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Soft Union Weak-interior Ideals of Semigroups

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Abstract

It has shown to be interesting and beneficial for mathematicians to generalize the ideals of an algebraic structure. In this context, the concept of weak-interior ideal was presented as a generalization of quasi-ideal, interior ideal, and (left/right) ideal of a semigroup. In this paper, we transfer this concept to soft set theory and introduce a novel type of soft union (S-uni) ideal called "soft union (S-uni) weak-interior ideal”. The main goal of this study is to obtain the relations between S-uni weak-interior ideals and other certain types of S-uni ideals of a semigroup. Our findings indicate that an S-uni weak-interior ideal is a generalization of an S-uni ideal and interior ideal; however, the converses are true under certain conditions. Furthermore, we demonstrate that the S-uni bi-ideals and S-uni quasi-ideals coincide with weak-interior ideals of a group. Our key theorem, which shows that if a subsemigroup of a semigroup is a weak-interior ideal, then its soft characteristic function is an S-uni weak-interior ideal, and vice versa, allows us to build a bridge between semigroup and soft set theory. Besides, we provide some conceptual analysis of the concept in terms of soft set operations, and the soft anti and soft inverse image by backing up our claims with informative examples.

Keywords: *Semigroup, Simple Semigroup, Soft Set, Soft Union Weak-interior Ideals, Weak-interior Ideals.*

1 Introduction

Semigroups play a fundamental role in many branches of mathematics as they give the abstract algebraic basis for "memoryless" systems, which restart on each iteration. In practical mathematics, semigroups-which were first investigated formally in the early 1900s-are essential models for linear time-invariant systems. Since finite semigroups are inextricably related to finite automata, studying them is critical in theoretical computer science. Additionally, in probability theory, semigroups and Markov processes are related. The concept of ideals is crucial to understanding the mathematical structures and their applications, thus many mathematicians have focused most of their research on generalizing ideals in algebraic structures. Namely, further study of algebraic structures requires the generalization of ideals in algebraic structures. Dedekind established the idea of ideals for the theory of algebraic numbers, and Noether expanded it to include associative rings. The concept of a one-sided ideal of any algebraic structure is an extension of the idea of an ideal, and the one-sided and two-sided ideals are still fundamental ideas in ring theory.

In 1952, the concept of bi-ideals was introduced by Good and Hughes [1] for semigroups. Steinfeld [2] first presented the notion of quasi-ideals for semigroups and then for rings. Quasi-ideals are generalizations of right ideals and left ideals whereas bi-ideals are generalizations of quasi-ideals. The concept of interior ideal was first demonstrated by Lajos [3] and further studied by Szasz [4,5]. Interior ideals are generalizations of ideals. Rao [6-9] has developed several novel new types of ideals of semigroup, which are generalizations of the ones that already exist, such as bi-interior ideals, bi-quasi ideals, quasi-interior ideals, weak-interior ideals, and bi-quasi-interior ideals. Furthermore, the idea of essential ideals in semigroups was proposed by Baupradist et al. [10]. As a more generalized concept of the different types of ideals, the concept of “almost” ideals was proposed, and their characteristics and their relations between the related ideals were thoroughly examined. In this context, in [11], the concept of almost ideals of semigroups was first put up. A subsequent paper [12] extended the concept of bi-ideals to almost bi-ideals of semigroups. While the notion of almost quasi-ideals was first introduced in [13], by proposing almost interior ideals and weakly almost interior ideals of semigroups, the ideas of almost ideals and interior ideals of semigroups were expanded and studied in [14]. Different types of almost ideals of semigroups were proposed by the authors in [15–18]. Furthermore, in [13, 15–20], several fuzzy almost ideal types for semigroups were investigated.

Molodtsov [21] introduced the “Soft Set Theory” in 1999 to understand problems involving uncertainty and to find suitable solutions for them. Since then, many significant studies have been conducted on concepts related to soft sets, especially concerning soft set operations. Maji et al. [22] presented some definitions related to soft sets and defined certain operations on soft sets. Pei and Miao [23] and Ali et al. [24] introduced various operations of soft sets. Sezgin and Atagün [25] worked on soft set operations. For more about soft set operations which have been popular since its inception, we refer to [26-37]. The concept and operations of soft sets were modified by Çağman and Enginoğlu [38]. Çağman et al. [39] developed the concept of soft intersection groups, which led to the investigation of various soft algebraic systems. Sezgin [40], using soft sets in the application of semigroup theory, defined soft union (S-uni) semigroups, left (right/two-sided) ideals, and bi-ideals of semigroups; Sezgin et al. [41] defined S-uni interior ideals, quasi-ideals, and generalized bi-ideals of semigroups, and thoroughly examined their fundamental properties. In terms of the S-uni substructures of semigroups, Sezer et al. [42] defined and classified certain kinds of semigroups. In [43] certain kinds of regularities of semigroups are characterized by soft union quasi-ideals, soft union (generalized) bi-ideals, and soft union semiprime ideals of a semigroup. As a generalization of different types of soft intersection ideals, soft intersection almost ideals were proposed and studied in [44-55]. The soft forms of various algebraic structures have been studied in [56-68].

Rao [9] introduced the notion of weak-interior ideals as a generalization of quasi-ideal, interior ideal, left (right) ideal, and ideal of semigroup and studied the properties of weak-interior ideals of semigroup. The concept of weak-interior ideals has also been studied by Rao [69] for Γ-semirings, Rao and Rao [70] for Γ-semigroups, and Rao [71] for semirings. In this paper, we transfer this concept to soft set theory and semigroups by presenting "soft union (S-uni) weak-interior ideals of semigroups". We obtain the relations between S-uni weak-interior ideals and other types of S-uni ideals of a semigroup. Our results show every S-uni weak-interior ideal of a regular semigroup is an S-uni subsemigroup, and S-uni weak-interior ideal is a generalization of S-uni ideal and S-uni interior ideal. We also show that every idempotent S-uni weak-interior ideal is both an S-uni ideal and S-uni interior ideal and every S-uni weak-interior ideal is both an S-uni ideal and S-uni interior ideal of a group. Furthermore, we show that S-uni bi-ideals and S-uni quasi-ideals both correspond to S-uni weak-interior ideals of a group. Our essential theorem, which states that if a subsemigroup of a semigroup is a weak-interior ideal, then its soft characteristic function is an S-uni weak-interior ideal, and vice versa, enables us to bridge the gap between semigroup theory and soft set theory. We demonstrate how this idea connects to the current algebraic structures in classical semigroup theory by using this theorem. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of soft set operations, soft anti image, and soft inverse image, supporting our assertions with particular, illuminating examples. The paper is organized into four sections. Section 1 provides an overview of the subject, while Section 2 delves into the basic concept of semigroup and soft set ideals, as well as their relevant definitions and consequences. In Section 3, we propose the concept of S-uni weak-interior ideals and analyze its properties as well as their relationships with other types of S-uni ideals using concrete examples. Section 4 summarizes our findings and discusses the potential future research.

2 Preliminaries

Throughout this paper, S denotes a semigroup. A nonempty subset L of S is called a subsemigroup of S if $LL \subseteq L$, is called a bi-ideal of S if $LL \subseteq L$ and $LSL \subseteq L$, is called an interior ideal of S if $LL \subseteq L$ and $SLS \subseteq L$, and is called a quasi-ideal of S if $LS \cap SL \subseteq L$. A subsemigroup L of S is called a left weak-interior ideal of S (left WI-ideal) if $SLL \subseteq L$, is called a right weak-interior ideal of S (right WI-ideal) if $LLS \subseteq L$, and is called a weak-interior ideal of S (WI-ideal) if it is both left WI-ideal and right WI-ideal [9]. If S is a regular semigroup, then for all $x \in S$, there exists an element $y \in S$ such that $x = xyx$. A semigroup S is called left simple (L-simple) if it contains no proper left ideal of S, is called right simple (R-simple) if it contains no proper right ideal of S, and is called simple if it contains no proper ideal.

Theorem 2.1 [72, 73]. *Let S be a semigroup. Then,*

(1) S is L-simple (R-simple) iff $S\text{ռ} = S$ $(\text{ռ}S = S)$ for all $\text{ռ} \in S$. That is, for every $\text{ռ}, \text{ϭ} \in S$, there exists $\text{ɷ} \in S$ such that $\text{ϭ} = \text{ɷռ}$ $(\text{ϭ} = \text{ռɷ})$

(2) S is both L-simple and R-simple iff S is a group.

Definition 2.2 [21, 38]. *Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and $\Upsilon \subseteq E$. The soft set (SS) f_Υ over U is a function such that $f_\Upsilon: E \rightarrow P(U)$, where for all $x \notin \Upsilon$, $f_\Upsilon(x) = \emptyset$. That is, $f_\Upsilon = \{(x, f_\Upsilon(x)): x \in E, f_\Upsilon(x) \in P(U)\}$.*

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.3 [38]. *Let $f_\mathcal{H} \in S_E(U)$. If $f_\mathcal{H}(x) = \emptyset$ for all $x \in E$, then $f_\mathcal{H}$ is called a null SS and indicated by $\emptyset_E$.*

Definition 2.4 [38]. *Let $f_\mathcal{H}, f_\aleph \in S_E(U)$. If $f_\mathcal{H}(x) \subseteq f_\aleph(x)$, for all $x \in E$, then $f_\mathcal{H}$ is a soft subset of $f_\aleph$ and indicated by $f_\mathcal{H} \,\tilde{\subseteq}\, f_\aleph$. If $f_\mathcal{H}(x) = f_\aleph(x)$, for all $x \in E$, then $f_\mathcal{H}$ is called soft equal to $f_\aleph$ and denoted by $f_\mathcal{H} = f_\aleph$.*

Definition 2.5 [38]. *Let $f_\mathcal{H}, f_\aleph \in S_E(U)$. The union (intersection) of $f_\mathcal{H}$ and $f_\aleph$ is the SS $f_\mathcal{H} \,\tilde{\cup}\, f_\aleph$ $(f_\mathcal{H} \,\tilde{\cap}\, f_\aleph)$, where $(f_\mathcal{H} \,\tilde{\cup}\, f_\aleph)(w) = f_\mathcal{H}(w) \cup f_\aleph(w)$ $\big((f_\mathcal{H} \,\tilde{\cap}\, f_\aleph)(w) = f_\mathcal{H}(w) \cap f_\aleph(w)\big)$, for all $w \in E$, respectively.*

Definition 2.6 [38]. *Let $f_\mathcal{H}, f_\aleph \in S_E(U)$. Then, $\vee$-product ($\wedge$-product) of $f_\mathcal{H}$ and $f_\aleph$, denoted by $f_\mathcal{H} \vee f_\aleph$ $(f_\mathcal{H} \wedge f_\aleph)$ is defined by $(f_\mathcal{H} \vee f_\aleph)(\eta, \text{v}) = f_\mathcal{H}(\eta) \cup f_\aleph(\text{v})$ $\big((f_\mathcal{H} \wedge f_\aleph)(\eta, \text{v}) = f_\mathcal{H}(\eta) \cap f_\aleph(\text{v})\big)$ for all $(\eta, \text{v}) \in E \times E$, respectively.*

Definition 2.7 [39]. *Let $f_\mathcal{H}, f_\aleph \in S_E(U)$ and ϕ be a function from $\mathcal{H}$ to $\aleph$. Then, soft anti image of $f_\mathcal{H}$ under ϕ, and soft pre-image (or soft inverse image) of $f_\aleph$ under ϕ are the SSs $\phi(f_\mathcal{H})$ and $\phi^{-1}(f_\aleph)$ such that*

$$\left(\phi^*(f_{\mathcal{H}})\right)(\mathrm{v}) = \begin{cases} \bigcup \{f_{\mathcal{H}}(\mathrm{e}) \mid \mathrm{e} \in \mathcal{H} \text{ and } \phi(\mathrm{e}) = \mathrm{v}\}, & \text{if } \phi^{-1}(\mathrm{v}) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $\mathrm{v} \in \aleph$ *and* $\left(\phi^{-1}(f_{\aleph})\right)(\mathrm{e}) = f_{\aleph}(\phi(\mathrm{e}))$ *for all* $\mathrm{e} \in \mathcal{H}$.

Definition 2.8 [40]. *Let* $f_{\mathcal{H}} \in S_E(U)$ *and* $\alpha \subseteq U$. *Then, lower* α*-inclusion of* $f_{\mathcal{H}}$, *denoted by* $\mathcal{L}(f_{\mathcal{H}}; \alpha)$, *is defined as* $\mathcal{L}(f_{\mathcal{H}}; \alpha) = \{x \in \mathcal{H} \mid f_{\mathcal{H}}(x) \subseteq \alpha\}$.

Definition 2.9 [40]. *Let* $\hbar_S, \mathrm{б}_S \in S_S(U)$. *S-uni product* $\hbar_S * \mathrm{б}_S$ *is defined by*

$$(\hbar_S * \mathrm{б}_S)(\eta) = \begin{cases} \bigcap_{\eta = \mathrm{wd}} \{\hbar_S(\mathrm{w}) \cup \mathrm{б}_S(\mathrm{d})\}, & \text{if } \exists \mathrm{w}, \mathrm{d} \in S \text{ such that } \eta = \mathrm{wd} \\ U, & \text{otherwise} \end{cases}$$

Theorem 2.10 [40]. *Let* $p_S, \omega_S, \mu_S \in S_S(U)$. *Then,*

i. $(p_S * \omega_S) * \mu_S = p_S * (\omega_S * \mu_S)$
ii. $p_S * \omega_S \neq p_S * \omega_S$, generally.
iii. $p_S * (\omega_S \tilde{\cup} \mu_S) = (p_S * \omega_S) \tilde{\cup} (p_S * \mu_S)$ and $(p_S \tilde{\cup} \omega_S) * \mu_S = (p_S * \mu_S) \tilde{\cup} (\omega_S * \mu_S)$
iv. $p_S * (\omega_S \tilde{\cap} \mu_S) = (p_S * \omega_S) \tilde{\cap} (p_S * \mu_S)$ and $(p_S \tilde{\cap} \omega_S) * \mu_S = (p_S * \mu_S) \tilde{\cap} (\omega_S * \mu_S)$
v. If $p_S \tilde{\subseteq} \omega_S$, then $p_S * \mu_S \tilde{\subseteq} \omega_S * \mu_S$ and $\mu_S * p_S \tilde{\subseteq} \mu_S * \omega_S$
vi. If $\mathfrak{H}_S, y_S \in S_S(U)$ such that $\mathfrak{H}_S \tilde{\subseteq} p_S$ and $y_S \tilde{\subseteq} \omega_S$, then $\mathfrak{H}_S * y_S \tilde{\subseteq} p_S * \omega_S$.

Definition 2.11 [40]. *Let* $\mathrm{B} \subseteq S$. *We denote by* ζ_{B^c} *the soft characteristic function of the complement* B *and it is defined as*

$$\zeta_{\mathrm{B}^c}(\upsilon) = \begin{cases} U, & \text{if } \upsilon \in S \backslash \mathrm{B} \\ \emptyset, & \text{if } \upsilon \in \mathrm{B} \end{cases}$$

Theorem 2.12 [40]. *Let* $\emptyset \neq \mathcal{H}, \mathcal{M} \subseteq S$. *Then,*

i. If $\mathcal{H} \subseteq \mathcal{M}$, then $\zeta_{\mathcal{H}^c} \tilde{\subseteq} \zeta_{\mathcal{M}^c}$.
ii. $\zeta_{\mathcal{H}^c} \tilde{\cap} \zeta_{\mathcal{M}^c} = \zeta_{\mathcal{H}^c \cap \mathcal{M}^c}$ and $\zeta_{\mathcal{H}^c} \tilde{\cup} \zeta_{\mathcal{M}^c} = \zeta_{\mathcal{H}^c \cup \mathcal{M}^c}$.

Definition 2.13 [40]. *An* SS $\varkappa_S$ *over* U *is called a soft union (S-uni) subsemigroup of* S *if* $\varkappa_S(xy) \subseteq \varkappa_S(x) \cup \varkappa_S(y)$ *for all* $x, y \in S$.

Here note that in [40], the definition of "soft union subsemigroup of *S*" is given as "soft union semigroup of *S*"; however in this paper, we prefer to use "soft union (S-uni) subsemigroup of *S*". Also, from now on, we prefer to use "S-uni" instead of "soft union".

Definition 2.14 [40, 41]. *An* SS $\varkappa_S$ *over* U *is called an S-uni left (right) ideal of* S *if* $\varkappa_S(\upsilon z) \subseteq \varkappa_S(z)$ $\left(\varkappa_S(\upsilon z) \subseteq \varkappa_S(\upsilon)\right)$ *for all* $\upsilon, z \in S$, *and is called an S-uni two-sided ideal (S-uni ideal) of* S *if it is both S-uni left ideal of* S *over* U *and S-uni right ideal of* S *over* U. *An S-uni subsemigroup* $\varkappa_S$ *is called an S-uni bi-ideal of* S *if*

$\varkappa_S(\upsilon\eta\hbar) \subseteq \varkappa_S(\upsilon) \cup \varkappa_S(\hbar)$ for all $\upsilon, \eta, \hbar \in S$. An SS $\varkappa_S$ over U is called an S-uni interior ideal of S if $\varkappa_S(\upsilon\eta\hbar) \subseteq \varkappa_S(\eta)$ for all $\upsilon, \eta, \hbar \in S$.

It is easy to see that if $\varkappa_S(x) = \emptyset$ for all $x \in S$, then $\varkappa_S$ is an S-uni subsemigroup (left ideal, right ideal, ideal, bi-ideal, interior ideal). We denote such a kind of S-uni subsemigroup (left ideal, right ideal, ideal, bi-ideal, interior ideal) by $\tilde{\theta}$. It is obvious that $\tilde{\theta} = \zeta_{S^C}$, that is, $\tilde{\theta}(x) = \emptyset$ for all $x \in S$ [40, 41].

Definition 2.15 [41]. *An SS $\varkappa_S$ over U is called an S-uni quasi-ideal of S over U if $(\tilde{\theta} * \varkappa_S) \,\tilde{\cup}\, (\varkappa_S * \tilde{\theta}) \,\tilde{\supseteq}\, \varkappa_S$.*

Theorem 2.16 [40]. *Let $\varkappa_S \in S_S(U)$. Then,*

i) $\tilde{\theta} * \tilde{\theta} \,\tilde{\supseteq}\, \tilde{\theta}$

ii) $\tilde{\theta} * \varkappa_S \,\tilde{\supseteq}\, \tilde{\theta}$ and $\varkappa_S * \tilde{\theta} \,\tilde{\supseteq}\, \tilde{\theta}$

iii) $\varkappa_S \,\tilde{\cap}\, \tilde{\theta} = \tilde{\theta}$ and $\varkappa_S \,\tilde{\cup}\, \tilde{\theta} = \varkappa_S$

Theorem 2.17 [40, 41]. *Let $\varkappa_S \in S_S(U)$. Then,*

(1) $\varkappa_S$ is an S-uni subsemigroup iff $\varkappa_S * \varkappa_S \,\tilde{\supseteq}\, \varkappa_S$

(2) $\varkappa_S$ is an S-uni left (right) ideal iff $\tilde{\theta} * \varkappa_S \,\tilde{\supseteq}\, \varkappa_S$ $(\varkappa_S * \tilde{\theta} \,\tilde{\supseteq}\, \varkappa_S)$

(3) $\varkappa_S$ is an S-uni bi-ideal iff $\varkappa_S * \varkappa_S \,\tilde{\supseteq}\, \varkappa_S$ and $\varkappa_S * \tilde{\theta} * \varkappa_S \,\tilde{\supseteq}\, \varkappa_S$

(4) $\varkappa_S$ is an S-uni interior ideal iff $\tilde{\theta} * \varkappa_S * \tilde{\theta} \,\tilde{\supseteq}\, \varkappa_S$

Theorem 2.18 [40, 41].

(1) Every S-uni left (right/two-sided) ideal is an S-uni subsemigroup (S-uni bi-ideal/S-uni quasi-ideal).

(2) Every S-uni ideal is an S-uni interior ideal.

(3) Every S-uni quasi-ideal is an S-uni subsemigroup (S-uni bi-ideal).

Proposition 2.19 [40]. *Let $f_S \in S_S(U)$, α be a subset of U, $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni subsemigroup, then $\mathfrak{F}(f_S;\alpha)$ is a subsemigroup.*

Proposition 2.20 [42]. *Every S-uni bi-ideal is an S-uni right ideal of an L-simple semigroup.*

Theorem 2.21 [40]. *$\emptyset \neq R \subseteq S$ is a subsemigroup iff the SS $\mathfrak{f}_S$ defined by*

$$\mathfrak{f}_S(m) = \begin{cases} \alpha, & if\ m \in S \setminus R \\ \beta, & if\ m \in R \end{cases}$$

is an S-uni subsemigroup, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

For more about soft int-groups and soft cosets, we refer to [74].

3 Soft Union Weak-interior Ideals of Semigroups

In this section, we introduced soft union weak-interior ideals of semigroups, gave examples, examined in detail their relations with other soft union ideals, and analyzed in terms of some soft set concepts and operations.

Definition 3.1. *An SS f_S over U is called soft union (S-uni) left (right) weak-interior ideal of S over U if $f_S(xyz) \subseteq f_S(y) \cup f_S(z)$ $\big(f_S(xyz) \subseteq f_S(x) \cup f_S(y)\big)$ for all $x, y, z \in S$. An SS over U is called an S-uni weak-interior ideal of S if it is both S-uni left weak-interior ideal and S-uni right weak-interior ideal of S over U.*

Hereafter, S-uni left (right) weak-interior ideal of S over U is denoted by S-uni left (right) WI-ideal for brevity.

Example 3.2. Consider the semigroup $S = \{$Ծ, ᖷ, Ѧ, Ꝏ$\}$ defined by the following table:

Table 1: Cayley table with "·" binary operation

·	Ծ	ᖷ	Ѧ	Ꝏ
Ծ	Ծ	Ծ	Ծ	Ծ
ᖷ	Ծ	Ծ	Ծ	Ծ
Ѧ	Ծ	Ծ	ᖷ	Ծ
Ꝏ	Ծ	Ծ	ᖷ	ᖷ

Let $\mathfrak{f}_S$ and g_S be SSs over $U = S_3$ as follows:

$\mathfrak{f}_S = \{($Ծ$, \{(1)\}), ($ᖷ$, \{(1), (12)\}), ($Ѧ$, \{(1), (13)\}), ($Ꝏ$, \{(1), (23)\})\}$

$g_S = \{($Ծ$, \{(1), (123), (132)\}), ($ᖷ$, \{(12)\}), ($Ѧ$, \{(13)\}), ($Ꝏ$, \{(23)\})\}$

Then, $\mathfrak{f}_S$ is an S-uni WI-ideal. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$\mathfrak{f}_S($ԾᖷѦ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($ᖷ$) \cup \mathfrak{f}_S($Ѧ$)$, $\mathfrak{f}_S($ꝎꝎꝎ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($Ꝏ$) \cup \mathfrak{f}_S($Ꝏ$)$

$\mathfrak{f}_S($ѦѦԾ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($Ѧ$) \cup \mathfrak{f}_S($Ծ$)$

It can be easily shown that the SS $\mathfrak{f}_S$ satisfies the S-uni left WI-ideal condition for all other element combinations of the set S. Similarly,

$\mathfrak{f}_S($ᖷѦꝎ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($ᖷ$) \cup \mathfrak{f}_S($Ѧ$)$, $\mathfrak{f}_S($ꝎꝎѦ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($Ꝏ$) \cup \mathfrak{f}_S($Ꝏ$)$

$\mathfrak{f}_S($ꝎѦᖷ$) = \mathfrak{f}_S($Ծ$) \subseteq \mathfrak{f}_S($Ꝏ$) \cup \mathfrak{f}_S($Ѧ$)$

It can be easily shown that the SS $\mathfrak{f}_S$ satisfies the S-uni right WI-ideal condition for all other element combinations of the set S, thus $\mathfrak{f}_S$ is an S-uni WI-ideal. However, since $g_S($ѦѦѦ$) = g_S($Ծ$) \nsubseteq g_S($Ѧ$) \cup g_S($Ѧ$)$, g_S is not an S-uni WI-ideal.

It is well known that a subsemigroup Ƀ of a semigroup S is a left (right) WI-ideal if SɃɃ $\subseteq$ Ƀ (ɃɃ$S \subseteq$ Ƀ). It is natural to extend this property to semigroup theory with Proposition 3.3, Proposition 3.4, and Theorem 3.5.

Proposition 3.3. *Let ꝑ$_S \in S_S(U)$. Then, ꝑ$_S$ is an S-uni left WI-ideal iff $\tilde{\theta} *$ ꝑ$_S *$ ꝑ$_S \,\tilde{\supseteq}\,$ ꝑ$_S$.*

Proof: Suppose that ꝥ_S is an S-uni left WI-ideal and $a \in S$. If $(\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S)(a) = U$, then $\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Otherwise, there exist elements $x, y, p, q \in S$ such that $a = xy$ and $y = pq$, for $a \in S$. Since ꝥ_S is an S-uni left WI-ideal, $\text{ꝥ}_S(a) = \text{ꝥ}_S(xy) = \text{ꝥ}_S((pq)y) \subseteq \text{ꝥ}_S(q) \cup \text{ꝥ}_S(y)$. Therefore,

$$(\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S)(a) = [(\tilde{\theta} * \text{ꝥ}_S) * \text{ꝥ}_S](a) = \bigcap_{a=xy} \{(\tilde{\theta} * \text{ꝥ}_S)(x) \cup \text{ꝥ}_S(y)\}$$

$$= \bigcap_{a=xy} \left\{ \bigcap_{x=pq} \{\tilde{\theta}(p) \cup \text{ꝥ}_S(q)\} \cup \text{ꝥ}_S(y) \right\} = \bigcap_{a=pqy} \{\text{ꝥ}_S(q) \cup \text{ꝥ}_S(y)\}$$

$$\supseteq \bigcap_{a=pqy} \{\text{ꝥ}_S(pqy)\} = \text{ꝥ}_S(xy) = \text{ꝥ}_S(a)$$

Thus, we have $\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Moreover, in the case where $a = xy$ and $x \neq pq$ for $a \in S$, since $(\tilde{\theta} * \text{ꝥ}_S)(x) = U$, $\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$ is satisfied.

Conversely, assume that $\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Let $a = xyz$ for $a, x, y, z \in S$. Then, we have

$$\text{ꝥ}_S(xyz) = \text{ꝥ}_S(a) \subseteq (\tilde{\theta} * \text{ꝥ}_S * \text{ꝥ}_S)(a) = [(\tilde{\theta} * \text{ꝥ}_S) * \text{ꝥ}_S](a)$$

$$= \bigcap_{a=mn} \{(\tilde{\theta} * \text{ꝥ}_S)(m) \cup \text{ꝥ}_S(n)\} \subseteq (\tilde{\theta} * \text{ꝥ}_S)(xy) \cup \text{ꝥ}_S(z)$$

$$= \bigcap_{xy=pq} \{\tilde{\theta}(p) \cup \text{ꝥ}_S(q)\} \cup \text{ꝥ}_S(z) \subseteq [\tilde{\theta}(x) \cup \text{ꝥ}_S(y)] \cup \text{ꝥ}_S(z)$$

$$= [\emptyset \cup \text{ꝥ}_S(y)] \cup \text{ꝥ}_S(z) = \text{ꝥ}_S(y) \cup \text{ꝥ}_S(z)$$

Hence, $\text{ꝥ}_S(xyz) \subseteq \text{ꝥ}_S(y) \cup \text{ꝥ}_S(z)$ implying that ꝥ_S is an S-uni left WI-ideal.

Proposition 3.4. *Let* $\text{ꝥ}_S \in S_S(U)$. *Then,* ꝥ_S *is an S-uni right WI-ideal iff* $\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta} \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$.

Proof: Assume that ꝥ_S is an S-uni right WI-ideal and $\text{v} \in S$. If $(\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta})(\text{v}) = \emptyset$, then $\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta} \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Otherwise, there exist elemenets $x, y, p, q \in S$ such that $\text{v} = xy$ and $y = pq$, for $\text{v} \in S$. Since ꝥ_S is an S-uni right WI-ideal, $\text{ꝥ}_S(\text{v}) = \text{ꝥ}_S(xy) = \text{ꝥ}_S(x(pq)) \subseteq \text{ꝥ}_S(x) \cup \text{ꝥ}_S(p)$. Thus,

$$(\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta})(\text{v}) = [\text{ꝥ}_S * (\text{ꝥ}_S * \tilde{\theta})](\text{v}) = \bigcap_{\text{v}=xy} \{\text{ꝥ}_S(x) \cup (\text{ꝥ}_S * \tilde{\theta})(y)\}$$

$$= \bigcap_{\text{v}=xy} \left\{ \text{ꝥ}_S(x) \cup \bigcap_{y=pq} \{\text{ꝥ}_S(p) \cup \tilde{\theta}(q)\} \right\} = \bigcap_{\text{v}=xpq} \{\text{ꝥ}_S(x) \cup \text{ꝥ}_S(p)\}$$

$$\supseteq \bigcap_{\text{v}=xpq} \{\text{ꝥ}_S(xpq)\} = \text{ꝥ}_S(xy) = \text{ꝥ}_S(\text{v})$$

Hence, we have $\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta} \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Moreover, in the case where $\text{v} = xy$ and $x \neq pq$ for $\text{v} \in S$, since $(\text{ꝥ}_S * \tilde{\theta})(y) = U$, $\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta} \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$ is satisfied.

Conversely, let $\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta} \mathrel{\tilde{\supseteq}} \text{ꝥ}_S$. Let $\text{v} = \text{пet}$ for $\text{v}, \text{п}, \text{e}, \text{t} \in S$. Then, we have

$$\text{ꝥ}_S(\text{пet}) = \text{ꝥ}_S(\text{v}) \subseteq (\text{ꝥ}_S * \text{ꝥ}_S * \tilde{\theta})(\text{v}) = [\text{ꝥ}_S * (\text{ꝥ}_S * \tilde{\theta})](\text{v})$$

$$= \bigcap_{\mathrm{v}=mn} \{\mathrm{p}_S(m) \cup (\mathrm{p}_S * \tilde{\theta})(n)\} \subseteq \mathrm{p}_S(\pi) \cup (\mathrm{p}_S * \tilde{\theta})(\mathrm{et})$$
$$= \mathrm{p}_S(\pi) \cup \bigcap_{\mathrm{et}=pq} \{\mathrm{p}_S(p) \cup \tilde{\theta}(q)\} \subseteq \mathrm{p}_S(\pi) \cup [\mathrm{p}_S(\mathrm{e}) \cup \tilde{\theta}(\mathrm{t})]$$
$$= \mathrm{p}_S(\pi) \cup [\mathrm{p}_S(\mathrm{e}) \cup \emptyset] = \mathrm{p}_S(\pi) \cup \mathrm{p}_S(\mathrm{e})$$
Therefore, $\mathrm{p}_S(\pi\mathrm{et}) \subseteq \mathrm{p}_S(\pi) \cup \mathrm{p}_S(\mathrm{e})$, implying that p_S is an S-uni right WI-ideal.

Theorem 3.5. *Let* $\mathrm{p}_S \in S_S(U)$. *Then,* p_S *is an S-uni WI-ideal iff* $\tilde{\theta} * \mathrm{p}_S * \mathrm{p}_S \,\tilde{\supseteq}\, \mathrm{p}_S$ *and* $\mathrm{p}_S * \mathrm{p}_S * \tilde{\theta} \,\tilde{\supseteq}\, \mathrm{p}_S$.
Proof: It follows from Proposition 3.3 and Proposition 3.4.

Corollary 3.6. $\tilde{\theta}$ is an S-uni WI-ideal.

Proposition 3.7. $\emptyset \neq R \subseteq S$ *is a left (right) WI-ideal iff the S-uni subsemigroup* $\mathfrak{f}_S$ *defined by*
$$\mathfrak{f}_S(m) = \begin{cases} \alpha, & if\ m \in S \setminus R \\ \beta, & if\ m \in R \end{cases}$$
is an S-uni left (right) WI-ideal, where $\alpha, \beta \subseteq U$ *such that* $\alpha \supseteq \beta$.
Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Suppose R is a left WI-ideal and $x, a, b \in S$. If $a, b \in R$, then $xab \in R$. Hence, $\mathfrak{f}_S(xab) = \mathfrak{f}_S(a) = \mathfrak{f}_S(b) = \beta$ and so $\mathfrak{f}_S(xab) \subseteq \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b)$. If $a \notin R$ and $b \notin R$ then, $xab \in R$ or $xab \notin R$. In this case, if $xab \in R$, then $\beta = \mathfrak{f}_S(xab) \subseteq \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \alpha$. If $xab \notin R$, then $\alpha = \mathfrak{f}_S(xab) \subseteq \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \alpha$. If $a \in R$ or $b \in R$, then $xab \in R$ or $xab \notin R$. Here, firstly note that, if $a \in R$ or $b \in R$, then either $\mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \beta$ (the case where $a \in R$ and $b \in R$) or $\mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \alpha$ (the case where $a \in R$ and $b \notin R$ (or if $a \notin R$ and $b \in R$)). Thus, either $xab \in R$ or $xab \notin R$, in any case $\mathfrak{f}_S(xab) \subseteq \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b)$, since $\alpha \supseteq \beta$. Hence, $\mathfrak{f}_S$ is an S-uni left WI-ideal. Conversely assume that S-uni subsemigroup $\mathfrak{f}_S$ is an S-uni left WI-ideal. Let $a, b \in R$, and $x \in S$. Then, $\mathfrak{f}_S(xab) \subseteq \mathfrak{f}_S(a) = \mathfrak{f}_S(b) = \beta$. Since $\beta \subseteq \alpha$ and the function is a two-valued function, $\mathfrak{f}_S(xab) \neq \alpha$, implying that $\mathfrak{f}_S(xab) = \beta$. Hence, $xab \in R$. By Theorem 2.21, R is a subsemigroup. Thus, R is a left WI-ideal.

Theorem 3.8. $\emptyset \neq R \subseteq S$ *is a WI-ideal iff the S-uni subsemigroup* $\mathfrak{f}_S$ *defined by*
$$\mathfrak{f}_S(m) = \begin{cases} \alpha, & if\ m \in S \setminus R \\ \beta, & if\ m \in R \end{cases}$$
is an S-uni WI-ideal, where $\alpha, \beta \subseteq U$ *such that* $\alpha \supseteq \beta$.

Proposition 3.9. *Let* H *be a subsemigroup. Then,* H *is a left (right) WI-ideal iff* ζ_{H^c} *is an S-uni left (right) WI-ideal.*
Proof: Since
$$\zeta_{\mathrm{H}^c}(\upsilon) = \begin{cases} U, & if\ \upsilon \in S \setminus \mathrm{H} \\ \emptyset, & if\ \upsilon \in \mathrm{H} \end{cases}$$
and $U \supseteq \emptyset$, the remainder of the proof is completed based on Proposition 3.7.

Theorem 3.10. *Let* H *be a subsemigroup. Then,* H *is a WI-ideal iff* ζ_{H^c} *is an S-uni WI-ideal.*

Example 3.11. We consider the semigroup in Example 3.2. One can show that Ᵽ $= \{$Ꝩ, ᖷ, Ѧ$\}$ is a WI-ideal. By the definition of the soft characteristic function, $\zeta_{Ᵽ^c} = \{$(Ꝩ, $\emptyset$), (ᖷ, $\emptyset$), (Ѧ, $\emptyset$), (ɷ, U)$\}$. Then, $\zeta_{Ᵽ^c}$ is an S-uni WI-ideal. Conversely, by choosing the S-uni WI-ideal as $f_S = \{$(Ꝩ, $\emptyset$), (ᖷ, $\emptyset$), (Ѧ, $\emptyset$), (ɷ, U)$\}$, which is the soft characteristic function of $K = \{$Ꝩ, ᖷ, Ѧ$\}$, one can show that K is a WI-ideal.

Now, we continue with the relationships between S-uni WI-ideals and other types of S-uni ideals of S.

Proposition 3.12. *Every S-uni left WI-ideal is an S-uni subsemigroup of a regular semigroup.*
Proof: Let f_S be an S-uni WI-ideal of a regular semigroup S and $y, r \in S$. By assumption, for all $y \in S$, there exists $\hbar \in S$ such that $y = y\hbar y$. Thus, $f_S(yr) = f_S((y\hbar y)r) = f_S((y\hbar)yr) \subseteq f_S(y) \cup f_S(r)$. Hence, f_S is an S-uni subsemigroup.

Proposition 3.13. *Every S-uni right WI-ideal is an S-uni subsemigroup of a regular semigroup.*
Proof: Let f_S be an S-uni right WI-ideal of a regular semigroup S and $\upsilon, \eta \in S$. Then, for all $\eta \in S$, there exists $x \in S$ such that $\eta = \eta x \eta$. Thus, $f_S(\upsilon\eta) = f_S(\upsilon(\eta x \eta)) = f_S(\upsilon\eta(x\eta)) \subseteq f_S(\upsilon) \cup f_S(\eta)$. Hence, f_S is an S-uni subsemigroup.

Theorem 3.14. *Every S-uni WI-ideal is an S-uni subsemigroup of a regular semigroup.*
Proof: The proof follows from Proposition 3.12 and Proposition 3.13.

Proposition 3.15. *Every S-uni left ideal is an S-uni left WI-ideal.*
Proof: Let f_S be an S-uni left ideal. Then, $\tilde{\theta} * f_S \tilde{\supseteq} f_S$ and $f_S * f_S \tilde{\supseteq} f_S$. Thus, $\tilde{\theta} * f_S * f_S \tilde{\supseteq} f_S * f_S \tilde{\supseteq} f_S$. Hence, f_S is an S-uni left WI-ideal.

We present a counterexample to demonstrate that the converse of Proposition 3.15 is not valid.

Example 3.16. Consider the SS f_S in Example 3.2. It was shown in Example 3.2 that f_S is an S-uni left WI-ideal. Since f_S(ѦѦ) $= f_S$(ᖷ) $\not\subseteq f_S$(Ѧ), f_S is not an S-uni left ideal.

Proposition 3.17 demonstrates that the converse of Proposition 3.15 is valid for the L-simple semigroups, and Proposition 3.18 demonstrates that the converse of Proposition 3.15 is valid for the idempotent SSs as well.

Proposition 3.17. *Let* $f_S \in S_S(U)$ *and* S *be an L-simple semigroup. Then, the following conditions are equivalent:*

1. f_S is an S-uni left ideal.
2. f_S is an S-uni left WI-ideal.

Proof: Proposition 3.15 demonstrates that (1) implies (2). Assume that f_S is an S-uni left WI-ideal and $\hbar, \eta \in S$. By assumption, there exists $x \in S$ such that $\hbar = x\eta$. Thus, $f_S(\hbar\eta) = f_S\big((x\eta)\eta\big) = f_S(x(\eta\eta)) \subseteq f_S(\eta) \cup f_S(\eta) = f_S(\eta)$. Thus, f_S is an S-uni left ideal.

Proposition 3.18. *Let f_S be an idempotent $\mathcal{SS}$ over U. Then, the following conditions are equivalent:*

1. f_S is an S-uni left ideal.
2. f_S is an S-uni left WI-ideal.

Proof: Proposition 3.15 demonstrates that (1) implies (2). Let f_S be an S-uni left WI-ideal. Since f_S is an idempotent S-uni left WI-ideal, $\tilde{\theta} * f_S = \tilde{\theta} * f_S * f_S \tilde{\supseteq} f_S$. Hence, f_S is an S-uni left ideal.

From here, it is obvious that any idempotent S-uni left WI-ideal coincides with the S-uni left ideal.

Proposition 3.19. *Every S-uni right ideal is an S-uni right WI-ideal.*
Proof: Let f_S be an S-uni right ideal. Then, $f_S * \tilde{\theta} \tilde{\supseteq} f_S$ and $f_S * f_S \tilde{\supseteq} f_S$. Thus, $f_S * f_S * \tilde{\theta} \tilde{\supseteq} f_S * f_S \tilde{\supseteq} f_S$. Therefore, f_S is an S-uni right WI-ideal.

We present a counterexample to demonstrate that the converse of Proposition 3.19 is not valid.

Example 3.20. Consider the $\mathcal{SS}$ $\mathfrak{f}_S$ in Example 3.2. It was shown in Example 3.2 that $\mathfrak{f}_S$ is an S-uni right WI-ideal. Since $\mathfrak{f}_S(\text{ѦѦ}) = \mathfrak{f}_S(\text{ꟻ}) \nsubseteq \mathfrak{f}_S(\text{Ѧ})$, $\mathfrak{f}_S$ is not an S-uni right ideal.

Proposition 3.21 demonstrates that the converse of Proposition 3.19 is valid for the R-simple semigroups, and Proposition 3.22 demonstrates that the converse of Proposition 3.19 is valid for the idempotent $\mathcal{SS}$s as well.

Proposition 3.21. *Let $f_S \in S_S(U)$ and S be an R-simple semigroup. Then, the following conditions are equivalent:*

1. f_S is an S-uni right ideal.
2. f_S is an S-uni right WI-ideal.

Proof: Proposition 3.19 demonstrates that (1) implies (2). Assume that f_S is an S-uni right WI-ideal and $\upsilon, \hbar \in S$. By assumption, there exists $x \in S$ such that $\hbar = \upsilon x$. Then, $f_S(\upsilon\hbar) = f_S\big(\upsilon(\upsilon x)\big) = f_S((\upsilon\upsilon)x) \subseteq f_S(\upsilon) \cup f_S(\upsilon) = f_S(\upsilon)$. Thereby, f_S is an S-uni right ideal.

Proposition 3.22. *Let f_S be an idempotent $\mathcal{SS}$ over U. Then, the following conditions are equivalent:*

1. f_S is an S-uni right ideal.

2. f_S is an S-uni right WI-ideal.

Proof: Proposition 3.19 demonstrates that (1) implies (2). Assume that f_S is an S-uni right WI-ideal. Since f_S is an idempotent S-uni right WI-ideal, $f_S * \tilde{\theta} = f_S * f_S * \tilde{\theta} \widetilde{\supseteq} f_S$. Thus, f_S is an S-uni right ideal.

From here, it is obvious that any idempotent S-uni right WI-ideal coincides with the S-uni right ideal.

Theorem 3.23. *Every S-uni ideal is an S-uni WI-ideal.*
Proof: It follows from Proposition 3.15 and Proposition 3.19.

Here, note that the converse of Theorem 3.23 is not true, following from Example 3.16 and Example 3.20. Theorem 3.24 demonstrates that the converse of Theorem 3.23 is valid for groups, and Theorem 3.25 demonstrates that the converse of Theorem 3.23 is valid for the idempotent $\mathcal{SS}$s as well.

Theorem 3.24. *Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*

1. f_S is an S-uni ideal.
2. f_S is an S-uni WI-ideal.

Proof: Theorem 3.23 demonstrates that (1) implies (2). Assume that f_S is an S-uni WI-ideal of a group S. Then, by Theorem 2.1 (2), S is both an L-simple and an R-simple semigroup. The remainder of the proof is completed based on Proposition 3.17, and Proposition 3.21.

Theorem 3.25. *Let f_S be an idempotent $\mathcal{SS}$ over U. Then, the following conditions are equivalent:*

1. f_S is an S-uni ideal.
2. f_S is an S-uni WI-ideal.

Proof: It follows from Proposition 3.18 and Proposition 3.22.

Proposition 3.26. *Every S-uni interior ideal is an S-uni left WI-ideal.*
Proof: Let f_S be an S-uni interior ideal. Then, $\tilde{\theta} * f_S * \tilde{\theta} \widetilde{\supseteq} f_S$. Thus, $\tilde{\theta} * f_S * f_S \widetilde{\supseteq} \tilde{\theta} * f_S * \tilde{\theta} \widetilde{\supseteq} f_S$. Hence, f_S is an S-uni left WI-ideal.

We present a counterexample to demonstrate that the converse of Proposition 3.26 is not valid.

Example 3.27. Consider the semigroup $S = \{\sigma, ω, ε\}$ defined by the following table:

Table 2: Cayley Table with "⊛" binary operation

⊛	σ	ω	ε
σ	σ	ω	ε
ω	σ	ω	ε
ε	σ	ω	ε

Let $ꝓ_S$ be an SS over $U = \{\Gamma, \Theta, \Lambda, \Pi, \Sigma\}$ as follows:

$$ꝓ_S = \{(\sigma, \{\Gamma, \Theta, \Lambda\}), (ɷ, \{\Pi\}), (Ɛ, \{\Sigma\})\}$$

Here, $ꝓ_S$ is an S-uni left WI-ideal. In fact,

$$(\tilde{\theta} * ꝓ_S * ꝓ_S)(\sigma) = \{\Gamma, \Theta, \Lambda\} \supseteq ꝓ_S(\sigma) = \{\Gamma, \Theta, \Lambda\}$$

$$(\tilde{\theta} * ꝓ_S * ꝓ_S)(ɷ) = \{\Pi\} \supseteq ꝓ_S(ɷ) = \{\Pi\}, (\tilde{\theta} * ꝓ_S * ꝓ_S)(Ɛ) = \{\Sigma\} \supseteq ꝓ_S(Ɛ) = \{\Sigma\}$$

thus, $ꝓ_S$ is an S-uni left WI-ideal. However, since $ꝓ_S(\sigma ɷ Ɛ) = ꝓ_S(Ɛ) \nsubseteq ꝓ_S(ɷ)$, $ꝓ_S$ is not an S-uni interior ideal.

Proposition 3.28 demonstrates that the converse of Proposition 3.26 is valid for L-simple semigroups.

Proposition 3.28. *Let $f_S \in S_S(U)$ and S be an L-simple semigroup. Then, the following conditions are equivalent:*

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni left WI-ideal.

Proof: Proposition 3.26 demonstrates that (1) implies (2). Assume that f_S is an S-uni left WI-ideal. Since S is an L-simple semigroup, by Proposition 3.17, f_S is an S-uni left ideal. Let $a, b, \eta \in S$. By assumption, there exists $x \in S$ such that $\eta = xb$. Thus, $f_S(ab\eta) = f_S(ab(xb)) = f_S((abx)b) \subseteq f_S(b)$. Hence, f_S is an S-uni interior ideal.

Proposition 3.29. *Every S-uni interior ideal is an S-uni right WI-ideal.*
Proof: Let $ƒ_S$ be an S-uni interior ideal. Then, $\tilde{\theta} * ƒ_S * \tilde{\theta} \,\tilde{\supseteq}\, ƒ_S$. Thus, $ƒ_S * ƒ_S * \tilde{\theta} \,\tilde{\supseteq}\, \tilde{\theta} * ƒ_S * \tilde{\theta} \,\tilde{\supseteq}\, ƒ_S$. Hence, $ƒ_S$ is an S-uni right WI-ideal.

We present a counterexample to demonstrate that the converse of Proposition 3.29 is not valid.

Example 3.30. Consider the semigroup $S = \{\rho, \nu, \tau\}$ defined by the following table:

Table 3: Cayley Table with "⊖" binary operation

$\ominus$	ρ	ν	τ
ρ	ρ	ρ	ρ
ν	ν	ν	ν
τ	τ	τ	τ

Let $ɓ_S$ be an SS over $U = \{\Gamma, \Theta, \Lambda, \Pi, \Sigma\}$ as follows:

$$ɓ_S = \{(\rho, \{\Gamma, \Theta\}), (\nu, \{\Lambda\}), (\tau, \{\Pi, \Sigma\})\}$$

Here, $ɓ_S$ is an S-uni right WI-ideal. In fact,

$$(ɓ_S * ɓ_S * \tilde{\theta})(\rho) = \{\Gamma, \Theta\} \supseteq ɓ_S(\rho) = \{\Gamma, \Theta\}, (ɓ_S * ɓ_S * \tilde{\theta})(\nu) = \{\Lambda\} \supseteq ɓ_S(\nu) = \{\Lambda\}$$

$$(ɓ_S * ɓ_S * \tilde{\theta})(\tau) = \{\Pi, \Sigma\} \supseteq ɓ_S(\tau) = \{\Pi, \Sigma\}$$

thus, $ɓ_S$ is an S-uni right WI-ideal. However, since $ɓ_S(\tau\nu\rho) = ɓ_S(\tau) \nsubseteq ɓ_S(\nu)$, $ɓ_S$ is not an S-uni interior ideal.

Proposition 3.31 demonstrates that the converse of Proposition 3.29 is valid for R-simple semigroups.

Proposition 3.31. *Let $f_S \in S_S(U)$ and S be an R-simple semigroup. Then, the following conditions are equivalent:*

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni right WI-ideal.

Proof: Proposition 3.29 demonstrates that (1) implies (2). Assume that f_S is an S-uni right WI-ideal. Since S is an R-simple semigroup, by Proposition 3.21, f_S is an S-uni right ideal. Let $\mathrm{v}, b, \hbar \in S$. By assumption, there exists $x \in S$ such that $\mathrm{v} = bx$. Thus, $f_S(\mathrm{v}b\hbar) = f_S\big((bx)b\hbar\big) = f_S\big(b(xb\hbar)\big) \subseteq f_S(b)$. Hence, f_S is an S-uni interior ideal.

Theorem 3.32. *Every S-uni interior ideal of S is an S-uni WI-ideal.*
Proof: The proof follows from Proposition 3.26 and Proposition 3.29.

Theorem 3.33 demonstrates that the converse of Theorem 3.32 is valid for the groups.

Theorem 3.33. *Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:*

1. f_S is an S-uni interior ideal.
2. f_S is an S-uni WI-ideal.

Proof: Theorem 3.32 clearly demonstrates that (1) implies (2). Assume that f_S is an S-uni WI-ideal and S is a group. By Theorem 2.1 (2), S is both an R-simple and an L-simple semigroup. The remainder of the proof is completed based on Proposition 3.28 and Proposition 3.31.

Moreover, it is obvious that every idempotent S-uni WI-ideal is an S-uni interior ideal.

Proposition 3.34. *Every S-uni bi-ideal is an S-uni right WI-ideal of an L-simple semigroup.*
Proof: Let f_S be an S-uni bi-ideal of an L-simple semigroup. Then, by Proposition 2.20, f_S is an S-uni right ideal. The remainder of the proof is clear from Proposition 3.19.

Proposition 3.35. *Every S-uni bi-ideal is an S-uni left WI-ideal of an R-simple semigroup.*
Proof: Let f_S be an S-uni bi-ideal of an R-simple semigroup S and $ȝ, ꭇ, ꞩ \in S$. By assumption, there exists $ɏ \in S$ such that $ȝ = ꭇɏ$. Then, $f_S(ȝꭇꞩ) = f_S\big((ꭇɏ)ꭇꞩ\big) = f_S\big((ꭇɏꭇ)ꞩ\big) \subseteq f_S(ꭇɏꭇ) \cup f_S(ꞩ) \subseteq \big(f_S(ꭇ) \cup f_S(ꭇ)\big) \cup f_S(ꞩ) = f_S(ꭇ) \cup f_S(ꞩ)$implying that f_S is an S-uni left WI-ideal.

Theorem 3.36. *Every S-uni bi-ideal is an S-uni WI-ideal for a group S.*
Proof: By Theorem 2.1 (2), S is both an R-simple and L-simple semigroup. The remainder of the proof is completed based on Proposition 3.34 and Proposition 3.35.

Proposition 3.37. *Every S-uni quasi-ideal is an S-uni right WI-ideal of an L-simple semigroup.*
Proof: Let f_S be an S-uni quasi-ideal of an L-simple semigroup S. Then by Theorem 2.18 (3), f_S is an S-uni bi-ideal. Since S is an L-simple semigroup, f_S is an S-uni right WI-ideal by Proposition 3.34.

Proposition 3.38. *Every S-uni quasi-ideal is an S-uni left WI-ideal of an R-simple semigroup.*
Proof: Let f_S be an S-uni quasi-ideal of an R-simple semigroup S. Then by Theorem 2.18 (3), f_S is an S-uni bi-ideal. Since S is an R-simple semigroup, f_S is an S-uni left WI-ideal by Proposition 3.35.

Theorem 3.39. *Every S-uni quasi-ideal is an S-uni WI-ideal for a group S.*
Proof: By Theorem 2.1 (2), S is both an R-simple and an L-simple semigroup. The remainder of the proof is completed based on Proposition 3.37 and Proposition 3.38.

Proposition 3.40. *Every S-uni left WI-ideal is an S-uni quasi-ideal of an L-simple semigroup.*
Proof: Let f_S be an S-uni left WI-ideal of an L-simple semigroup S. Since S is an L-simple semigroup, f_S is an S-uni left ideal by Proposition 3.17. Then, by Theorem 2.18 (1), f_S is an S-uni quasi-ideal.

Moreover, it is obvious that every idempotent S-uni left WI-ideal is an S-uni quasi-ideal.

Proposition 3.41. *Every S-uni right WI-ideal is an S-uni quasi-ideal of an R-simple semigroup.*
Proof: Let f_S be an S-uni right WI-ideal of an R-simple semigroup S. Since S is an R-simple semigroup, f_S is an S-uni right ideal by Proposition 3.21. Then, by Theorem 2.18 (1), f_S is an S-uni quasi-ideal.

Moreover, it is obvious that every idempotent S-uni right WI-ideal is an S-uni quasi-ideal.

Theorem 3.42 demonstrates that the converse of Theorem 3.39 valid as well.

Theorem 3.42. *Every S-uni WI-ideal is an S-uni quasi-ideal for a group S.*
Proof: By Theorem 2.1 (2), S is both an R-simple and an L-simple semigroup. The remainder of the proof is completed based on Proposition 3.40 and Proposition 3.41.

Moreover, it is obvious that every idempotent S-uni WI-ideal is an S-uni quasi-ideal.

Proposition 3.43. *Every S-uni left WI-ideal is an S-uni bi-ideal of an L-simple semigroup.*

Proof: Let f_S be an S-uni left WI-ideal of an L-simple semigroup S. Since S is an L-simple semigroup, f_S is an S-uni quasi-ideal by Proposition 3.40. Then by Theorem 2.18 (3), f_S is an S-uni bi-ideal.

Moreover, it is obvious that every idempotent S-uni left WI-ideal is an S-uni bi-ideal.

Proposition 3.44. *Every S-uni right WI-ideal is an S-uni bi-ideal of an R-simple semigroup.*
Proof: Let f_S be an S-uni right WI-ideal of an R-simple semigroup S. Since S is an R-simple semigroup, f_S is an S-uni quasi-ideal by Proposition 3.37. Then, by Theorem 2.18 (3), f_S is an S-uni bi-ideal.

Moreover, it is obvious that every idempotent S-uni right WI-ideal is an S-uni bi-ideal.

Theorem 3.45 demonstrates that the converse of Theorem 3.36 valid as well.

Theorem 3.45. *Every S-uni WI-ideal is an S-uni bi-ideal for a group* S.
Proof: By Theorem 2.1 (2), S is both an R-simple and an L-simple semigroup. The remainder of the proof is completed based on Proposition 3.43 and Proposition 3.44.

Moreover, it is obvious that every idempotent S-uni WI-ideal is an S-uni bi-ideal.

Proposition 3.46. *Let* f_S *and* f_T *be S-uni left (right) WI-ideals of* S *and* T*, respectively. Then,* $f_S \vee f_T$ *is an S-uni left (right) WI-ideal of* $S \times T$.
Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Let $(\varsigma_1, ŧ_1), (\varsigma_2, ŧ_2), (\varsigma_3, ŧ_3) \in S \times T$. Then,

$$f_{S\vee T}\big((\varsigma_1, ŧ_1)(\varsigma_2, ŧ_2)(\varsigma_3, ŧ_3)\big) = f_{S\vee T}(\varsigma_1\varsigma_2\varsigma_3, ŧ_1ŧ_2ŧ_3) = f_S(\varsigma_1\varsigma_2\varsigma_3) \cup f_T(ŧ_1ŧ_2ŧ_3)$$
$$\subseteq \big(f_S(\varsigma_2) \cup f_S(\varsigma_3)\big) \cup \big(f_T(ŧ_2) \cup f_T(ŧ_3)\big) = \big(f_S(\varsigma_2) \cup f_T(ŧ_2)\big) \cup \big(f_S(\varsigma_3) \cup f_T(ŧ_3)\big)$$
$$= f_{S\vee T}(\varsigma_2, ŧ_2) \cup f_{S\vee T}(\varsigma_3, ŧ_3)$$

Thus, $f_S \vee f_T$ is an S-uni left WI-ideal of $S \times T$.

Theorem 3.47. *Let* f_S *and* f_T *be S-uni WI-ideals of* S *and* T*, respectively. Then,* $f_S \vee f_T$ *is an S-uni WI-ideal of* $S \times T$.

Proposition 3.48. *Let* $ʄ_S$ *and* $\wp_S$ *be S-uni left (right) WI-ideals. Then,* $ʄ_S \,\tilde{\cup}\, \wp_S$ *is an S-uni left (right) WI-ideal.*
Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Let $ʄ_S$ and $\wp_S$ be S-uni left WI-ideals. Then, $\tilde{\theta} * ʄ_S * ʄ_S \,\tilde{\supseteq}\, ʄ_S$ and $\tilde{\theta} * \wp_S * \wp_S \,\tilde{\supseteq}\, \wp_S$. Thus, $\tilde{\theta} * (ʄ_S \,\tilde{\cup}\, \wp_S) * (ʄ_S \,\tilde{\cup}\, \wp_S) \,\tilde{\supseteq}\, \tilde{\theta} * ʄ_S * ʄ_S \,\tilde{\supseteq}\, ʄ_S$ and $\tilde{\theta} * (ʄ_S \,\tilde{\cup}\, \wp_S) * (ʄ_S \,\tilde{\cup}\, \wp_S) \,\tilde{\supseteq}\, \tilde{\theta} * \wp_S * \wp_S \,\tilde{\supseteq}\, \wp_S$. Hence, $\tilde{\theta} * (ʄ_S \,\tilde{\cup}\, \wp_S) * (ʄ_S \,\tilde{\cup}\, \wp_S) \,\tilde{\supseteq}\, ʄ_S \,\tilde{\cup}\, \wp_S$. Thus, $ʄ_S \,\tilde{\cup}\, \wp_S$ is an S-uni left WI-ideal.

Theorem 3.49. *Let $\mathfrak{f}_S$ and $\wp_S$ be S-uni WI-ideals. Then, $\mathfrak{f}_S \tilde{\cup} \wp_S$ is an S-uni WI-ideal.*
Proposition 3.50. *Let $\mathfrak{f}_S$ be an S-uni left and η_S be an S-uni right ideal. Then, $\mathfrak{f}_S * \eta_S$ is an S-uni WI-ideal.*
Proof: Let $\mathfrak{f}_S$ be an S-uni left and η_S be an S-uni right ideal. Then, $\tilde{\theta} * \mathfrak{f}_S \tilde{\supseteq} \mathfrak{f}_S$, $\eta_S * \tilde{\theta} \tilde{\supseteq} \eta_S$, and $\mathfrak{f}_S * \mathfrak{f}_S \tilde{\supseteq} \mathfrak{f}_S$, $\eta_S * \eta_S \tilde{\supseteq} \eta_S$. Thus, $\tilde{\theta} * (\mathfrak{f}_S * \eta_S) * (\mathfrak{f}_S * \eta_S) \tilde{\supseteq} \mathfrak{f}_S * \eta_S * \mathfrak{f}_S * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \tilde{\theta} * \mathfrak{f}_S * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \mathfrak{f}_S * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \eta_S$. Hence, $\mathfrak{f}_S * \eta_S$ is an S-uni left WI-ideal. Similarly, since $(\mathfrak{f}_S * \eta_S) * (\mathfrak{f}_S * \eta_S) * \tilde{\theta} \tilde{\supseteq} \mathfrak{f}_S * \eta_S * \mathfrak{f}_S * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \eta_S * \tilde{\theta} * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \eta_S * \eta_S \tilde{\supseteq} \mathfrak{f}_S * \eta_S$, thus $\mathfrak{f}_S * \eta_S$ is an S-uni right WI-ideal. Therefore, $\mathfrak{f}_S * \eta_S$ is an S-uni WI-ideal.

Corollary 3.51. Let $\mathfrak{f}_S$ and h_S be S-uni ideals. Then, $\mathfrak{f}_S * h_S$ is an S-uni WI-ideal.

Proposition 3.52. *Let f_S be an S-uni subsemigroup over U, α be a subset of U,and $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni left (right) WI-ideal, then $\mathcal{S}(f_S;\alpha)$ is a left (right) WI-ideal.*
Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Since $f_S(\upsilon) = \alpha$ for some $\upsilon \in S$, $\emptyset \neq \mathcal{S}(f_S;\alpha) \subseteq S$. Let $k, n \in \mathcal{S}(f_S;\alpha)$ and $\upsilon \in S$. Then, $f_S(k) \subseteq \alpha$ and $f_S(n) \subseteq \alpha$. It is needed to show that $\upsilon kn \in \mathcal{S}(f_S;\alpha)$ for all $k, n \in \mathcal{S}(f_S;\alpha)$ and $\upsilon \in S$. Since f_S is an S-uni left WI-ideal, it follows that $f_S(\upsilon kn) \subseteq f_S(k) \cup f_S(n) \subseteq \alpha \cup \alpha = \alpha$ implying that $\upsilon kn \in \mathcal{S}(f_S;\alpha)$. Moreover, since f_S is an S-uni subsemigroup over U, by Proposition 2.19, $\mathcal{S}(f_S;\alpha)$ is a subsemigroup. Thus, $\mathcal{S}(f_S;\alpha)$ is a left WI-ideal. Therefore, the proof is completed.

Theorem 3.53. *Let f_S be an S-uni subsemigroup over U, α be a subset of U, and $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-uni WI-ideal, then $\mathcal{S}(f_S;\alpha)$ is a WI-ideal.*

We illustrate Theorem 3.53 with Example 3.54.

Example 3.54. Consider the semigroup S in Example 3.2. Let $\wp_S$ be an SS over $U = S_3$ as follows:

$$\wp_S = \{(\text{♉}, \{(1)\}), (\text{ᖷ}, \{(1), (12)\}), (\text{₳}, \{(1), (12), (123)\}), (\text{Ꝗ}, \{(1), (12), (132)\})\}$$

Here, $\wp_S$ is an S-uni WI-ideal. Firstly, $\wp_S$ is an S-uni subsemigroup. In fact,

$$(\wp_S * \wp_S)(\text{♉}) = \{(1)\} \supseteq \wp_S(\text{♉}) = \{(1)\}$$
$$(\wp_S * \wp_S)(\text{ᖷ}) = \{(1), (12)\} \supseteq \wp_S(\text{ᖷ}) = \{(1), (12)\}$$
$$(\wp_S * \wp_S)(\text{₳}) = U \supseteq \wp_S(\text{₳}) = \{(1), (12), (123)\}$$
$$(\wp_S * \wp_S)(\text{Ꝗ}) = U \supseteq \wp_S(\text{Ꝗ}) = \{(1), (12), (132)\}$$

thus, $\wp_S$ is an S-uni subsemigroup. Similarly, $\wp_S$ is an S-uni left WI-ideal. In fact,

$$(\tilde{\theta} * \wp_S * \wp_S)(\text{♉}) = \{(1)\} \supseteq \wp_S(\text{♉}) = \{(1)\}$$
$$(\tilde{\theta} * \wp_S * \wp_S)(\text{ᖷ}) = U \supseteq \wp_S(\text{ᖷ}) = \{(1), (12)\}$$
$$(\tilde{\theta} * \wp_S * \wp_S)(\text{₳}) = U \supseteq \wp_S(\text{₳}) = \{(1), (12), (123)\}$$
$$(\tilde{\theta} * \wp_S * \wp_S)(\text{Ꝗ}) = U \supseteq \wp_S(\text{Ꝗ}) = \{(1), (12), (132)\}$$

thus, $\wp_S$ is an S-uni left WI-ideal. Similarly, $\wp_S$ is an S-uni right WI-ideal. In fact,

$$(\wp_S * \wp_S * \tilde{\theta})(\text{Ծ}) = \{(1)\} \supseteq \wp_S(\text{Ծ}) = \{(1)\}$$
$$(\wp_S * \wp_S * \tilde{\theta})(\text{ᖷ}) = U \supseteq \wp_S(\text{ᖷ}) = \{(1),(12)\}$$
$$(\wp_S * \wp_S * \tilde{\theta})(\text{Ѧ}) = U \supseteq \wp_S(\text{Ѧ}) = \{(1),(12),(123)\}$$
$$(\wp_S * \wp_S * \tilde{\theta})(\text{Ꙍ}) = U \supseteq \wp_S(\text{Ꙍ}) = \{(1),(12),(132)\}$$

thus, $\wp_S$ is an S-uni right WI-ideal. Thus, $\wp_S$ is an S-uni WI-ideal. By considering the image set of $\wp_S$, that is,

$$Im(\wp_S) = \{\{(1)\},\{(1),(12)\},\{(1),(12),(123)\},\{(1),(12),(132)\}\}$$

we obtain the following:

$$\text{Ꞩ}(\wp_S;\alpha) = \begin{cases} \{\text{Ծ},\text{ᖷ},\text{Ѧ}\}, & \alpha = \{(1),(12),(123)\} \\ \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}, & \alpha = \{(1),(12),(132)\} \\ \{\text{Ծ},\text{ᖷ}\}, & \alpha = \{(1),(12)\} \\ \{\text{Ծ}\}, & \alpha = \{(1)\} \end{cases}$$

Here, $\{\text{Ծ},\text{ᖷ},\text{Ѧ}\}$, $\{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}$, $\{\text{Ծ},\text{ᖷ}\}$ and $\{\text{Ծ}\}$ are all WI-ideals. In fact, since

$$\{\text{Ծ},\text{ᖷ},\text{Ѧ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ѧ}\} = \{\text{Ծ},\text{ᖷ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ѧ}\}, \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ꙍ}\} = \{\text{Ծ},\text{ᖷ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}$$
$$\{\text{Ծ},\text{ᖷ}\}\cdot\{\text{Ծ},\text{ᖷ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ}\}, \{\text{Ծ}\}\cdot\{\text{Ծ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ}\}$$

each $\text{Ꞩ}(\wp_S;\alpha)$ is a subsemigroup. Similarly, since

$$S\cdot\{\text{Ծ},\text{ᖷ},\text{Ѧ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ѧ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ѧ}\}, S\cdot\{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ꙍ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}$$
$$S\cdot\{\text{Ծ},\text{ᖷ}\}\cdot\{\text{Ծ},\text{ᖷ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ}\}, S\cdot\{\text{Ծ}\}\cdot\{\text{Ծ}\} = \{\text{Ծ}\} \subseteq \{\text{Ծ}\}$$

each $\text{Ꞩ}(\wp_S;\alpha)$ is a left WI-ideal. Similarly, since

$$\{\text{Ծ},\text{ᖷ},\text{Ѧ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ѧ}\}\cdot S = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ѧ}\}, \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}\cdot\{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}\cdot S = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ},\text{Ꙍ}\}$$
$$\{\text{Ծ},\text{ᖷ}\}\cdot\{\text{Ծ},\text{ᖷ}\}\cdot S = \{\text{Ծ}\} \subseteq \{\text{Ծ},\text{ᖷ}\}, \{\text{Ծ}\}\cdot\{\text{Ծ}\}\cdot S = \{\text{Ծ}\} \subseteq \{\text{Ծ}\}$$

each $\text{Ꞩ}(\wp_S;\alpha)$ is a right WI-ideal, and thus each of $\text{Ꞩ}(\wp_S;\alpha)$ is a WI-ideal.

Now, consider the SS g_S in Example 3.2. By taking into account

$$Im(g_S) = \{\{(12)\},\{(13)\},\{(23)\},\{(1),(123),(132)\}\}$$

we obtain the following:

$$\text{Ꞩ}(g_S;\alpha) = \begin{cases} \{\text{ᖷ}\}, & \alpha = \{(12)\} \\ \{\text{Ѧ}\}, & \alpha = \{(13)\} \\ \{\text{Ꙍ}\}, & \alpha = \{(23)\} \\ \{\text{Ծ}\}, & \alpha = \{(1),(123),(132)\} \end{cases}$$

Here, $\{\text{Ꙍ}\}$ is not a WI-ideal. In fact, since $\{\text{Ꙍ}\}\cdot\{\text{Ꙍ}\} = \{\text{ᖷ}\} \nsubseteq \{\text{Ꙍ}\}$, one of the $\text{Ꞩ}(g_S;\alpha)$ is not a subsemigroup, hence it is not a WI-ideal. It is seen that each of $\text{Ꞩ}(g_S;\alpha)$ is not a WI-ideal. On the other hand, in Example 3.2 it was shown that g_S is not an S-uni WI-ideal.

Definition 3.55. *Let f_S be an S-uni subsemigroup and S-uni left (right) WI-ideal. Then, the left (right) WI-ideals $\text{Ꞩ}(f_S;\alpha)$ are called lower α-left (right) WI-ideals of f_S.*

Proposition 3.56. *Let $f_S \in S_S(U)$, $\text{Ꞩ}(f_S;\alpha)$ be the lower α-left (right) WI-ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-uni left (right) WI-ideal.*

Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Let $a, x, y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) =$

α_2. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathfrak{S}(f_S;\alpha_1)$ and $y \in \mathfrak{S}(f_S;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x, y \in \mathfrak{S}(f_S;\alpha_2)$ and since $\mathfrak{S}(f_S;\alpha)$ is a left WI-ideal for all $\alpha \subseteq U$, it follows that $axy \in \mathfrak{S}(f_S;\alpha_2)$. Hence, $f_S(axy) \subseteq \alpha_2 = \alpha_1 \cup \alpha_2 = f_S(x) \cup f_S(y)$. Thus, f_S is an S-uni left WI-ideal.

Theorem 3.57. *Let $f_S \in S_S(U)$, $\mathfrak{S}(f_S;\alpha)$ be the lower α-WI-ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-uni WI-ideal.*

Proposition 3.58. *Let $f_S, f_T \in S_E(U)$, and ψ be a semigroup isomorphism from S to T. If f_S is an S-uni left (right) WI-ideal of S, then $\psi(f_S)$ is an S-uni left (right) WI-ideal of T.*

Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Let $t_1, t_2, t_3 \in T$. Since ψ is surjective, there exist $\mathscr{s}_1, \mathscr{s}_2, \mathscr{s}_3 \in S$ such that $\psi(\mathscr{s}_1) = t_1$, $\psi(\mathscr{s}_2) = t_2$ and $\psi(\mathscr{s}_3) = t_3$. Then,

$$\begin{aligned}
\big(\psi^*(f_S)\big)(t_1t_2t_3) &= \bigcap \{f_S(\mathscr{s}) : \mathscr{s} \in S, \psi(\mathscr{s}) = t_1t_2t_3\} \\
&= \bigcap \{f_S(\mathscr{s}) : \mathscr{s} \in S, \mathscr{s} = \psi^{-1}(t_1t_2t_3)\} \\
&= \bigcap \big\{f_S(\mathscr{s}) : \mathscr{s} \in S, \mathscr{s} = \psi^{-1}\big(\psi(\mathscr{s}_1\mathscr{s}_2\mathscr{s}_3)\big) = \mathscr{s}_1\mathscr{s}_2\mathscr{s}_3\big\} \\
&= \bigcap \{f_S(\mathscr{s}_1\mathscr{s}_2\mathscr{s}_3) : \mathscr{s}_i \in S, \psi(\mathscr{s}_i) = t_i, i = 1,2,3\} \\
&\subseteq \bigcap \{f_S(\mathscr{s}_2) \cup f_S(\mathscr{s}_3) : \mathscr{s}_2, \mathscr{s}_3 \in S, \psi(\mathscr{s}_2) = t_2 \text{ and } \psi(\mathscr{s}_3) = t_3\} \\
&= \big(\psi(f_S)\big)(t_2) \cup \big(\psi(f_S)\big)(t_3)
\end{aligned}$$

Hence, $\psi(f_S)$ is an S-uni left WI-ideal of T.

Theorem 3.59. *Let $f_S, f_T \in S_E(U)$, and ψ be a semigroup isomorphism from S to T. If f_S is an S-uni WI-ideal of S, then $\psi^*(f_S)$ is an S-uni WI-ideal of T.*

Proposition 3.60. *Let $f_S, f_T \in S_E(U)$ and ψ be a semigroup isomorphism from S to T. If f_T is an S-uni left (right) WI-ideal of T, then $\psi^{-1}(f_T)$ is an S-uni left (right) WI-ideal of S.*

Proof: The proof is presented only for S-uni left WI-ideal, as the proof for S-uni right WI-ideal can be shown similarly. Let $\eta_1, \eta_2, \eta_3 \in S$. Then, $\big(\psi^{-1}(f_T)\big)(\eta_1\eta_2\eta_3) = f_T\big(\psi(\eta_1\eta_2\eta_3)\big) = f_T\big(\psi(\eta_1)\psi(\eta_2)\psi(\eta_3)\big) \subseteq f_T\big(\psi(\eta_2)\big) \cup f_T\big(\psi(\eta_3)\big) = \big(\psi^{-1}(f_T)\big)(\eta_2) \cup \big(\psi^{-1}(f_T)\big)(\eta_3)$. Thus, $\psi^{-1}(f_T)$ is an S-uni left WI-ideal of S.

Theorem 3.61. *Let $f_S, f_T \in S_E(U)$, and ψ be a semigroup isomorphism from S to T. If f_T is an S-uni WI-ideal of T, then $\psi^{-1}(f_T)$ is an S-uni WI-ideal of S.*

Proposition 3.62. *For a semigroup S, the following conditions are equivalent:*

1. S is regular.
2. $f_S = \tilde{\theta} * \mathfrak{f}_S * \mathfrak{f}_S$ for every idempotent S-uni left WI-ideal.

Proof: Suppose that (1) holds. Let S be a regular, $ʃ_S$ be an idempotent S-uni left WI-ideal, and $x \in S$. Then, $\tilde{\theta} * ʃ_S * ʃ_S \,\tilde{\supseteq}\, ʃ_S$, $ʃ_S * ʃ_S = ʃ_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\big(\tilde{\theta} * ʃ_S * ʃ_S\big)(x) = \big(\tilde{\theta} * ʃ_S\big)(x) = \bigcap_{x=ab} \big\{\tilde{\theta}(a) \cup ʃ_S(b)\big\}$$
$$\subseteq \tilde{\theta}(xy) \cup ʃ_S(x) = \emptyset \cup ʃ_S(x) = ʃ_S(x)$$

Therefore, $\tilde{\theta} * ʃ_S * ʃ_S \,\tilde{\subseteq}\, ʃ_S$ implying that $ʃ_S = \tilde{\theta} * ʃ_S * ʃ_S$.
Conversely, let $ʃ_S = \tilde{\theta} * ʃ_S * ʃ_S$, where $ʃ_S$ is an S-uni left WI-ideal. To prove that S is regular, we need to show that $Ƀ = SɃɃ$ for every left WI-ideal $Ƀ$. It is clear that $SɃɃ \subseteq Ƀ$. Thus, it suffices to prove that $Ƀ \subseteq SɃɃ$. On the contrary, let there exist $ϭ \in Ƀ$ such that $ϭ \notin SɃɃ$. By Proposition 3.9, $\zeta_{Ƀ^c}$ is an S-uni left WI-ideal. Since $ϭ \in Ƀ$, thus, $\zeta_{Ƀ^c}(ϭ) = \emptyset$. On the other hand, since $ϭ \notin SɃɃ$, this implies that there do not exist $y, z \in Ƀ$ and $x \in S$ such that $ϭ = xyz$. Thus, $\big(\zeta_{S^c} * \zeta_{Ƀ^c} * \zeta_{Ƀ^c}\big)(ϭ) = \big(\tilde{\theta} * \zeta_{Ƀ^c} * \zeta_{Ƀ^c}\big)(ϭ) = U$. However, this conflicts with our hypothesis. Thus, $Ƀ \subseteq SɃɃ$ and so $Ƀ = SɃɃ$. Therefore, S is regular.

Proposition 3.63. *For a semigroup S, the following conditions are equivalent:*

1. S is regular.
2. $ʃ_S = ʃ_S * ʃ_S * \tilde{\theta}$ for every idempotent S-uni right WI-ideal.

Proof: Suppose that (1) holds. Let S be a regular semigroup, $ʃ_S$ be an idempotent S-uni right WI-ideal, and $x \in S$. Then, $ʃ_S * ʃ_S * \tilde{\theta} \,\tilde{\supseteq}\, ʃ_S$, $ʃ_S * ʃ_S = ʃ_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\big(ʃ_S * ʃ_S * \tilde{\theta}\big)(x) = \big(ʃ_S * \tilde{\theta}\big)(x) = \bigcap_{x=ab} \big\{ʃ_S(a) \cup \tilde{\theta}(b)\big\}$$
$$\subseteq ʃ_S(x) \cup \tilde{\theta}(yx) = ʃ_S(x) \cup \emptyset = ʃ_S(x)$$

Therefore, $ʃ_S * ʃ_S * \tilde{\theta} \,\tilde{\subseteq}\, ʃ_S$ implying that $ʃ_S = ʃ_S \circ ʃ_S \circ \tilde{\theta}$.
Conversely, let $ʃ_S * ʃ_S * \tilde{\theta} = ʃ_S$, where $ʃ_S$ is an S-uni right WI-ideal. To prove that S is regular, we need to show that $Ƀ = ɃɃS$ for every right WI-ideal $Ƀ$. It is clear that $ɃɃS \subseteq Ƀ$. Thus, it suffices to prove that $Ƀ \subseteq ɃɃS$. On the contrary, let there exist $ϭ \in Ƀ$ such that $ϭ \notin ɃɃS$. By Proposition 3.9, $\zeta_{Ƀ^c}$ is an S-uni right WI-ideal. Since $ϭ \in Ƀ$, thus, $\zeta_{Ƀ^c}(ϭ) = \emptyset$. On the other hand, since $ϭ \notin$ ɃɃS, this implies that there do not exist $x, y \in Ƀ$ and $z \in S$ such that $ϭ = xyz$. Thus, $\big(\zeta_{Ƀ^c} * \zeta_{Ƀ^c} * \zeta_{S^c}\big)(ϭ) = \big(\zeta_{Ƀ^c} * \zeta_{Ƀ^c} * \tilde{\theta}\big)(ϭ) = U$. However, this conflicts with our hypothesis. Thus, $Ƀ \subseteq$ ɃɃS and so $Ƀ = ɃɃS$. Therefore, S is regular.

Theorem 3.64. *For a semigroup S, the following conditions are equivalent:*

1. S is regular.
2. $ʃ_S = \tilde{\theta} * ʃ_S * ʃ_S = ʃ_S * ʃ_S * \tilde{\theta}$ for every idempotent S-uni WI-ideal.

4 Conclusion

As a generalization of the quasi-ideal, interior ideal, left (right) ideal, and ideal of semigroup, Rao [9] developed the idea of weak-interior ideals (WI-ideals) and studied the characteristics of weak-interior ideals (WI-ideals) of a semigroup. By introducing "S-uni weak-interior ideals (S-uni WI-ideals) of semigroups", we applied this idea to SS theory and semigroup theory in this study. The relationships between S-uni WI-ideals and various varieties of S-uni ideals of a semigroup were derived. We showed that an S-uni ideal and S-uni interior ideal is an S-uni WI-ideal, however, the converses are not true with counterexamples. For the converses, we show that the semigroup should be group or the S-uni WI-ideal should be idempotent. Besides, we show that in a group, S-uni bi-ideals and S-uni quasi-ideals coincide with S-uni WI-ideals. With our key theorem that shows if a subsemigroup of a semigroup is a WI-ideal, then its soft characteristic function is an S-uni WI-ideal, and vice versa, we show how this notion relates to the existing algebraic structures in classical semigroup theory and thus, we construct a relation between semigroup theory and SS theory. In addition, we give conceptual characterizations of the novel idea in terms of soft anti image, soft inverse image, and SS operations, providing specific and insightful examples to back up our claims. In future studies, S-uni WI-ideals can be characterized more by certain types of semigroups.

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where A → B denotes that A is B but B may not always be A.

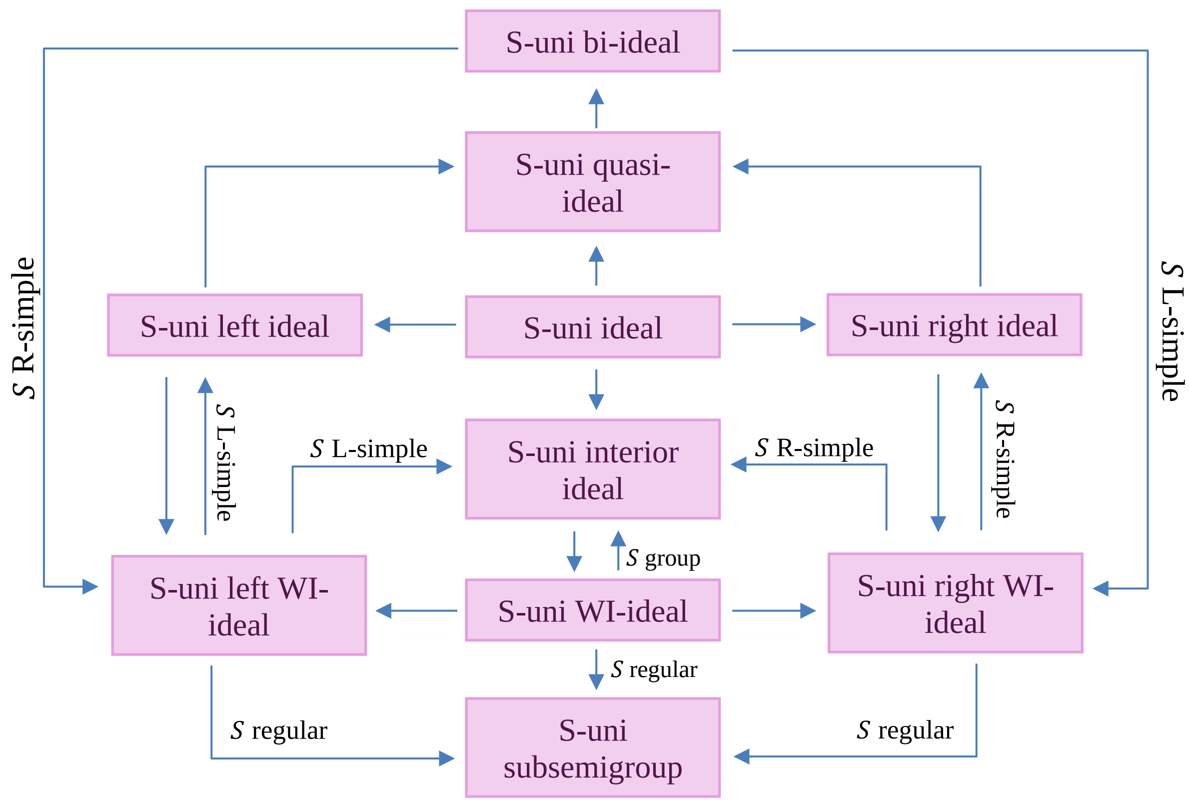

Fig 1: The relation between several S-uni ideals and their generalized ideals of semigroups.

5 Open Problem

How is the characterization of the S-uni weak-interior ideals, if the semigroup is semisimple, duo, right (left) zero or intra/completely/quasi/weakly regular?

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