Int. J. Open Problems Compt. Math., Vol. 18, No. 4, December 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376

On Spectral Properties of Orthogonal Isometries

Jacob Gachago, Benard Okelo, Willy Kangogo

Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology,
Box 210-40601, Bondo-Kenya.
e-mail: benard@aims.ac.za

Received 3 May 2025; Accepted 15 July 2025

Abstract

Isometries are special mappings with a property of distance preservation. This property makes them unique and interesting in-terms of characterization. However, it is very difficult to do characterizations of properties of isometires in a general Banach space setting due to the intricate underlying structures in Banach spaces. In this paper, we characterize spectral properties of isometries and in particular when they are orthogonal. We show that orthogonal isometries preserve distance even when they are subjected to direct sum decomposition.

Keywords: *Spectrum, Operator, Isometry, Orthogonality.*
2010 Mathematics Subject Classification: Primary 47A30; secondary 46B10.

1 Introduction

Studies on isometries have been carried out by many researchers with nice results obtained [1]. The work of [46] investigated m-isometric operators and there properties. In the investigation they established conditions for an operator to preserve orthogonality given that there are distinct eigenvalues. They also established orthogonality of diagonal Jordan block that corresponds to certain distinct eigenvalues. Additionally, it was established that if $S \in B(H)$ and H is non empty then the spectrum of S is contained in the topological boundary ($\mathcal{T}$) or the spectrum of S is in the closure of the unit disc ($\overline{\mathcal{D}}$). This was further investigated by [2] in operator spaces. Research on orthogonality of generalized eigenvector and isometries has also been done by [3]. The research was based on the fact that if T is a Jordan block and $L_S(x)$ is a linear span, then $\delta(S_{L_S(x)}) = y$ and in this case y is eigenvalue of S. This was extend the research of [4] which considered the link between spectral properties of the direct sum of a finite number of Jordan blocks and further consider the infinite number of Jordan blocks. The study of [5] also consider S^n and its direct sum of both finite and infinite number of Jordan blocks. In [3] the work established the link between an isometry in $B(H)$ and shift operators in W^*-algebra. In the work, it was established that if y is an isometry and ξ and ζ are mutually orthogonal and commute with y and if $\xi + \zeta = 1$, then $\xi y \xi$ and $\zeta y \zeta$ are unitary shift and unitary in W^*-algebra respectively. In [6], the investigation established that $\xi y \xi$ and y share the orthogonal shift spectrum. Additionally, a p-shift sequence spectrum of y was developed with the initial projection in [7]. It was further established that if y is a unilateral shift then there is orthogonal shift spectrum of y with total summation 1 as seen in [8]. The work of [44] researched on the link between the intersection of $\Gamma_j = \{x \in \mathcal{C} :\mid X \mid = y\}$ and the $\delta(S)$. In the work, they established that if $\Gamma_j \cap \delta(S)$ where S is a Lamperti operator, then the spectral projection belongs to $\delta_T(S)$. It was further established in [9] that if S and $S^{'}$ are Lampert operators then $\delta_T(S)$ is a band projection. In [10] established $S \oplus S^{'}$ also has a band projection. Furthermore, it has been established that $S^n : n \in \mathcal{N}$ also has a projection band in $\delta_T(S^n)$ see ([11]-[15] and the references therein).

The study of [37] investigated properties of quasi-isometries. In the research it was established that if S is quasi-isometry, then the approximate point spectrum is contained in S^1, further if $\overline{\xi} \in \delta_p \in (S^*)$ and $\xi \in \delta_p \in (S)$, $\overline{\xi} \in a \in (S^*)$, then $\xi \in a \in (S)$ as seen in [16] and finally it was concluded that eigenspace of non-zero eigenvalues of S are orthogonal [17]. The work of [42] extended the research on partial isometries. In the research a link between the spectrum for decomposably regular operator in a Banach space and the closed disk was established. The link was further extended by [19] to spectral radius and power bounded operators S^n $\forall n \in \mathcal{N}$. Moreover, research was further extended to both left and right decomposable regular operators and it was established in ([20]- [29]) that the point spectrum and the spectrum of these partial isometries do not share any element with the $\mathcal{D}$. It was further established that if S_1 and S_2 are partial isometries and $\|S_1 - S_2\| < 1$ then $\delta(S_1) = \delta(S_2) = \overline{\mathcal{D}}$ as discussed in [30]. An investigation on the orthogonality of isometries and particularly $N(S_1)$ and the inverse S_2 of S_1 was established in [31]. In the investigation it was established that if $N(S_1) \neq \emptyset$ then $N(S_1) \perp S_2(X)$ which implies both ways that $\| I - S_1 S_2 \| = 1$. This was further characterized in a general set up by [32]. Authors in [36] researched on tuples of commuting partial isometries.

In the investigation a link was established between d-tuple of operators in a Hilbert space and the point spectrum and the approximate point spectrum. In the research the spectral properties for single variable partial operators were also researched as seen in [22]. It was also established in [24] that the approximate point spectrum in contained in S^1. In [23], the study established that if $S = (S_1, ..., S_n)$ has eigenvalues $\beta = (\beta_1, ..., \beta_n$ and $\gamma = (\gamma_1, ..., \gamma_n)$ such that $\sum_{(1 \leq i \leq n)} \beta_i \overline{\gamma_i} \neq 1$ then they are orthogonal. This result was extended by [35], [33] and [34] to norm convergency of $\| (S_i - \beta_i)x_m$ and $\| (S_i - \gamma_i)y_m \|$ to zero and approximate point spectrum and it was established that if those conditions are satisfied then $\langle x_m, y_m \rangle \to 0$ as $(m \to \infty)$. The investigation was extended to invertible tuples by [38]. In the research it was established that [0] is not contained in $\delta_{ap}(S)$. Further if $\beta = (\beta_1, ..., \beta_n \in \delta_{ap}(T)$, then $1 \in \delta_{ap}(\sum_{1 \leq i \leq n} \beta_i T_i$ and finally if $\beta = (\beta_1, ..., \beta_n \in \delta_p(T)$, then $1 \in \delta_p(\sum_{1 \leq i \leq n} \beta_i T_i$ as given by [39]. In [40], the author researched on isometries and established that an isometry is a partial isometry if there is a projection onto $S \in B(X)$ of norm 1. The link between non-zero partial isometry and contractive inverse has also been investigated by [40], [41] and [43]. In the researches it was established that the norms of the two are equal and the two norms are generally equal to 1. The spectral properties of the two operators were further investigated in [47]. In the research it was established that if T is the contractive generalized inverse of S, in the closed unit disc $\mathcal{D}$ then $\delta(T) \subseteq \delta(\mathcal{D})$. Further, it was established in [44] that if the intersection of the resolvent set of S or T and $\mathcal{D}$ is a non empty set, then the spectrum of T and the spectrum of S are all equal to $\overline{\mathcal{D}}$. The investigation further established that $\mathcal{D} \subseteq \delta(T)$ and $\mathcal{D} \subseteq \delta(S)$ then it follows that $\overline{\mathcal{D}} \subseteq \delta(T)$ and $\overline{\mathcal{D}} \subseteq \delta(S)$.

The investigation was extended to holomorphic isometries and in this case it was established that $\delta(S) \subseteq \overline{\mathcal{D}}$. A link between power bounded operator of partial isometry and its generalized contractive inverse is $T^m S^m T^m$ for an integer m has been established [45]. Additionally, it was established that if S is either right or left invertible but not generally invertible then the spectrum of both T and S are equal to $\overline{\mathcal{D}}$. A link between the point spectrum and the two operators was also established [46]. In the investigation it was established that $\mathcal{D}$ is contained in the point spectrum of S and T. Further it was noted that the intersection of $\mathcal{D}$ and $\delta(S)$ is empty. For holomorphic operators it was further established that if S is not in $B(X)^{-1}$ then the spectrums of T and S are all equal to $\overline{\mathcal{D}}$. Additionally, for two partial isometries whose difference in norm is less that 1 it was also established that the spectrums of this two operators is also equal to $\overline{\mathcal{D}}$. Finally it was established that for the kernel of S is orthogonal to $T(x)$ and also the kernel of T is orthogonal to $S(X)$. In [4], the authors researched on the spectral properties of (A, m)-isometries and established conditions that ensures that (A, m)-isometries are N-supercyclic and (A, m)-isometries are not N-supercyclic. In the investiga-

tion, it was established that spectra of (A, m)-isometries of $A \neq 0$ intersects S^1. Furthermore, it was established that if T is a compact subset of a complex space and $T \cap \partial\mathcal{D} \neq \emptyset$ then for a finite dimensional H and $S, A \in B(H)$, and if $A \geq 0$ intersect with S then $\delta(S) = T$. Furthermore it was established that power bounded isometries are not supercyclic. The investigation was further directed to the relationship between the approximate spectrum of A and $S \in B(H)$ and it was established that if 0 is not a member of $\delta_{ap}(A)$ then for an (A, m)-isometry S can never be supercyclic. The research further considered the point spectrum and it was established that if 0 is not in the point spectrum then $\|AS^k\| \leq M$ for all integers $k > 0$ and $AS^k \to 0$ as $k \to \infty$. Conditions for an isometry not to be N-supercyclic were also investigated.
In [33], the work researched on partial isometries. In the investigation, the link between properties of partial isometries, orthogonality and subspaces were analyzed. Moreover, it was established that if S is a normal partial isometry and $S = SS^*S$ then $\beta = \beta \mid \beta \mid^2$, for all $\beta \in \delta(S)$. Furthermore, it was established that if S is a partial isometry then SS^* and S^*S are orthogonal projections into $ker(S)^{\perp}$ and ran(S) respectively. Also, the study established that partial isometries can be obtained by find the product of orthogonal projection with unitary operators. It was established the partial isometry can be obtained by even commuting the orthogonal projection with the unitary operator. The investigation further proved that if S and T are partial isometries, that ST is also a partial isometry provided that S^*S and TT^* also commute. It was further established that square contractions can be factored into partial isometries if they unitary or singular [34]. The research further proved that for square partial isometries, the tensor product of two partial isometries is also an isometry. Thereafter, they established the link between partial isometries and the spectrum in the $\mathcal{D}^-$ and the investigation was extended to polynomials and it was established that if the solution of the monic polynomials lies in $\mathcal{D}^-$ then zero is the characteristic polynomial of partial isometry. A link between the spectrum partial isometry and $\mathcal{D}^-$ was also established.
Recently, [10] investigated properties of properties of diagonal partial isometries. In the investigation it was established that if $S = [S_{jk}]^d_{j,k}$ is a block upper triangular matrix and each $S = [S_{jk}]^d_{j,k}$ is upper triangular then $S \sim S_{11} \oplus S_{22} \oplus ... \oplus S_{dd}$. An isomorphism between partial isometries was also established showcasing the extent to which eigenvalues obtained form the polynomials of partial isometries. Furthermore, it was established that if $S, T \in M_n$, then $M(S)M(T)$ is a partial isometry if and only if S is a partial isometry.
Additionally, if $A \cong S \oplus A$ where S is upper triangular partial isometry then the spectrum of S is in $\mathcal{D}$. The research also considered the relationship between partial isometries and their adjoints. It was established that $S \cong S^T$ and the result also holds for unitary and composition operators on H^2 in open unit disk that have orthogonal power [26]. In the research it was established that if f is a

non-automorphic symbol with orthogonal power then the essential spectrum of T_f is equal to $\| T_f \|_e \overline{\mathcal{U}}$. Based on this finding, it was established in [18] that it is possible to obtain the spectrum $\delta(T_f)$ of T_f. Furthermore, it was established that $\delta(T_f) = r_e(T_f)\overline{\mathcal{U}} \cup \{(T^{'}(0))^k : k = 1, 2, ...\} \cup \{1\}$. Koldobsky [22] researched on isometries in the Banach space X. In the investigation, it was established that operators for X into itself that preserve orthogonality are isometries that scalar multiples. In the research it further proved that if $\xi \in D(x, y)$ and $\lambda, \beta \in \mathcal{R}$ then $x + \lambda y \perp \lambda x + \beta y$ iff $x^*(\lambda x + \beta y) = 0\ \forall x \in T(x + \lambda y)$. Further an investigation was extended to a line segment and it was established that $x + \xi y \perp y$ on a closed interval in $\mathcal{R}$ if $\| x + \xi y \| = \| x + ny \|\ \forall \xi \in [n, M] \subseteq \mathcal{R}$.

2 Preliminaries

We outline some preliminary concepts are useful in this study. We give them under this section for ease of understanding of the work.

Definition 2.1 *([16]) Consider $\mathcal{X}$ and $\mathcal{Y}$ to be normed spaces. An orthogonal isometry $T : \mathcal{X} \to \mathcal{Y}$ is a map such that for all $x, y \in \mathcal{X}$, $x \perp y$ implies $T(x) \perp T(y)$, where $\langle x, y \rangle = 0$ in inner product spaces and for all $x \in \mathcal{X}$, $\|T(x)\| = \|x\|$.*

Definition 2.2 *([28]) The spectrum $\sigma(A) = \{\lambda \in \mathcal{C} : \lambda I - A \notin A^{-1}\}$, where A is an operatorA^{-1} is the set of all invertible element and I is the multiplicative identity.*

3 Literature review

We discuss literature on orthogonal isometries in this work. We consider various studies, their relevance and critical contributions to this study. The work of [36] researched on tuples of commuting partial isometries. In the investigation a link was established between d-tuple of operators in a Hilbert space and the point spectrum and the approximate point spectrum. In the research the spectral properties for single variable partial operators were also researched on. It was established that the approximate point spectrum in contained in S^1 as seen in the next theorem.

Theorem 3.1 *Let $S = (S_1, ..., S_n) \in B(H)^n$ be a joint $(m; (r_1, ..., r_n)$-partial isometry of n-tuples such that $N(S^n)$ is a reducing subspace for each $S_i (1 \leq i \leq n)$. Then $\delta_{ap}(S) \subset \partial(S^1)^n \cup [0]$ where $[0] = \{(\beta_1, ..., \beta_n \in \mathcal{C}^n : \Pi_{(1 \leq i \leq n)} \beta_k = 0\}$.*

It was further established that if $S = (S_1, ..., S_n)$ has eigenvalues $\beta = (\beta_1, ..., \beta_n$ and $\gamma = (\gamma_1, ..., \gamma_n)$ such that $\sum_{(1 \leq i \leq n)} \beta_i \overline{\gamma_i} \neq 1$ are orthogonal. This result was extended to norm convergency of $\| (S_i - \beta_i)x_m$ and $\| (S_i - \gamma_i)y_m \|$ to zero and approximate point spectrum and established that if those conditions are satisfied then $\langle x_m, y_m\rangle \to 0$ as $(m \to \infty)$. The investigation was extended to invertible tuples. In the research it was established that [0] is not contained in $\delta_{ap}(S)$. Further if $\beta = (\beta_1, ..., \beta_n \in \delta_{ap}(T)$, then $1 \in \delta_{ap}(\sum_{1\leq i\leq n} \beta_i T_i$ and finally if $\beta = (\beta_1, ..., \beta_n \in \delta_p(T)$, then $1 \in \delta_p(\sum_{1\leq i\leq n} \beta_i T_i$. In there investigation, they established conditions for an operator to preserve orthogonality given that there are distinct eigenvalues. They established orthogonality of diagonal Jordan block that corresponds to certain distinct eigenvalues. Additionally, it was established that if $S \in B(H)$ and H is non empty then the spectrum of S is contained in the topological boundary $(\mathcal{T})$ or the spectrum of S is in the closure of the unit disc $(\overline{\mathcal{D}})$. The orthogonality of generalized eigenvector was summarized by the following theorem.

Theorem 3.2 *([46], Theorem 5.3) Suppose λ_1 and λ_2 distinct elements of $\mathcal{T}$. Suppose $S \in B(M)$ and $k_i \in V_{S;\lambda_i}$ for i=1,2. Then the following holds:*

(i). $Re\langle S^n x_1, S^n x_2\rangle = 0 \forall n \in \mathcal{Z}_+$ provided $\lambda_1 = -\lambda_2$ and $\| S^n(k_1 + k_2 \|^2$ is a polynomial in n;

(ii). $\langle S^n x_1, S^n x_2\rangle = 0 \forall n \in \mathcal{Z}_+$ provided $\lambda_1 = -\lambda_2$ and there exists $(\epsilon_1, \epsilon_2) \in \mathcal{U}$ such that $\| S^n(\epsilon k_1 + k_2) \|^2$ is a polynomial in n for $i = 1, 2$;

(iii). $\langle S^n x_1, S^n x_2\rangle = 0 \forall n \in \mathcal{Z}_+$ provided $\lambda_1 = -\lambda_2$ and $\| S^n(k_1 + k_2) \|^2$ is a polynomial in n.

The research was based on the fact that if T is a Jordan block, then $L_S(x)$ is a linear span, then $\delta(S_{L_S(x)}) = y$ and in this case y is eigenvalue of S. In the our investigation we will extend the research and consider the link between spectral properties of the direct sum of a finite number of Jordan blocks and further consider the infinite number of Jordan blocks. We will also consider S^n and its direct sum of both finite and infinite number of Jordan blocks of isometries.

In [3], the authors established the link between an isometry in $B(H)$ and shift operators in W^*-algebra. In the research it was established that if y is an isometry and ξ and ζ are mutually orthogonal and commute with y and if $\xi + \zeta = 1$, then $\xi y \xi$ and $\zeta y \zeta$ are unitary shift and unitary in W^*-algebra respectively. In the investigation it was established that $\xi y \xi$ and y share the orthogonal shift spectrum. Additionally, a p-shift sequence spectrum of y was developed with the initial projection. It was further established that if y is a unilateral shift then there is orthogonal shift spectrum of y with total summation 1.

In [44], the author researched on the link between the intersection of $\Gamma_j = \{x \in \mathcal{C} :| X |= y\}$ and the $\delta(S)$. In the research they established that if $\Gamma_j \cap \delta(S)$

where S is a Lamperti operator, then the spectral projection belongs to $\delta_T(S)$. It was further established that if S and $S^{'}$ are Lampert operators then $\delta_T(S)$ is a band projection. In our research we established $S \oplus S^{'}$ also has a band projection. Further we established that $S^n : n \in \mathcal{N}$ also has a projection band in $\delta_T(S^n)$.
The study of [37] investigated on properties of quasi-isometries. In the research it was established that if S is quasi-isometry, then the approximate point spectrum is contained in S^1, further if $\overline{\xi} \in \delta_p \in (S^*)$, then $\xi \in \delta_p \in (S)$, $\overline{\xi} \in a \in (S^*)$, then $\xi \in a \in (S)$ and finally concluded that eigenspace of non-zero eigenvalues of S are orthogonal. The following theorem provided a wide range of properties of quasi-isometries that establishes the link between approximated point spectrum, the spectrum and orthogonality of mutually orthogonal quasi-isometries.

Theorem 3.3 *Suppose S is quasi-isometry. Then*

(i). $\delta_{ap}(S) \sim 0$ *is a subset of* S^1;

(ii). $\overline{\beta} \in \delta(S^*)$ *whenever* $\beta \in \delta(S^*)$;

(iii). $\overline{\beta} \in \delta_{ap}(S^*)$ *whenever* $\beta \in \delta_{ap}(S^*)$;

(iv). the eigen space corresponds to distinct non-zero eigenvalues of S are mutually orthogonal;

(v). isolated points of $\delta(S)$ are eigenvalues of S.

In our work, it is interesting to investigate properties of quasi-isometries under different technical approaches which includes direct sum and power bounded operators. We also investigated spectral properties of UP in S^1. In [42], the researchers extended the research on partial isometries. In the research a link between the spectrum for decomposable regular operator in a Banach space and the closed disk was established. The link was further extended to spectral radius and power bounded operators S^n $\forall n \in \mathcal{N}$ The research was further extended to both left and right decomposable regular operators and established that the point spectrum and the spectrum of this partial isometries do not share any element with the $\mathcal{D}$. It was further established that if S_1 and S_2 are partial isometries and $\|S_1 - S_2\| < 1$ then $\delta(S_1) = \delta(S_2) = \overline{\mathcal{D}}$.
The investigation on the orthogonality of $N(S_1)$ and the inverse S_2 of S_1 was established. In the investigation it was established that if $N(S_1) \neq \emptyset$ then $N(S_1) \perp S_2(X)$ which implies both ways that $\| I - S_1S_2 \|= 1$.
In this study it is interesting to investigate power isometries. We have also investigated the spectral properties of the direct sun of finite number of partial isometries and also the direct sum of a partial isometry with its adjoint. We also investigated the properties of direct sum of convergent sequence under

the norms and in the direct sum of a finite number of operators and there point spectrum and the approximate point spectrum of those operators under review.
Many authors researched on isometries and established that an isometry is a partial isometry if there is a projection onto $S \in B(X)$ of norm 1. The link between non-zero partial isometry and contractive inverse. In the research it was established the norms of the two are equal and the two norms are generally equal to 1. The spectral properties of the two operators were further investigated. In the research it was established that if T is the contractive generalized inverse of S, in the closed unit disc $\mathcal{D}$ then $\delta(T) \subseteq \delta(\mathcal{D})$.
Researchers have worked on the spectral properties of (A, m)-isometries and established conditions that ensures that (A, m)-isometries are N-supercyclic and (A, m)-isometries not to be N-supercyclic. In the investigation it was established that spectrum of (A, m)-isometries if $A \neq 0$ intersects S^1. Furthermore it was established that if T is a compact subset of a complex space and $T \cap \partial\mathcal{D} \neq \emptyset$ then for a finite dimensional H and $S, A \in B(H)$, and if $A \geq 0$ with S then $\delta(S) = T$. Furthermore it was established that power bounded isometries are not supercyclic.
The investigation was further directed to the relationship between the approximate spectrum of A and $S \in B(H)$ and it was established that if 0 is not a member of $\delta_{ap}(A)$ then for an (A, m)-isometry S can never be supercyclic. The research further considered the point spectrum and it was established that if 0 is not in the point spectrum then $\|AS^k\| \leq M$ for all integers $k > 0$ and $AS^k \to 0$ as $k \to \infty$. Conditions for an isometry not to be N-supercyclic were also investigated. Fore instance it was established that the operator satisfies the following properties cannot be supercyclic.

Theorem 3.4 *If $S \in B(H)$ is an (A, m)-isometry with dim $(Hker(A_S)) > N$ for some $N \geq 1$, then S is not N-supercyclic.*

In our research it has been interesting to establish that N-supercyclicity does not exist in the direct sum of $S_i \in B(H)$ by investigating the point spectrum of direct sum of the S_i. Further we also investigated the point spectrum of $(\oplus_{i=1}^{n} A_{\oplus_{i=1}^{n} S})$. We extended our investigation to the direct sum of infinite number of bounded linear operators. In [7] utilizing the fact that positive operators with the spectrum $\delta(S)$ and given that the norm of S is in the spectrum of S formulated the orthogonality conditions as summarized in the lemma below:

Lemma 3.5 *Suppose that $\| T \| = 1 = \| S \|$ with $T, S \in B(H, K)$. Then $T \perp_P S$ iff $\forall \beta \in \mathcal{C}$ the operator $(1+ \mid \beta \mid^2)I - (T + \beta S)^*(T + \beta S)$ is positive and not invertible.*

An investigation was further extended to commuting normal operators and it was established that $\delta(T, S)$ is not contained in $\{(\gamma, x) \in \mathcal{C} \times \mathcal{R} :\mid \gamma \mid^2 + x^2 =$

$1, x \geq 0\}$. Further research was done on column orthogonal operators. It a set of operators is column orthogonal if its finite subsets are also column orthogonal.
The work of [23] researched on approximate preservation of orthogonality. In the investigation it was established that for reflexive Banach spaces X, if ξ, α are smooth points in X and $J(\xi) = \{\omega\}$ and $J(\alpha) = \{\beta\}$ with mutually orthogonal elements ω, β, then for $0 \leq \epsilon < \frac{1}{2}$ then a hyperspace contained in $\xi^{\perp_B^\epsilon} \cap \alpha^{\perp_B^\epsilon}$ does not exist. Further it was established that for normed linear spaces X, Y if $S \in B(X, Y)$ preserves ϵ-orthogonality at each point $\xi \in X \forall 0 \leq \epsilon < 1$, then S is $1-1$.
More work by [45] researched on approximation by partial isometries. In the investigation they established that if A and B are normal operators then norm is preserved as seen in the theorem below.

Theorem 3.6 *Let A and B be normal operators then the following inequality holds:* $\| A - B \| \geq \sup \inf_{x \in \delta(A) y \in \delta(B)} | x - y |$.

Furthermore for all S and $P_S = (S^*S)^{\frac{1}{2}}$ and $\widetilde{S}$ which is a 2×2 matrix with S in a_{12} and S^* in a_{21} position and zero elsewhere, then $\delta(\widetilde{S}) \supseteq \{\pm\beta : \beta \in \delta(P_S)\}$ and finally, for any arbitrary operator and a partial isometry S $\| A - S \| \geq min\{\beta, | \beta - 1 |\} \forall \beta \in \delta((A^*A)^{\frac{1}{2}})$ In [32], the authors established that for self-adjoint operators $S \in B(H)$, $\delta(S) \subset \mathcal{R}$ and the eigenvector associated with every unique eigenvalues of S admit orthogonality. In our investigation we established the direct sum of such operators also preserve orthogonality between the eigenvector and the distinct eigenvalues. On partial isometries, [1] researched on partial A-isometric and left polynomially partial isometry. In the investigation, it was established that if $S \in B(H)$ is left p-partial isometry such that the kernel of $p(S)$ is invariant under T, $\ker(p(S)) \subseteq \ker(S^*TS)$ and $\operatorname{ran}(Ap(S)) \subseteq \operatorname{ran}(\overline{p}(S^*))$. if 0 is not in approximate spectrum of T, then $\delta_{ap}(S) \subseteq \partial\mathcal{D} \cup \mathcal{R}(p)$ in particular $\overline{\delta_p(S)} \subseteq \partial\mathcal{D} \cup \mathcal{R}(p)$, further for $\overline{\beta} \in \delta_{ap}(S^*) \setminus \mathcal{R}(\overline{p})$ whenever $\beta \in \delta_{ap}(S) \setminus \mathcal{R}(p)$, $\overline{\beta} \in \delta_p(S^*) \setminus \mathcal{R}(\overline{p})$ whenever $\beta \in \delta_p(S) \setminus \mathcal{R}(p)$ and finally if p is a monomial, then the eigenspaces of S corresponding to eigenvalues are mutually orthogonal in $(H, \| \,.\, \|_T)$.

4 Research methodology

Theorem 4.1 ***(Closed Graph Theorem)****[15] It states that if the graph of an operator T is closed then T is bounded and continuous.*

This theorem is a useful tool in functional analysis, the Closed Graph Theorem connects the topological characteristics of linear operators with their algebraic structure. If the graph of a projection operator in a Hilbert space (a particular kind of Banach space) is closed, then the boundedness of the projection can be deduced. It guarantees that linear operators between Banach spaces are continuous given appropriate conditions (closed graph), making study and implementation of such operators easier in a variety of academic and physical settings. The Closed Graph Theorem can be used to prove the boundedness of operators that are not initially known to be continuous in real-world applications. This is very helpful for different applications in quantum physics, differential equations, and functional analysis.

Theorem 4.2 ***(Hartwig-Katz Theorem*** *[46]) Let U and V be $n \times n$ EP matrices. The following conditions are equivalent:*
(i). Let U and V.
(ii). $R(UV) = R(U) \cap R(V)$ and $RS(UV) = RS(U) \cap RS(V)$, where RS means row space.
(iii). $Ran(UV) \subseteq Ran(U)$ and $RS(UV) \subseteq Ran(U)$.

Theorem 4.3 ***(Open Mapping Theorem*** *[37]) Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach spaces and $J : \mathcal{A} \to \mathcal{B}$ a continuous linear operator. If J is surjective, then J is an open map, that is, for every open set $F \subset \mathcal{A}$, the set $J(F)$ is open in $\mathcal{B}$.*

Theorem 4.4 ***(Inverse Mapping Theorem)****[4] States that every continuously differentiable operator is closed and bounded.*

It is a fundamental concept in multivariable calculus, the Inverse Mapping Theorem sheds light on the local invertibility of differentiable functions. The inverse function's existence and differentiability are made possible by the requirement that the Jacobian be invertible, which guarantees the function's good behaviour close to the point of interest. This theorem helps solve challenging mathematical puzzles in a variety of academic fields and expands our knowledge of the local structure of functions.

5 Technical Approaches

Tensor product: This is a technical approach that is useful in tensor analysis of the operators. We consider TP of EOs and check the underlying structures whether they are well defined and if they possess the properties of operators. Then NR and Sp are characterized for the TP of these operators.
Direct sum decomposition: This is a technical approach that is useful in analysis of matricial operators. We consider DSD of EOs and check the underlying structures whether they are well defined and if they possess the properties of matricial operators. Then Sp are characterized for the DSD of these matricial operators.

6 Main results

In this section, we investigate spectral properties of orthogonal isometries. We carry out our investigation under different technical approaches. We begin by providing the necessary condition that guarantee $(A_1 \oplus ... \oplus A_n, m)-$isometry.

Proposition 6.1 *Let* $A_i \in B(H)$ $i = 1, 2, ..., n$ *and* $A_i \neq 0$, *then the spectrum of* $(A_1 \oplus ... \oplus A_n, m)-$*isometry intersects the unit circle.*

Proof. Since $(A_1 \oplus ... \oplus A_n, m)$ is an $(A_1 \oplus ... \oplus A_n)'$-isometry where A_i' are A_i-covariance of T_i. Suppose $T_1 \oplus ... \oplus T_n$ is an $A_1 \oplus ... \oplus A_n$-isometry such that $\delta(T_1 \oplus ... \oplus T_n) \cap \partial\mathcal{D} = \phi$. We show that $(A_1 \oplus ... \oplus A_n)$ is a null operator. Now $H \oplus ... \oplus H = (H \oplus ... \oplus H)_1 + (H \oplus ... \oplus H)_2$, $T_1 \oplus ... \oplus T_n = (T_1 \oplus ... \oplus T_n)_1 + (T_1 \oplus ... \oplus T_n)_2$ with $T_1 \oplus ... \oplus T_n) = (T_1 \oplus ... \oplus T_n) \mid_{(H\oplus...\oplus H)_1 + (H\oplus...\oplus H)_2}$ such that $\delta_1 = \delta(T_1 \oplus ... \oplus T_n)_1 = \delta(T_1 \oplus ... \oplus T_n) \cap \mathcal{D}$ and $\delta_2 = \delta(T_1 \oplus ... \oplus T_n)_2 = \delta(T_1 \oplus ... \oplus T_n) \cap \overline{\mathcal{D}}^C$. Let us show that $(A_1 \oplus ... \oplus A_n) \mid_{(H_1\oplus...\oplus H_n)} = 0$. Using that $(H_1 \oplus ... \oplus H_n)_1$ is an invariant subspace of $(T_1 \oplus ... \oplus T_n)$ and $(T_1 \oplus ... \oplus T_n)$ is $(A_1 \oplus ... \oplus A_n)-$ isometry then we have $(T_1 \oplus ... \oplus T_n)_1$ is $(P_1 \oplus ... \oplus P_n)_{(H_1\oplus...\oplus H_n)_1}(A_1 \oplus ... \oplus A_n) \mid_{(H_1\oplus...\oplus H_n)_1}$ $-$isometry and with spectral radius less than 1. Since $\delta(T_1 \oplus ... \oplus T_n)_1 \subset \mathcal{D}$. Hence $(T_1 \oplus ... \oplus T_n)_1^m (h_1 + ... + h_n)_1 \to 0$ for $(h_1 + ... + h_n)_1 \in (H_1 \oplus ... \oplus H_n)_1$. Thus for any $(h_1 + ... + h_n)_1 \in (H_1 \oplus ... \oplus H_n)_1$ we have that $\langle (A_1 \oplus ... \oplus A_n)(h_1 + ... + h_n), (h_1 + ... + h_n) \rangle = \langle (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^m (h_1 + ... + h_n), (T_1 \oplus ... \oplus T_n)^m (h_1 + ... + h_n) \rangle = \langle (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)_1^M (h_1 + ... + h_n), (T_1 \oplus ... \oplus T_n)_1^m (h_1 + ... + h_n) \rangle \to 0$ as $m \to \infty$. Thus $(A_1 \oplus ... \oplus A_n) \mid_{(H_1\oplus...\oplus H_n)_1} = 0$. Further, since $\delta(T_1 \oplus ... \oplus T_n) \cap \overline{\mathcal{D}}^C$, then $(T_1 \oplus ... \oplus T_n)_2$ is invertible and $(P_1 \oplus ... \oplus P_n)_{(H_1\oplus...\oplus H_n)}(A_1 \oplus ... \oplus A_n) \mid_{(H_1\oplus...\oplus H_n)_2}$ $-$ is an isometry hence $\langle (A_1 \oplus ... \oplus A_n)(h_1 + ... + h_n), (h_1 + ... + T_n) \rangle = 0$.

We now establish the link between compact subsets and the spectral properties of $(T_1 \oplus ... \oplus T_n)$ that is $(A_1 \oplus ... \oplus A_n)-$isometry.

Lemma 6.2 *Let* K_i *be a compact subset of* $\mathcal{C}_i$ *such that* $(K_1 \oplus ... \oplus K_n) \cap \partial(\mathcal{D}_1 \oplus ... \oplus \mathcal{D}_n) \neq \phi$ *then there exists* H_i *and* $T_i, A_i \in B(H)$, *with* $A_i \geq 0$ *such that* $(T_1 \oplus ... \oplus T_n)$ *is* $(A_1 \oplus ... \oplus A_n)-$*isometry with* $\delta(T_1 \oplus ... \oplus T_n) = (K_1 \oplus ... \oplus K_n)$.

Proof. Let $(H_1 \oplus ... \oplus H_n) = l_2(\mathcal{N} \oplus \mathcal{C})_1 \oplus ... \oplus l_2(\mathcal{N} \oplus \mathcal{C})_n$. Let A be a $2x2$ matrix with $\alpha_i > 0$ in the second row and second column position, since $(K_1 \oplus ... \oplus K_n) \cap \partial(\mathcal{D}_1 \oplus ... \oplus \mathcal{D}_n) \neq \phi$, let T_i be a linear operator an H as T_i a $2x2$ matrix with $(D_1 \oplus ... \oplus D_n)_i$ is the first row first column position and $(\lambda_1 \oplus ... \oplus \lambda_n)_i$ in the second row and second column position where $(\lambda_1 \oplus ... \oplus \lambda_n)_i \in (K_1 \oplus ... \oplus K_n) \cap \partial(\mathcal{D}_1 \oplus ... \oplus \mathcal{D}_n)$ and $(D_1 \oplus ... \oplus D_n)((x_1 + ... + x_n)_1, (x_1 + ... + x_n)_1, ...) = (\beta_1 + ... + \beta_n)_{i1}(x_1 + ... + x_n)_1, (\beta_1 + ... + \beta_n)_{i2}(x_1 + ... + x_n)_2, ...)$

with $\overline{\{(\beta_1 + ... + \beta_n)_{in} : n \in \mathcal{N}\}} = (K_1 \oplus ... K_n)_i$, then we get $(T_1 + ... + T_n)$ is an $(A_1 \oplus ... \oplus A_n)-$isometry and $\delta(T_1 \oplus ... \oplus T_n) = \delta(D_1 \oplus ... \oplus D_n) \cup (\lambda_1 + ... + \lambda_n = K_1 \oplus ... \oplus K_n$.

Lemma 6.3 *Suppose $A_i, T_i \in B(H_i)$ such that $(T_1 \oplus ... \oplus T_n)$ is $(A_1 \oplus ... \oplus A_n)-$isometry and $(0, 0, ..., 0)$ to not in $\delta_p(T_1 \oplus ... \oplus T_n)$, then there exists $M > 0$ such that $\| (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k \| \leq M$, for all positive integer k.*

Proof. If $(A_1 \oplus ... \oplus A_n)$ is an $(A_1 \oplus ... \oplus A_n)-$isometry. Then $\| (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k(x_1 + ... + x_n) \| = \| (A_1 \oplus ... \oplus A_n) \| \leq \|(A_1 \oplus ... \oplus A_n)\| \|(x_1 + ... + x_n)\|$. Therefore, $\| (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k \| \leq M$, for all positive integer k.

Theorem 6.4 *Let $A_i, T_i \in B(H_i)$ such that $(T_1 \oplus ... \oplus T_n)$ is $(A_1 \oplus ... \oplus A_n)-$isometry and $(0, 0, ..., 0)$ is not in $\delta_p(T_1 \oplus ... \oplus T_n)$, then for nonzero $x_1 + ... + x_n \in (H_1 \oplus ... \oplus H_n), (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k(x_1 + ... + x_n) \to (0, 0, ..., 0)$.*

Proof. Suppose $x_1 + ... + x_n \in (H_1 \oplus ... \oplus H_n)$ is anonzero vector, since $(0, 0, ..., 0)$ is not in $\delta_p(T_1 \oplus ... \oplus T_n)$, then we have $\| (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k(x_1 + ... + x_n) \| = \| (A_1 \oplus ... \oplus A_n)(x_1 + ... + x_n) \| \neq 0$. Therefore, $\| (A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n)^k(x_1 + ... + x_n) \| \to \| (A_1 \oplus ... \oplus A_n)(x_1 + ... + x_n) \| \neq 0$ as $k \to \infty$. Hence, $(A_1 \oplus ... \oplus A_n)(T_1 \oplus ... \oplus T_n) \to 0$ as $k \to \infty$ for all nonzero $x_1 + ... + x_n \in (H_1 \oplus ... \oplus H_n)$.

We now provide conditions for the direct sum of an isometry and its adjoint to be an isometry.

Proposition 6.5 *Let T be isometric. Suppose n is a positive integer such that $n \geq n_0$, $N(T^*)^{n_0} = N(T^*)^n$ then $S = T \oplus T^*$ is isometric. Moreover, $\overline{\alpha + \overline{\alpha}} \in \delta(S^*)$ whenever $\alpha + \overline{\alpha} \in \delta(S)$.*

Proof. Let $\alpha, \alpha \in \delta_P(T \oplus T^*)$. Suppose that $\alpha = 0$ and $\overline{\alpha} = 0$. If $0 \in \mathcal{C} \setminus \delta_P(T \oplus T^*)^*$, then $(T \oplus T^*)^{*2}(T \oplus T^*)^2 = (T \oplus T^*)^*(T \oplus T^*), (T \oplus T^*)(T \oplus T^*)^2 = (T \oplus T^*)$ or $(T \oplus T^*)^{*2}(T \oplus T^*) = (T \oplus T^*)^*$. Now suppose α and $\overline{\alpha}$ are all nonzero, put x be such that $Tx = \alpha x$ and y be such that $T^*y = \overline{\alpha}y$, then since $(T \oplus T^*)^{*2}(T \oplus T^*)^2 = (T \oplus T^*)^*(T \oplus T^*)$, then $\alpha + \overline{\alpha}(T \oplus T^*)(x + y) = (\alpha + \overline{\alpha})^2(T \oplus T^*)^2(x + y)$. Now $| \alpha + \overline{\alpha} |= 1$ and therefore $(T \oplus T^*)^* - (\alpha + \overline{\alpha})(I \oplus I)(T \oplus T^*)^*)(x + y) = 0$. To show that $\overline{\alpha + \overline{\alpha}} \in \delta(S^*)$, we need to show that $(T \oplus T^*)^*)(x + y) \neq 0$. Suppose $((T \oplus T^*) = 0$, then $0 = \langle x + y, (T \oplus T^*)^*(x + y) \rangle = \langle (T \oplus T^*)x + y, x + y \rangle = \alpha + \overline{\alpha}\langle x + y, x + y \rangle$ and hence $\alpha + \overline{\alpha} = 0$ since $x + y$ is nonzero. This is a contradiction since $|\alpha + \overline{\alpha}| = 1$.

We extend the above properties to the approximate spectral properties.

Proposition 6.6 *Let T be isometric. Suppose n is a positive integer such that $n \geq n_0$, $N(T^*)^{n_0} = N(T^*)^n$ then $S = T \oplus T^*$ is isometric. Moreover, $\overline{\alpha + \overline{\alpha}} \in \delta_{ap}(S^*)$ whenever $\alpha + \overline{\alpha} \in \delta_{ap}(S)$.*

Proof. Let $\alpha + \overline{\alpha} \in \delta_{ap}(T \oplus T^*)$. Suppose $\alpha + \overline{\alpha} = 0$ then it follows that $0 \in \delta_{ap}(T \oplus T^*)^*$. Now suppose $\alpha + \overline{\alpha} \neq 0$, and if $x_n + y_n$ is a unit vector such that $(T \oplus T^*) - (\alpha + \overline{\alpha})(I \oplus I)(x_n + y_n) \to 0$ then $-(\alpha + \overline{\alpha})^2 (T \oplus T^*)^{*2}(x_n + y_n) - (\alpha + \overline{\alpha})(T \oplus T^*)^*)(x_n + y_n) = (T \oplus T^*)^{*2}((T \oplus T^*)^2(x_n + y_n) + (\alpha + \overline{\alpha})^2(x_n + y_n)) - (T \oplus T^*)^*(T \oplus T^*)(x_n + y_n) - (\alpha + \overline{\alpha})(x_n + y_n) \to 0$ as $n \to \infty$ or $((\alpha + \overline{\alpha})(T \oplus T^*)^* - (I \oplus I)(x_n + y_n) \to 0$. Now since $(\alpha + \overline{\alpha}) = lim\langle (T \oplus T^*)(x_n + y_n), x_n + y_n \rangle = lim\langle (x_n + y_n), (T \oplus T^*)^*(x_n + y_n) \rangle$ and $(\alpha + \overline{\alpha}) \neq 0, (T \oplus T^*)(x_n + y_n)$ does not converge to zero. Now choose $(T \oplus T^*)^*(x_n + y_n)_k$ be a subsequence of $(T \oplus T^*)(x_n + y_n)$ such that $\| (T \oplus T^*)(x_n + y_n)_k \| \geq M$ for some positive number M. Set $(z_k + w_k) = \frac{(T \oplus T^*)(x_n + y_n)_k}{\|(T \oplus T^*)^*(x_n + y_n)_k\|}$, then $(z_k + w_k)$ is a sequence of unit vectors such that $(\alpha + \overline{\alpha})(T \oplus T^*) - (I \oplus I)(z_k + w_k)(z_k + w_k) \to 0$ and $| \alpha + \overline{\alpha}) | = 1$.

Lemma 6.7 *Let T be isometric. Suppose n is a positive integer such that $n \geq n_0$, $N(T^*)^{n_0} = N(T^*)^n$ then $S = T \oplus T^*$ is isometric. Moreover, the eigenspace corresponding to distinct nonzero eigenvalue of S are mutually orthogonal.*

Proof. Let $\alpha + \overline{\alpha}$ and $\beta + \overline{\beta}$ be distinct nonzero eigenvalues of $T \oplus T^*$. Suppose $(T \oplus T^*)(x + y) = (\alpha + \overline{\alpha})(x + y)$ and $(T \oplus T^*)(w + z) = (\beta + \overline{\beta})(w + z)$ then $0 = \langle (T \oplus T^*)^2(x + y), \overline{(T \oplus T^*)}(w + z) \rangle - \langle (T \oplus T^*)(x + y), (T \oplus T^*)(w + z) \rangle = (\alpha + \overline{\alpha})\overline{(\beta + \overline{\beta})}((\alpha + \overline{\alpha})\overline{(\beta + \overline{\beta})} - 1)(x + y, w + z)$. Since $(\alpha + \overline{\alpha}) \neq 0$ and $(\beta + \overline{\beta}) \neq 0$ then $(\alpha + \overline{\alpha})\overline{(\beta + \overline{\beta})} \neq 0$ and $| \beta + \overline{\beta} | = 1$. Additionally, $\alpha + \overline{\alpha} \neq \beta + \overline{\beta}$. Therefore, $\alpha + \overline{\alpha} \neq \frac{1}{\overline{\beta + \overline{\beta}}}$ or $(\alpha + \overline{\alpha})\overline{(\beta + \overline{\beta})} \neq 0$. Hence, $\langle x + y, w + z \rangle = 0$.

Next, we characterize conditions under which the direct sum of two partial isometries is an isometry.

Lemma 6.8 *Let $A_i \in M_n, i = 1, 2$ be partial isometries and $M_i \subseteq H, i = 1, 2, 3, 4$. Suppose $M_1 \oplus M_2$ and $M_3 \oplus M_4$ are orthogonal and $A_1 \oplus A_2 : M_1 \oplus M_2 \to M_3 \oplus M_4$, then $\delta(A_1 \oplus A_2) \subseteq \mathcal{D}$.*

Proof. Suppose $A_i x_i = \lambda_i x_i$ and $\| x_1 + x_2 \|$, then $| \lambda_1 + \lambda_2 | = \| (A_1 \oplus A_2)(x_1 + x_2) \| \leq \| (A_1 \oplus A_2) \| \| (x_1 + x_2) \| < 1$. Since $A_1 \oplus A_2$ preserves the partial isometry properties then $\| (A_1 \oplus A_2) \| = 1$. Thus $\delta(A_1 \oplus A_2) = \delta(A_1) \cup \delta(A_2)$. Since A_i are partial isometries, then $\delta(A_1) \subseteq \mathcal{D}^-$ and $\delta(A_2) \subseteq \mathcal{D}^-$, thus $\delta(A_1) \cup \delta(A_2) \subseteq \mathcal{D}^-$.

We now establish the link between partial isometries and the direct sum of upper triangular partial isometries and unitary operators.

Theorem 6.9 *Let $A_i \in M_n, i = 1, 2$ be partial isometries and $A = T \oplus U$, and $A_1 \oplus A_2 : M_1 \oplus M_2 \to M_3 \oplus M_4$, then $\delta(T_i) \subseteq \overline{\mathcal{D}}$ if T_i are upper triangular partial isometries.*

Proof. Since $A_i \in M_n$ are partial isometries and $\delta(T) \subseteq \mathcal{D}$ and U is upper triangular with $\delta(U) \subseteq \mathcal{T}$, then since $A = T \oplus U$, then A_i are contractions. Additionally, since A_i are contractions then $A_1 \oplus A_2$ is also a contraction hence $\delta(A_1 \oplus A_2) = \delta(A_1) \cup \delta(A_2)$. But if A_i are contractions then $\delta(A_1) \subseteq \overline{\mathcal{D}}$ and $\delta(A_2) \subseteq \overline{\mathcal{D}}$, therefore $\delta(A_1 \oplus A_2) = \delta(A_1) \cup \delta(A_2) \subseteq \overline{\mathcal{D}}$.

We now extend our investigations to power isometries.

Theorem 6.10 *Let $A_i \in M_n$ be a partial isometry, suppose that $A_1 \oplus A_2$ has orthogonal subspaces in the initial and final spaces, then $(A_1 \oplus A_2) \to 0$ if A_i are completely non unitary.*

Proof. Suppose A_i are partial isometries and suppose that the initial and final subspaces of $A_1 \oplus A_2$ are orthogonal, then $A_1 \oplus A_2$ is a partial isometry. Since the direct sum preserves Jordan canonical form properties, then $(A_1 \oplus A_2)^n \to 0$. Therefore, $\delta(A_1 \oplus A_2) = \delta(A_1) \cup \delta(A_2) \subseteq \mathcal{D}$.

We now investigate the spectral properties of the product of two projections.

Corollary 6.11 *Let $P_1, P_2 \in M_n$ be two projections, then $\delta(P_1P_2) \subseteq \mathcal{D}$.*

Proof. Let P_1 and P_2 be orthogonal projections, then P_1P_2 is a partial isometry. Now, if P_1 and P_2 are projections, then $\delta(P_1) \subseteq \mathcal{D}$ and also $\delta(P_2) \subseteq \mathcal{D}$. Therefore, $\delta(P_1P_2) = \delta(P_1)\delta(P_2) \subseteq \mathcal{D}$.

We now establish conditions that preserves isometry properties between and isometry and its adjoint.

Proposition 6.12 *Let T be a surjective isometry, then $\delta(TT^*) \subseteq \mathcal{D}$.*

Proof. Let T be a surjective isometry, then $TT^* = T^*T = I$. Now $\delta(TT^*) = \delta(T^*T) = \delta(I)$. Since $\delta(T) \subseteq \mathcal{D}$ and $\delta(T^*) \subseteq \mathcal{D}$, then $\delta(T)\delta(T^*) \subseteq \mathcal{D}$. Also $\delta(T^*)\delta(T) \subseteq \mathcal{D}$. Therefore, the spectrum of T^*T lies in the complex unit plane.

We now provided conditions for the square of an isometry to be an isometry.

Proposition 6.13 *Let $T \in B(H)$ be an isometry such that $T^*T = TT^*$, then $\delta(T^2) \subseteq \mathcal{T}$ or $\delta(T^2) = \overline{\mathcal{D}}$.*

Proof. Let $T \in B(H)$ be such that $T^*T = TT^*$, then T^2 is an isometry, hence $\forall \mid \beta \mid< 1, \quad T^2 - \beta$ is an upper semi-Fredholm. Suppose that $(T^2)^{-1}$ exists, then ind$T^2 = 0$, thus by continuity of index, ind$(T^2 - \beta) = 0$ $\forall \beta$. Thus,

$(T^2-\beta)^{-1}$ exists $\forall \mid \beta \mid < 1$. Hence, $\delta(T^2) \subseteq \mathcal{T}$. Suppose $(T^2)^{-1}$ does not exist then $ind(T^2) = -codimR(T^2)^{-1} < 0$ and by continuity of the index then ind$(T^2-\beta) < 0, \forall \mid \beta \mid < 1$. So, $\delta(T^2) \supseteq \mathcal{D}$ hence $\delta(T^2) = \overline{\mathcal{D}}$.

We now provide conditions for the square of an isometry to be an isometry under direct sum.

Lemma 6.14 *Let $T, S \in B(H)$ be orthogonal isometries, then for $T^2 \oplus S^2$ and $x+y \in H_1 \oplus H_2$, then either $x+y \in \cap_{i=1}^{n} R(T^2 \oplus S^2)^n$ and $\delta_{(T^2 \oplus S^2)}(x+y) \subset \mathcal{T}$ or $x+y$ is not in $\cap_{i=1}^{n} R(T^2 \oplus S^2)^n$ and then $\delta_{(T^2 \oplus S^2)}(x+y) = \overline{\mathcal{D}}$.*

Proof. Let $M_1 \subseteq H_1$ and Let $M_2 \subseteq H_2$, then if $M_1 \oplus M_2 = \cap_{i=1}^{\infty} R(T^2 \oplus S^2)$, let $x+y \in M_1 \oplus M_2$, then $\delta_{(T^2 \oplus S^2)}(x+y) \subseteq \delta(T^2 \oplus S^2) \mid_{(M_1 \oplus M_2)} \subseteq \mathcal{T}$. Now suppose $x+y$ is not in $M_1 \oplus M_2$, then $x+y$ is not in $\cap_{n=1}^{\infty} R((T^2 \oplus S^2 - (\lambda+\beta))^n) \forall \lambda, \beta \in \mathcal{D}$. Thus $(\lambda+\beta) \in \delta_{(T^2 \oplus S^2)}(x+y) \forall \lambda, \beta \in \mathcal{D}$. Therefore $\delta_{(T^2 \oplus S^2)}(x+y) = \overline{\mathcal{D}}$.

We now establish the link between the spectrum, local spectrum and approximate spectrum.

Theorem 6.15 *Let $T, S \in B(H)$ be orthogonal invertible isometries. Then $\delta(T^2 \oplus S^2) = \delta_{loc}(T^2 \oplus S^2) = \delta_{ap}(T^2 \oplus S^2) \subseteq \mathcal{T}$.*

Proof. Since isometries have single value extension property, then $\delta_{loc}(T^2 \oplus S^2) \subseteq \delta_{ap}(T^2 \oplus S^2)$. Since $(T^2 \oplus S^2)$ is m-isometry, then it is decomposible. Thus, $\delta(T^2 \oplus S^2) = \delta_{loc}(T^2 \oplus S^2)$. Thus $\delta_{loc}(T^2 \oplus S^2) \subseteq \delta_{ap}(T^2 \oplus S^2) \subseteq \mathcal{T}$.

7 Open Problems

Isometries are special mappings with a property of distance preservation. This property makes them unique and interesting in-terms of characterization. However, it is very difficult to do characterizations of properties of isometires in a general Banach space setting due to the intricate underlying structures in Banach spaces. In this paper, we have characterized spectral properties of isometries and in particular when they are orthogonal. We have shown that orthogonal isometries preserve distance even when they are subjected to direct sum decomposition. This leaves two open problems that should be tackled.
Problem 1: Can one develop an efficient algorithm for analyzing the distance preservation conditions for orthogonal isometries as characterized in this work?
Problem 2: Do orthogonal isometries preserve distance even when they are subjected to tensor products?

References

[1] Abe T., Akiyama S., Hatori O., Isometries of the special orthogonal group, Linear Algebra Appl., 439(2013), 174-188.

[2] Aouichaoui M. A., Mosic D., On polynomially partial-A-isometric operators, Turkish journal of Mathematics, 47(7), (2023), 2122-2138.

[3] Bagheri-Bardi G. A., On the decomposition of contraction and Isometries, Scientia Mathematicae Japonicae, 1(2014), 1-8.

[4] Bermudez T., Saddi A., Zaway H., (A, m)-Isometries on Hilbert spaces, Bull. AMS, 102(2017), 1-12.

[5] Bourgin D. G., Approximate isometries, Bull. Amer. Math. Soc., 52(1946), 288-292.

[6] Bourgin D. G., Approximately isometric and multiplicative transformations on continuous function rings, Duke Math. J., 16(1949), 385-397.

[7] Bourgin D. G., Two dimensional ε-isometries, Trans. Amer. Math. Soc., 244(1978), 85-102.

[8] Bourgin D. G., Approximate isometries on finite dimensional Banach spaces, Trans. Amer. Math. Soc., 207(1975), 309-328.

[9] Cheng L., Dong Y., A note on the stability of nonsurjective ε-isometries of Banach spaces, Proc. Amer. Math. Soc., 148(2020), 4837-4844.

[10] Cheng L., Dong Y., Zhang J., On stability of nonlinear non-surjective ε-isometries of Banach spaces, J. Funct. Anal., 264(2013), 713-734.

[11] Cheng L., Zhou Y., On perturbed metric-preserved mappings and their stability characterizations, J. Funct. Anal., 266(2014), 4995-5015.

[12] Coburn L. A., The C^*-algebra generated by an isometry, Bull. Amer. Math. Soc., 73(1967), 8-19.

[13] Curto R. E., The spectra of elementary operators, Indian Uni. Math. J., 32(1983), 193-197.

[14] De Jeu M., Pinto P. R., The structure of doubly non-commuting isometries, Adv. Math., 368(2020), 107-149.

[15] Dilworth S. J., Approximate isometries on finite-dimensional normed spaces, Bull. London Math. Soc., 31(1999), 704-714.

[16] Embry M., Rosenbulm M., Spectra tensor products and linear operator equations, Pacific J. Math., 53(1974), 95-107.

[17] Figiel T., On non linear isometric embeddings of normed linear spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 16(1968), 185-188.

[18] Guesba M., Symmetric properties of elementary operators, Novi Sad J. Math., 61(2012), 1-6.

[19] Gevirtz J., Stability of isometries on Banach spaces, Proc. Amer. Math. Soc., 89(1983), 633-636.

[20] Gruber P. M., Stability of isometries, Trans. Amer. Math. Soc., 245(1978), 263-277.

[21] Guralnick R.M., On isometry groups of self-adjoint traceless and skew-symmetric matrices, arXiv:1709.04507.

[22] Koldobsky A., Operators preserving orthogonality are isometries, arXiv: math/9212203v1[math.FA]., (1992), 1-4.

[23] Matache V., Composition operators whose symbols have orthogonal powers, Houston journal of Mathematics, 37(3), (2011), 845-857.

[24] Manuel G., Mostafa M., Linear maps that preserve Semi- Fredholm operators acting on Banach Spaces, Acta scientiarum Mathematicarum, 84(12), (2018), 137-149.

[25] Hatori O., Isometries of the unitary groups in C^*-algebras, Studia Math., 221(2014), 61-86.

[26] Hatori O., Molnár P., Generalized isometries of the special unitary group, Arch. Math., 106(2016), 155-163.

[27] Hatori O., Molnár P., Isometries of the unitary group, Proc. Amer. Math. Soc., 140(2012), 2141-2154.

[28] Hyers D. H., Ulam S. M., On approximate isometries, Bull. Amer. Math. Soc., 51(1945), 288-292.

[29] Hyers D. H., Ulam S. M., On approximate isometries on the space of continuous functions, Ann. Math., 48(1947), 285-289.

[30] Kiratu B. N., On the Spectral properties of 2-isometries and related operators on a Hilbert space, Masters Thesis, School of mathematics, University of Nairobi, 2011.

[31] Koehler D., Rosenthal P., On isometries of normed linear spaces, Studia Mathematica, 36(1970), 213-216.

[32] Koldobsky A., Operators preserving orthogonality are isometries, Proc. R. Soc. Edinburgh Sect. A, 123(1993), 835-837.

[33] Omladi M., Semrl P., On non linear perturbations of isometries, Math. Ann., 303(1995), 617-628.

[34] ørgensen P.J., Proskurin D., On C^*-algebras generated by pairs of q-commuting isometries, J. Phys. A, 38(12), (2005), 266-780.

[35] Okelo. N. B, Norms of self-adjoint two-sided multiplication operators in norm-attainable class, MathLAB journal, 2(2020), 12-22.

[36] Ould A. M., On the joint (m, q)-partail isometries and the joint m-invertible tuples of commuting operators on a Hilbert space, Italian journal of Pure and Applied Mathematics, 4(2018), 438-463.

[37] Patel S. M., A note on quasi-isometries, Glasnik Matematicki, 35(55), (2000), 307-312.

[38] Qian S., ε-Isometric embeddings, Proc. Amer. Math. Soc., 123(1995), 1797-1803.

[39] Sain D., Manna J., Paul K., On local preservation of orthogonality and its application to isometries, Linear Algebra Appl., 690(2024), 112-131.

[40] Sarkar J., Wold decomposition for doubly commuting isometries, Linear Algebra Appl. 445(2014), 289-314.

[41] Schaefer H., Some spectral properties of positive linear operator, Pacific journal of Mathematics, 10(3), (1960), 1009-1019.

[42] Schmoeger C., Partial Isometries on Banach spaces, Bul. AMS, 67(2000), 1-13.

[43] Wang C., Wu X., On the spectral theory of positive operators and pde applications, Discrete and continuous dynamical systems, 40(6),(2020), 3171-3200.

[44] Wolfgang A., Spectral properties of Lamperti operators, Indiana University Mathematics journal, 32(2), (1983), 199-215.

[45] Wu P. Y., Approximation by partial isometries, Proceedings of Edinburgh Mathematical society, 29(1986), 255-261.

[46] Zenon J. J,Bong. J, and Jan S., m-isometric operator and their local properties, arXiv:1906.05215v1[math]., (2019), 1-19.

[47] Weber M., On C^*-algebras generated by isometries with twisted commutation relations, J. Func. Anal., 264 (2013), 1975-2004.