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Some new parametric integral formulas depending on an adjustable function

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Abstract

This article presents new integral formulas based on a single adjustable function and one or two minimally restricted parameters. These formulas are not found in previous literature. They are notable for their simplicity and their links to key mathematical constants and special functions.

Keywords: *Parametric integral formulas, Logarithmic function, Catalan constant, Pi, Special functions.*

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1 Introduction

Integral formulas play a central role in science. They are among the most useful mathematical tools. For a comprehensive overview of the most useful of these formulas, see the book [5]. The continued interest in discovering new integral formulas stems from their ability to solve emerging problems in a variety of fields. Recent developments and contributions in this area can be found in [6, 7, 8, 9, 2, 3, 4].

In this article, we contribute to this field by establishing several new tractable results. The integral formulas obtained depend on a single adjustable function and one or two tuning parameters. Notably, these parameters are subject to minimal restriction; one of them may even span the entire real line. These formulas also do not appear in [5], suggesting their potential for new and diverse practical applications. They are of particular interest due to their simplicity and broad applicability in various mathematical contexts. Furthermore, some of these formulas are connected to fundamental mathematical constants and functions, such as π, the Catalan constant and various special functions, including the sine integral function and the gamma integral function.

The structure of the article is as follows: Section 2 presents two main theorems. Section 3 discusses several corollaries derived from the second theorem. Section 4 formulates an open problem. Section 5 offers concluding remarks and perspectives on future work.

2 Two key theorems

2.1 First theorem

The first theorem provides a one-parameter integral formula that depends on a single adjustable function. This formula always yields a result of zero. The proof employs a fundamental change of variable approach.

Theorem 2.1 *Let $f : [0, +\infty)^2 \mapsto [0, +\infty)$ be a function such that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. Suppose that, for any $\alpha \in \mathbb{R}$, the integral*

$$I_\alpha := \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{x^\alpha \log(x)}{(x + x^\alpha)^2} dx$$

converges. Then, for any $\alpha \in \mathbb{R}$, we have

$$I_\alpha = 0.$$

Proof. For any $\alpha \in \mathbb{R}$, performing the change of variables $x = 1/y$, and using the properties of f and of the logarithmic function, we get

$$\begin{aligned} I_\alpha &= \int_{+\infty}^0 f\left(\frac{1}{y}, y\right) \frac{(1/y)^\alpha \log(1/y)}{[1/y + (1/y)^\alpha]^2} \left(-\frac{1}{y^2} dy\right) \\ &= -\int_0^{+\infty} f\left(y, \frac{1}{y}\right) \frac{y^\alpha \log(y)}{(y^\alpha + y)^2} dy = -I_\alpha. \end{aligned}$$

As a result, we have $I_\alpha = 0$. This concludes the proof. □

Some examples of applications of this theorem are given below.

Example 1. For any $x, y > 0$, and $\beta > 0$, let us consider

$$f(x, y) := \frac{1}{x^\beta + y^\beta}.$$

It is obvious that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. For any $\alpha \in \mathbb{R}$, Theorem 2.1 applied to this function gives

$$\int_0^{+\infty} \frac{x^{\alpha+\beta} \log(x)}{(x^{2\beta} + 1)(x + x^\alpha)^2} dx = 0.$$

Example 2. For any $x, y > 0$, and $\beta > 0$, let us consider

$$f(x, y) := \frac{1}{x^\beta + y^\beta + x^\beta y^\beta}.$$

It is clear that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. For any $\alpha \in \mathbb{R}$, Theorem 2.1 applied to this function gives

$$\int_0^{+\infty} \frac{x^{\alpha+\beta} \log(x)}{(x^{2\beta} + x^\beta + 1)(x + x^\alpha)^2} dx = 0.$$

Example 3. For any $x, y > 0$, and $\beta > 0$, let us consider

$$f(x, y) := \min(x^\beta, y^\beta).$$

It is obvious that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. For any $\alpha \in \mathbb{R}$, Theorem 2.1 applied to this function gives

$$\int_0^{+\infty} \min\left(x^\beta, \frac{1}{x^\beta}\right) \frac{x^\alpha \log(x)}{(x + x^\alpha)^2} dx = 0.$$

Example 4. For any $x, y > 0$, let us consider

$$f(x, y) := \arctan[\min(x, y)].$$

It is clear that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. For any $\alpha \in \mathbb{R}$, Theorem 2.1 applied to this function gives

$$\int_0^{+\infty} \arctan\left[\min\left(x, \frac{1}{x}\right)\right] \frac{x^\alpha \log(x)}{(x + x^\alpha)^2} dx = 0.$$

Note that all of these integral formulas have the property that α belongs to the set of real numbers without restriction, which makes them particularly flexible and applicable.

2.2 Second theorem

The second theorem is more technical than the first. It provides a one-parameter integral formula that depends on a single adjustable function. The result is a simple integral that is independent of the parameter. Two different proofs are provided.

Theorem 2.2 *Let $f : [0, +\infty)^2 \mapsto [0, +\infty)$ be a function such that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. Suppose that, for any $\alpha \in \mathbb{R}$, the integral*

$$J_\alpha := \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha} dx$$

converges. Then, for any $\alpha \in \mathbb{R}$, *we have*

$$J_\alpha = \frac{1}{2}\int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx.$$

Proof. To demonstrate the versatility of this theorem, two proofs are presented: Proof 1 employs Theorem 2.1, while Proof 2 uses a suitable decomposition of the integrand.

Proof 1: Using Theorem 2.1. For any $x > 0$ and $\alpha \in \mathbb{R}$, using standard differentiation rules, we have

$$\frac{\partial}{\partial \alpha}\left[f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha}\right] = -f\left(x, \frac{1}{x}\right) \frac{x^\alpha \log(x)}{(x + x^\alpha)^2}.$$

It follows from the Leibniz integral rule and Theorem 2.1 that

$$\begin{aligned}\frac{\partial}{\partial \alpha} J_\alpha &= \frac{\partial}{\partial \alpha}\left[\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha} dx\right] \\ &= \int_0^{+\infty} \frac{\partial}{\partial \alpha}\left[f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha}\right] dx \\ &= -\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{x^\alpha \log(x)}{(x + x^\alpha)^2} dx = 0.\end{aligned}$$

As a result, for any $\alpha \in \mathbb{R}$, we can write

$$J_\alpha = C,$$

where C denotes a certain constant independent of α.

In particular, taking $\alpha = 1$, we find that

$$C = J_1 = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x} dx = \frac{1}{2}\int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx.$$

Therefore, for any $\alpha \in \mathbb{R}$, we have

$$J_\alpha = \frac{1}{2}\int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx.$$

Proof 2: Direct approach. For any $\alpha \in \mathbb{R}$, performing the change of variables $x = 1/y$ and using the property of f, we have

$$\begin{aligned}J_\alpha &= \int_{+\infty}^0 f\left(\frac{1}{y}, y\right) \frac{1}{1/y + (1/y)^\alpha}\left(-\frac{1}{y^2} dy\right) \\ &= \int_0^{+\infty} f\left(y, \frac{1}{y}\right) \frac{1}{y + y^{2-\alpha}} dy = \int_0^{+\infty} f\left(y, \frac{1}{y}\right) \frac{y^{\alpha-1}}{y^\alpha + y} dy \\ &= \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{x^{\alpha-1}}{x + x^\alpha} dx.\end{aligned}$$

Exploiting this new expression of J_α and factorizing, we have

$$2J_\alpha = J_\alpha + J_\alpha = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha} dx + \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{x^{\alpha-1}}{x + x^\alpha} dx$$
$$= \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1 + x^{\alpha-1}}{x(1 + x^{\alpha-1})} dx = \int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx.$$

We conclude that

$$J_\alpha = \frac{1}{2} \int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx.$$

This completes the proof. □

To the best of our knowledge, this integral formula is a novel addition to the existing body of literature. It enables us to derive various new results that would otherwise be intractable using standard methods such as finding primitives, performing changes of variables or integrating by parts. These features are explored in detail in the following section.

3 Corollaries

3.1 Consequences of Theorem 2.2

The corollary below uses the framework of Theorem 2.2 to present a two-parameter integral formula involving π.

Corollary 3.1 *For any $\alpha \in \mathbb{R}$ and $\beta > 0$, we have*

$$\int_0^{+\infty} \frac{x^\beta}{(x^{2\beta} + 1)(x + x^\alpha)} dx = \frac{\pi}{4\beta}.$$

Proof. For any $x, y > 0$ and $\beta > 0$, we set

$$f(x, y) := \frac{1}{x^\beta + y^\beta}.$$

It is obvious that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. Moreover, we can write

$$\int_0^{+\infty} \frac{x^\beta}{(x^{2\beta} + 1)(x + x^\alpha)} dx = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^\alpha} dx. \quad (1)$$

It follows from Theorem 2.2 and the arctangent primitive that, for any $\alpha \in \mathbb{R}$,

$$\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^{\alpha}} dx = \frac{1}{2} \int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx$$
$$= \frac{1}{2} \int_0^{+\infty} \frac{1}{x} \times \frac{x^{\beta}}{x^{2\beta} + 1} dx = \frac{1}{2\beta} \int_0^{+\infty} \frac{\beta x^{\beta - 1}}{x^{2\beta} + 1} dx$$
$$= \frac{1}{2\beta} \left[\arctan(x^{\beta})\right]_{x=0}^{x \to +\infty} = \frac{1}{2\beta} \times \frac{\pi}{2} = \frac{\pi}{4\beta}. \quad (2)$$

Based on Equations (1) and (2), for any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \frac{x^{\beta}}{(x^{2\beta} + 1)(x + x^{\alpha})} dx = \frac{\pi}{4\beta}.$$

This ends the proof. □

The corollary below is a modification of Corollary 3.1, with an additional power term in the denominator. It continues to use the framework of Theorem 2.2 to present a two-parameter integral formula involving π.

Corollary 3.2 *For any $\alpha \in \mathbb{R}$ and $\beta > 0$, we have*

$$\int_0^{+\infty} \frac{x^{\beta}}{(x^{2\beta} + x^{\beta} + 1)(x + x^{\alpha})} dx = \frac{\pi}{3\sqrt{3}\beta}.$$

Proof. For any $x, y > 0$ and $\beta > 0$, we set

$$f(x, y) := \frac{1}{x^{\beta} + y^{\beta} + x^{\beta} y^{\beta}}.$$

Then, it is obvious that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. Moreover, we can write

$$\int_0^{+\infty} \frac{x^{\beta}}{(x^{2\beta} + x^{\beta} + 1)(x + x^{\alpha})} dx = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^{\alpha}} dx. \quad (3)$$

It follows from Theorem 2.2, and the arctangent primitive and formulas that, for any $\alpha \in \mathbb{R}$,

$$\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^{\alpha}} dx = \frac{1}{2} \int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx$$
$$= \frac{1}{2} \int_0^{+\infty} \frac{1}{x} \times \frac{x^{\beta}}{x^{2\beta} + x^{\beta} + 1} dx = \frac{1}{2\beta} \int_0^{+\infty} \frac{\beta x^{\beta - 1}}{x^{2\beta} + x^{\beta} + 1} dx$$
$$= \frac{1}{2\beta} \left[\frac{2}{\sqrt{3}} \arctan\left(\frac{2x^{\beta} + 1}{\sqrt{3}}\right)\right]_{x=0}^{x \to +\infty}$$
$$= \frac{1}{\sqrt{3}\beta} \times \left\{\frac{\pi}{2} - \arctan\left[\frac{1}{\sqrt{3}}\right]\right\}$$
$$= \frac{1}{\sqrt{3}\beta} \arctan\left[\sqrt{3}\right] = \frac{1}{\sqrt{3}\beta} \times \frac{\pi}{3} = \frac{\pi}{3\sqrt{3}\beta}. \quad (4)$$

Based on Equations (3) and (4), for any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \frac{x^\beta}{(x^{2\beta}+x^\beta+1)(x+x^\alpha)}dx = \frac{\pi}{3\sqrt{3}\beta}.$$

This concludes the proof. □

The corollary below provides a simple consequence of Theorem 2.2, resulting in a new one-parameter integral formula.

Corollary 3.3 *For any $\alpha \in \mathbb{R}$ and $\beta > 0$, we have*

$$\int_0^{+\infty} \min\left(x^\beta, \frac{1}{x^\beta}\right) \frac{1}{x+x^\alpha}dx = \frac{1}{\beta}.$$

Proof. For any $x, y > 0$ and $\beta > 0$, we set

$$f(x,y) := \min(x^\beta, y^\beta).$$

It is clear that, for any $x, y > 0$, we have $f(x,y) = f(y,x)$. Moreover, we can write

$$\int_0^{+\infty} \min\left(x^\beta, \frac{1}{x^\beta}\right) \frac{1}{x+x^\alpha}dx = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x+x^\alpha}dx. \tag{5}$$

It follows from Theorem 2.2 and standard power primitives that, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x+x^\alpha}dx = \frac{1}{2}\int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx \\
&= \frac{1}{2}\int_0^{+\infty} \frac{1}{x} \times \min\left(x^\beta, \frac{1}{x^\beta}\right) = \frac{1}{2\beta}\left[\int_0^1 \beta x^{\beta-1}dx + \int_1^{+\infty} \beta x^{-\beta-1}dx\right] \\
&= \frac{1}{2\beta}\left([x^\beta]_{x=0}^{x=1} + \left[-x^{-\beta}\right]_{x=1}^{x\to+\infty}\right) = \frac{1}{2\beta}(1+1) = \frac{1}{\beta}.
\end{aligned} \tag{6}$$

Based on Equations (5) and (6), for any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \min\left(x^\beta, \frac{1}{x^\beta}\right) \frac{1}{x+x^\alpha}dx = \frac{1}{\beta}.$$

This completes the proof. □

The corollary below determines a one-parameter integral that is always equal to the well-known Catalan constant. Once again, there are no particular restrictions on the parameter α, which belongs to the entire real line.

Corollary 3.4 *For any $\alpha \in \mathbb{R}$, we have*

$$\int_0^{+\infty} \arctan\left[\min\left(x, \frac{1}{x}\right)\right] \frac{1}{x + x^{\alpha}} dx = G,$$

where G denotes the Catalan constant.

Proof. For any $x, y > 0$ and $\beta > 0$, we set

$$f(x, y) := \arctan[\min(x, y)].$$

It is obvious that, for any $x, y > 0$, we have $f(x, y) = f(y, x)$. Moreover, we can write

$$\int_0^{+\infty} \arctan\left[\min\left(x, \frac{1}{x}\right)\right] \frac{1}{x + x^{\alpha}} dx = \int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^{\alpha}} dx. \quad (7)$$

It follows from Theorem 2.2 and well-known arctangent integral value of the Catalan constant (see [1, Equation (2)]) that, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^{+\infty} f\left(x, \frac{1}{x}\right) \frac{1}{x + x^{\alpha}} dx = \frac{1}{2} \int_0^{+\infty} \frac{1}{x} f\left(x, \frac{1}{x}\right) dx \\
&= \frac{1}{2} \int_0^{+\infty} \frac{1}{x} \times \arctan\left[\min\left(x, \frac{1}{x}\right)\right] \\
&= \frac{1}{2} \left[\int_0^1 \frac{1}{x} \arctan(x) dx + \int_1^{+\infty} \frac{1}{x} \arctan\left(\frac{1}{x}\right) dx\right] \\
&= \frac{1}{2}(G + G) = G. \qquad (8)
\end{aligned}$$

Using Based on Equations (7) and (8), for any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \arctan\left[\min\left(x, \frac{1}{x}\right)\right] \frac{1}{x + x^{\alpha}} dx = G.$$

This ends the proof. □

To the best of our knowledge, this formula is not documented in any of the specialized literature on the Catalan constant. In particular, it is not included in the list of integrals representing the Catalan constant in [1].

3.2 Diverse integral formulas

For brevity, several other results based on Theorem 2.2 are given below without details. Some of these involve known special functions. Further details on these functions can be found in [5].

- For any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \log\left[1+\min\left(x,\frac{1}{x}\right)\right]\frac{1}{x+x^\alpha}dx = \frac{\pi^2}{12}.$$

- For any $\alpha \in \mathbb{R}$ and $\beta > 0$, we have

$$\int_0^{+\infty} \left(\frac{x}{x^2+1}\right)^\beta \frac{1}{x+x^\alpha}dx = 2^{-\beta-1}\sqrt{\pi}\frac{\Gamma(\beta/2)}{\Gamma[(1+\beta)/2]},$$

  where $\Gamma(x)$ denotes the gamma integral function.

- For any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \exp\left(-x^2-\frac{1}{x^2}\right)\frac{1}{x+x^\alpha}dx = \frac{1}{2}K_0(2),$$

  where $K_0(x)$ denotes the modified Bessel function of the second kind.

- For any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \sin\left[\min\left(x,\frac{1}{x}\right)\right]\frac{1}{x+x^\alpha}dx = \mathrm{Si}(1),$$

  where $\mathrm{Si}(x)$ denotes the sine integral function.

- For any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \sin\left[\max\left(x,\frac{1}{x}\right)\right]\frac{1}{x+x^\alpha}dx = \frac{\pi}{2} - \mathrm{Si}(1).$$

- For any $\alpha \in \mathbb{R}$, we have

$$\int_0^{+\infty} \sinh\left[\min\left(x,\frac{1}{x}\right)\right]\frac{1}{x+x^\alpha}dx = \mathrm{Shi}(1),$$

  where $\mathrm{Shi}(x)$ denotes the hyperbolic sine integral function.

4 Open problem

Corollaries 3.1, 3.2, 3.3, and 3.4, along with the formulas presented in Subsection 3.2, are all derived from Theorem 2.2. However, given the variety of integral techniques available, alternative systematic approaches may also lead to these or similar formulas. This raises the following open question:

Is there a broader unifying framework or alternative scheme that can systematically generate these integral formulas, independent of the method based on Theorem 2.2?

Answering this could unify diverse techniques and extend the range of applicable integral formulas in theory and practice.

5 Conclusion

In conclusion, we have presented general integral formulas that are innovative in their dependence on a single adjustable function and one or two tuning parameters. One of these parameters, α, can belong to the entire real line. Our main result, Theorem 2.2, can be applied to numerous mathematical scenarios, as exemplified in Corollaries 3.1, 3.2, 3.3, and 3.4 and Subsection 3.2. This can lead to formulas with results involving well-known mathematical constants and special functions. In particular, we have established a simple one-parameter integral formula for the Catalan constant. Future work will include exploring other applications of Theorem 2.2, as well as addressing the open problem formulated in Section 4. This will require a deeper investigation into alternative frameworks for deriving such integral formulas and a systematic study of their implications across various fields of mathematics.

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