 \oplus ... \oplus T_n)$.

Next, we examine the spectral properties of STSMOs which are binormal.

Theorem 3.13 *Let $M_{T,S}$ be a STSMO. Suppose that for $\mid \lambda \mid= 1$ and $TS^{*2}T = \lambda T^*S^2T^*$. Then $\beta^4 \in \delta(T^*S^2T)$.*

Proof. Let $\beta \in \delta(T)$, then by self adjointedness of T we get $T^*S^2T = TS^2T = T^4$. Furthermore, since $\mid \lambda \mid = 1$ then also $\lambda T^*S^2T^* = T^4$. Now if $\beta \in \delta(T)$ then for any $n \in \mathcal{N}$, we have $\beta^n \in \delta(T^n)$. Therefore, $\beta^4 \in \delta(T^*S^2T)$.

4 Open Problems

Characterization of Symmetrized Two-Sided Multiplication Operators have been done over years in-terms of their properties which include numerical ranges and norms among others. However, characterizing the spectrum and norms have not been exhausted still remains interesting. There exists an open question requiring the determination of the norms of elementary operators in a general Banach space setting. Since there is a strong relation ship between the norm and spectral radius, therefore it was of great significance for this study to characterize spectral properties of Symmetrized Two-Sided Multiplication Operators as an avenue that helps in solving the open problem.
Problem 1: Can one develop an efficient algorithm for determining spectra Symmetrized Two-Sided Multiplication Operators as characterized here?
Problem 2: Do Symmetrized Two-Sided Multiplication Operators preserve distance even when they are subjected to tensor norms?

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