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New two-parameter integral formulas proved via the digamma function

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Abstract

This paper presents new integral formulas involving two parameters: an integer parameter and a real parameter. These formulas are notable for their generality, as well as for the method of proof, which relies on the properties of the digamma function. Detailed derivations are provided, offering a fresh perspective compared to more conventional techniques. Additionally, the paper proposes an open problem concerning the possibility of proving these results using standard integral calculus methods.

Keywords: *Integral formulas, Digamma function, Changes of variables, Open problem.*

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1 Introduction

General integral formulas play a fundamental role in mathematics, physics and engineering. They are often used as intermediate tools to solve sophisticated problems. A substantial body of such formulas already exists, many of which are comprehensively compiled in [7]. Nevertheless, developing new, more general integral formulas is an active area of ongoing research, as demonstrated by the recent contributions in [8, 9, 10, 11, 2, 3, 4, 6, 1].

In this paper, we first derive a new integral formula involving two adjustable parameters: an integer parameter, n, and a real parameter, x. The integral in question is simply equal to $n\ln n$. The proof is innovative in that it is based on the properties of the digamma function. This demonstrates the analytical power of this special function in evaluating definite integrals. From our main result, we deduce two additional integral formulas. Notably, two classical integral formulas from [7] appear as special cases, illustrating the broader applicability of our approach.

The remainder of the paper is organized as follows: Section 2 presents the main result. Two further results are given in Section 3. An open problem is formulated in Section 4. Finally, concluding remarks and perspectives are provided in Section 5.

2 Main result

Our main integral formula is presented below. We emphasize its simplicity and the flexibility of the parameter choices.

Proposition 2.1 *For any $n \in \mathbb{N}\backslash\{0,1\}$ and $x > 0$, we have*

$$\int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1-e^{-t}} dt = n\ln n,$$

where

$$f_n(x,t) = \sum_{k=0}^{n-1} e^{-(x+k/n)t}.$$

Proof. The key to the proof lies in introducing the digamma function and exploiting its well-known properties. First, we recall that the digamma function at $x > 0$ is denoted by $\psi(x)$ and is defined as the derivative of the logarithm of the gamma function. More precisely, defining the gamma function as $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, we have

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The known properties of this function are listed in [7, Section 8.36]. See also [5]. In particular, it has the following integral representation:

$$\psi(x) = \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt. \tag{1}$$

See [7, Formula 8.3611].

Moreover, it satisfies a specific multiplication formula, which is described as follows: For any $n \in \mathbb{N}\backslash\{0,1\}$, we have

$$\psi(nx) = \frac{1}{n}\sum_{k=0}^{n-1} \psi\left(x + \frac{k}{n}\right) + \ln n.$$

See [7, Formula 8.3656]. As the first step of this proof, we propose to rearrange this formula as follows:

$$n\psi(nx) - \sum_{k=0}^{n-1} \psi\left(x + \frac{k}{n}\right) = n \ln n. \tag{2}$$

The parameters n and x will correspond to those involved, and the term on the right-hand side will be the desired result.

Specifically, substituting Equation (1) into Equation (2), we derive

$$n \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-nxt}}{1-e^{-t}}\right) dt - \sum_{k=0}^{n-1} \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-(x+k/n)t}}{1-e^{-t}}\right) dt = n \ln n,$$

so that

$$\int_0^{+\infty} \left(n\frac{e^{-t}}{t} - \frac{ne^{-nxt}}{1-e^{-t}}\right) dt - \int_0^{+\infty} \left(n\frac{e^{-t}}{t} - \frac{f_n(x,t)}{1-e^{-t}}\right) dt = n \ln n,$$

which is equivalent to

$$\int_0^{+\infty} \left[\left(n\frac{e^{-t}}{t} - \frac{ne^{-nxt}}{1-e^{-t}}\right) - \left(n\frac{e^{-t}}{t} - \frac{f_n(x,t)}{1-e^{-t}}\right)\right] dt = n \ln n,$$

so that, after simplifications,

$$\int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1-e^{-t}} dt = n \ln n.$$

This ends the proof of Proposition 2.1. □

In particular,

- taking $n = 2$, Proposition 2.1 gives

$$\int_0^{+\infty} \frac{e^{-xt} + e^{-(x+1/2)t} - 2e^{-2xt}}{1-e^{-t}} dt = \int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1-e^{-t}} dt$$
$$= n \ln n = 2 \ln 2,$$

- taking $n = 3$, Proposition 2.1 ensures that

$$\int_0^{+\infty} \frac{e^{-xt} + e^{-(x+1/3)t} + e^{-(x+2/3)t} - 3e^{-3xt}}{1-e^{-t}} dt = \int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1-e^{-t}} dt$$
$$= n \ln n = 3 \ln 3,$$

- taking $n = 4$, Proposition 2.1 gives

$$\int_0^{+\infty} \frac{e^{-xt} + e^{-(x+1/4)t} + e^{-(x+2/4)t} + e^{-(x+3/4)t} - 4e^{-4xt}}{1 - e^{-t}} dt$$
$$= \int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1 - e^{-t}} dt = n \ln n = 4 \ln 4.$$

Similar examples can be given for larger values of n. To the best of our knowledge, both the general integral formula and the special cases listed above are new. They do not appear in [7], at least not in this form.

3 Other results

Thanks to its functional simplicity and adjustable parameters n and x, Proposition 2.1 is flexible. The proposition below illustrates this by presenting two additional integral formulas that deviate from an exponential nature.

Proposition 3.1 *For any $n \in \mathbb{N}\backslash\{0,1\}$ and $x > 0$,*

1. *we have*
$$\int_0^1 \frac{g_n(x,t) - nt^{nx-1}}{1 - t} dt = n \ln n,$$
where
$$g_n(x,t) = \sum_{k=0}^{n-1} t^{x+k/n-1},$$

2. *we have*
$$\int_0^{+\infty} \frac{1}{t} \left[h_n(x,t) - \frac{n}{(1+t)^{nx}} \right] dt = n \ln n,$$
where
$$h_n(x,t) = \sum_{k=0}^{n-1} \frac{1}{(1+t)^{x+k/n}}.$$

Proof. The proofs of the two items are based on Proposition 2.1, i.e., for any $n \in \mathbb{N}\backslash\{0,1\}$ and $x > 0$, we have

$$\int_0^{+\infty} \frac{f_n(x,t) - ne^{-nxt}}{1 - e^{-t}} dt = n \ln n, \qquad (3)$$

where

$$f_n(x,t) = \sum_{k=0}^{n-1} e^{-(x+k/n)t},$$

and appropriate changes of variables. The details are given below.

1. Making the change of variables $y = e^{-t}$, i.e., $t = -\ln y$, into Equation (3), we obtain

$$\int_1^0 \frac{f_n(x, -\ln y) - ne^{-nx(-\ln y)}}{1 - e^{-(-\ln y)}} \left(-\frac{1}{y}dy\right) = n \ln n,$$

so that

$$\int_0^1 \frac{f_n^{\dagger}(x, y) - ny^{nx-1}}{1 - y} dy = n \ln n,$$

where

$$f_n^{\dagger}(x, y) = \frac{1}{y} f_n(x, -\ln y) = \sum_{k=0}^{n-1} y^{x+k/n-1} = g_n(x, y).$$

The desired formula is established.

2. Making the change of variables $z = e^t - 1$, i.e., $t = \ln(1+z)$, into Equation (3), we get

$$\int_0^{+\infty} \frac{f_n(x, \ln(1 + z)) - ne^{-nx \ln(1+z)}}{1 - e^{-\ln(1+z)}} \left(\frac{1}{1 + z}dz\right) = n \ln n,$$

so that

$$\int_0^{+\infty} \frac{1}{z} \left[f_n^{\ddagger}(x, z) - \frac{n}{(1 + z)^{nx}}\right] dz = n \ln n,$$

where

$$f_n^{\ddagger}(x, z) = f_n(x, \ln(1 + z)) = \sum_{k=0}^{n-1} \frac{1}{(1 + z)^{x+k/n}} = h_n(x, z).$$

The desired formula is obtained.

This ends the proof of Proposition 3.1. □

In particular,

- taking $n = 2$, the first item in Proposition 3.1 ensures that

$$\int_0^1 \frac{t^{x-1} + t^{x-1/2} - 2t^{2x-1}}{1 - t} dt = \int_0^1 \frac{g_n(x, t) - nt^{nx-1}}{1 - t} dt$$
$$= n \ln n = 2 \ln 2,$$

which is also a formula indicated in [7, Formula 3.2721],

- taking $n = 3$, the first item in Proposition 3.1 gives

$$\int_0^1 \frac{t^{x-1} + t^{x-2/3} + t^{x-1/3} - 3t^{3x-1}}{1-t} dt = \int_0^1 \frac{g_n(x,t) - nt^{nx-1}}{1-t} dt$$
$$= n \ln n = 3 \ln 3,$$

which is also a formula indicated in [7, Formula 3.2722],

- taking $n = 4$, the first item in Proposition 3.1 implies that

$$\int_0^1 \frac{t^{x-1} + t^{x-3/4} + t^{x-2/4} + t^{x-1/4} - 4t^{4x-1}}{1-t} dt$$
$$= \int_0^1 \frac{g_n(x,t) - nt^{nx-1}}{1-t} dt = n \ln n = 4 \ln 4,$$

- taking $n = 2$, the second item in Proposition 3.1 ensures that

$$\int_0^{+\infty} \frac{1}{t} \left[\frac{1}{(1+t)^x} + \frac{1}{(1+t)^{x+1/2}} - \frac{2}{(1+t)^{2x}} \right] dt$$
$$= \int_0^{+\infty} \frac{1}{t} \left[h_n(x,t) - \frac{n}{(1+t)^{nx}} \right] dt = n \ln n = 2 \ln 2,$$

- taking $n = 3$, the second item in Proposition 3.1 gives

$$\int_0^{+\infty} \frac{1}{t} \left[\frac{1}{(1+t)^x} + \frac{1}{(1+t)^{x+1/3}} + \frac{1}{(1+t)^{x+2/3}} - \frac{3}{(1+t)^{3x}} \right] dt$$
$$= \int_0^{+\infty} \frac{1}{t} \left[h_n(x,t) - \frac{n}{(1+t)^{nx}} \right] dt = n \ln n = 3 \ln 3,$$

- taking $n = 4$, the second item in Proposition 3.1 ensures that

$$\int_0^{+\infty} \frac{1}{t} \left[\frac{1}{(1+t)^x} + \frac{1}{(1+t)^{x+1/4}} + \frac{1}{(1+t)^{x+2/4}} + \frac{1}{(1+t)^{x+3/4}} \right.$$
$$\left. - \frac{4}{(1+t)^{4x}} \right] dt$$
$$= \int_0^{+\infty} \frac{1}{t} \left[h_n(x,t) - \frac{n}{(1+t)^{nx}} \right] dt = \int_0^{+\infty} \frac{1}{t} \left[h_n(x,t) - \frac{n}{(1+t)^{nx}} \right] dt$$
$$= n \ln n = 4 \ln 4.$$

Similar examples can be given for larger values of n. Thus, there are two special cases available in [7, Formulas 3.2721 and 3.2722]; the others are new additions to the literature.

4 Open problem

The proof of Proposition 2.1 is based on the use of the digamma function and several of its key properties. This approach is relatively uncommon when it comes to deriving integral formulas, and thus offers a novel perspective within the field. The following question naturally arises:

Can Proposition 2.1 be proved using more traditional or elementary techniques in integral calculus?

For certain specific values of the parameters, such as $n = 2$ and $x = 2$, the answer is affirmative: the result can be derived using a sum of standard primitives. However, in the general case, the situation remains unclear and further investigation is required to establish whether a fully elementary proof exists.

A similar question applies to the integral identities established in Proposition 3.1. Investigating alternative proofs of this kind could deepen our understanding of the structure and origin of these formulas.

5 Conclusion and perspectives

In this paper, we introduced a new integral formula that depends on two parameters, for which we provide a proof based on the properties of a special function: the digamma function. From this main result, we derived two additional integral formulas, some of which include classical formulas as special cases. Future work may involve exploring whether these formulas can be derived using more traditional integration techniques, as well as investigating potential generalizations involving other special functions or higher-dimensional analogues.

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