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On Analysis of Growth and Distortion Criteria for Certain Univalent Functions on the Unit Disk

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Abstract

Studies on growth and distortion conditions (GDC) for univalent functions (UF) in the unit disk (UD) have been conducted over decades with interesting findings obtained for various functions like the conformal mappings and slice regular functions with nice and very crucial in applications in various fields. However, complete analysis of these conditions has not been done. Recently, researchers gave an open question on the growth and distortion theorems asking whether the family of regular slice mappings is the largest subfamily of the unit ball of UF. In this paper, we analyze the GDC for UF in the UD. In particular, we consider the Koebe function (KF) and establish the its GDC by establishing its minimal and maximal extremal boundary points.

Keywords: *Growth, Distortion, Univalent function, Unit Disk.*
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1 Introduction

Schlicht functions or univalent functions are analytic functions which map a domain in $\mathcal{C}$ onto another domain in an injective manner. Growth and Distortion analysis of UF in the UD [34] is one of the key areas of complex analysis, it has attracted extensive research from a geometric perspective on account of its application to conformal mappings. The starting point of the inquiry is on the consideration of the unit disk being satisfactory for generalization of the findings regarding path-connected domains. Take into account a Schlicht function $\varphi : \mathcal{D} \to \mathcal{C}$. It is evident that if φ is mapped conformally from the domain $\mathcal{D}$ in $\mathcal{C}$, then the image domain is path-connected and not the entire $\mathcal{C}$, if that is not the case then even φ^{-1} would have been a constant by Liouville theorem (see [25]. [34] and [36]). Consequently, it is clear that Riemann's theorem establishes a bijection between non-empty open and proper path-connected subset $\mathcal{U} \subset \mathcal{C}$, $U \neq \mathcal{C}$, and $\varphi : \mathcal{D} \to \mathcal{C}$. In this study we concentrated on conditions that make the univalent function $\varphi : \mathcal{D} \to \mathcal{C}$ to map the origin to itself ($\varphi(0) = 0$) and guarantee normalization of the derivative of the function at the origin to 1 ($\varphi'(0) = 1$) [34], thus making analysis easier by getting rid of inappropriate constants. These unique (injective and holomorphic) univalent functions which conforms with these normalization conditions form what we call the class of $\mathcal{S}$. The key properties of the class of maps of $\mathcal{S}$: it is locally bounded, which means for every closed and bounded subset $\mathcal{A} \subset \mathcal{D}$ $\exists$ a constant $M_A : \mid \varphi(z) \mid \leq M_A \ \forall z \in \mathcal{A}$ and for all $\varphi \in \mathcal{S}$, and it is closed, meaning that if a sequence of functions $\{\varphi_n\}$ in $\mathcal{S}$ uniformly converges on closed and bounded subsets of $\mathcal{D}$ to φ, then φ is also in $\mathcal{S}$ [14]. These behaviors of the functions in $\mathcal{S}$ are validated by growth and distortion theorems which provide bounds on the modulus of the derivative of $|\varphi'(z)|$, for φ in class $\mathcal{S}$ ($\varphi \in \mathcal{S}$) [34]. Precisely for z in a unit disk $\mathcal{D}$, $\frac{1-|z|}{(1+|z|)^3} \leq \mid \varphi'(z) \mid \leq \frac{1+|z|}{(1-|z|)^3}$, for $r = |z| < 1$, we have $\frac{1-r}{(1+r)^3} \leq \mid \varphi'(z) \mid \leq \frac{1+r}{(1-r)^3}$. The distortion of the geometry of $\mathcal{D}$ by the function φ is controlled by these inequalities in this theorem. Again, the bounds on the modulus of φ is given by growth theorem [34]. That is, for any $\varphi \in \mathcal{S}$ and $r = |z| < 1$, we have growth inequalities involved. These inequalities of the growth theorem gives a description on how faster a function $\varphi(z)$ can grow as the magnitude of z tends to 1. It is important to note any univalent function f in $\mathcal{D}$ can undergo transformation to fit in the class of maps $\mathcal{S}$ and the other way round [14]. We can verify normalization in the class of S to ascertain that this claim actually works. Therefore, φ satisfies the standardization conditions $\varphi(0) = 0, \varphi'(0) = 1$ in class of maps $\mathcal{S}$[10]. Consequently, with regard to the above case, we have verified beyond reasonable doubt that numerous findings associated with univalent functions have been cited from the valuable characteristics of the class of maps $\mathcal{S}$. The class of maps $\mathcal{S}$ is the Kobe function [34]: $k(z) = \frac{1}{4}[(\frac{1+z}{1-z})^2 - 1] = z + 2z^2 + 3z^3 + 4z^4 + ...$ is the most extraordinary case. This functions is extremely vital in numerous findings regarding univalent functions. Now, with respect to the above findings we have uncovered, including normalization and ideal cases of class of $\mathcal{S}$, we are able to characterize conditions for growth and distortion of these functions $\mathcal{D}$. For numerical analysis to be done, development of algorithms is indispensable. In this regard, many studies have developed various algorithms. Computational methods adopted by [20] pioneered the idea of numerically approximating conformal maps and univalent functions. His work gave early algorithms for computing images of UF using iterative and polynomial approximation techniques. Moreover, the authors of [13] introduced robust algorithmic method for conformal mappings using the Schwarz-Christofel formula, a key tool in approximating univalent maps from the upper half-plane to polygonal regions, and their MATLAB SC toolbox has remained influential over the century. The research work by [19] portrayed estimates of higher-order coefficients using iterative and recursive methods. The algorithms from their work were later adapted into symbolic computation software like Maple to automatically compute sharp bounds. Now, numerical analysis of growth and distortion condition of univalent functions in a unit disk has been done over decades [30]. To analyze, we find continuity and compare with the bounds given by growth and distortion theorem. The results are then visualized by plotting the magnitudes of $\varphi(z)$ and $\varphi'(z)$ against the modulus of z and then overlap the bounds to visualize how the univalent function behaves within the unit disk $\mathcal{D}$. Precise plotting can be done by running a python code using numpy and matplotlib [45].

2 Preliminaries

Certain preliminary concepts are instrumental in this study. We give them under this section for ease of understanding of the work.

Definition 2.1 *([14], Definition 3.2) An open unit disk $\mathcal{D}$ is defined as a set of all points z in a complex plane $\mathcal{C}$ whose modulus is less than* 1.

Definition 2.2 *([44], Definition 2.4) A map is said to be univalent in a UD if it is analytic and one-to-one.*

Definition 2.3 *([1], Definition 3.1) Growth analysis is the exploration of the behavior of $\mid \varphi(z) \mid$ as $z \to 1$.*

Definition 2.4 *([38], Definition 1.3) Distortion analysis studies how the geometrical properties of a univalent function $\varphi(z)$ are distorted as $\varphi : \mathcal{D} \to \mathcal{C}$.*

3 Literature review

Related literature with fundamental results are reviewed in this chapter. We consider various studies and give a critique of the same.

3.1 Growth and distortion criteria

The origin of this work is based on [39] that extended standard definitions by the application of differential operators. This study highlights the derivatives and coefficient bounds of the GD theorems relevant to mappings in this class, mainly revealing the extreme points and convolution properties that contribute to our understanding of the functional landscape in the unit disk $\mathcal{D}$. Their perspective forms a basis for understanding the connection between growth and distortion, which can be further leveraged to formulate specific algorithms. In addition, this analysis is further enriched by the coefficients' estimates for bi-univalent classes, which consider functions that maintain bijection even in their inverses[39]. Remarkably, [37] investigations give outcomes about covering, distortion, and rotation theorems relevant to such classes of functions, thus aiding a deeper comprehension of functional implications when different conditions are applied.

Building upon this foundation, the author of [35] clearly established that growth and distortion theorems can also be applied to slice monogenic functions within the context of Clifford algebras. They came up with rigorous proofs for the comparable growth and distortion theorems in quaternionic representations, covering the applicability of these theorems beyond baroque complex analysis to more comprehensive settings. Their study demonstrates the adaptive nature of growth and distortion conditions across diverse mathematical outlines consequently forming part of algorithm development.

The authors of [31] reveals that growth theorems describe how the magnitude of a univalent function grows as the argument tends to the boundary of the UD. For instance, the growth theorem for univalent functions states that $\varphi \in \mathcal{S}$ and $r = |z| < 1$, $\frac{r}{(1+r)^2} \leq \mid \varphi(z) \mid \leq \frac{r}{(1-r)^2}$ and it has been given in a general context to harmonic mappings and other subclasses of univalent functions with sharp bounds established for specific families of functions [34].

In a more recent approach by [26], they scrutinized a family of functions which are star in nature by formulating structural results that comprise of growth theorems and coefficient estimates. Their study utilizes the established results by ascertaining clear properties that can be employed into algorithmic procedures. Moreover, the study by [37] also examined bi-univalent functions, which maintain bijection when both direct mappings and their inverses are considered. These investigations have resulted into effective distortion estimates and theorems pertinent within this context.

Several studies have revealed that the phenomenon of distortion in univalent functions discloses intricate connections to their growth behavior. For example the authors of [4] scrutinize (q, δ) - neighborhoods, easing probe of inclusion ramifications and inequalities among various subclasses of holomorphic univalent functions, suggesting opinions where outmoded estimates may be adapted or developed under definite operator frameworks. Moreover, with respect to

[26] studies, the connection between coefficient estimates and geometric properties such as starlikeness is crucial. They emphasize that certain structural properties enable functions to maintain strict distortion controls when subjected to analytical transformations like conformal mapping.

Another notable avenue of research concerns the coefficient conditions of univalent and harmonic mappings. The writers of [32] reveals that coefficient conditions of univalent and harmonic mappings basically from the Taylor series expansions is very fundamental in determining bijection and growth rates. Studies of [7] also discovered the relations between coefficient bounds and geometric properties like starlikeness through rigorous criteria involving harmonic functions and their derivatives. These studies have also clarified that harmonic version of the coefficients suggests a distinct understanding of the basic structures thereby allowing philosophers to craft subclasses of univalent functions possessing unique growth properties.

The research work of [22] also demonstrates key contributions by proposing specific properties of standardized univalent harmonic mappings. Their work revolve around estimates of functions and bounds that help in understanding the Bloch constant in these mappings thereby providing alternative insight on growth conditions crucial for creating algorithms intended for studying harmonic univalent functions. They ascertained that the algorithms can be constructed on structural properties of classes of functions, foundational coefficient bounds and geometrical constrains within complex analysis. They further emphasized that an effective algorithm should incorporate the various coefficients, properly apply distortion theorems and comprehend how diverse subclasses interrelate within the unit disk.

The authors of [16] and [23] separately investigated the harmonic mappings obtained by shear construction, highlighting essential conditions for these mappings to be convex. These studies cooperatively clarify the elaborate interplay between harmonicity and bijection in so doing emphasizing the importance of conformal mappings in this realm.

Studies have also shown that the systematic examination of the inverses of such functions also reveals growth conditions [27]. These studies have revealed how convex functions and their inverses yield significant insights into the growth behaviors of univalent functions by defining bounds related to Hankel determinants associated with these classes. They further spell out the complex nature of comparable constructs between univalent functions and geometric properties, which allows for a thorough understanding of distortion results thereby broadening our understanding to include transformations and mappings beyond traditional boundaries.

Numerous studies have proclaimed that the study of harmonic univalent functions highlights an entirely diverse perception on growth and distortion. A recent study by [41] illustrates how specific properties influence coefficient bounds and distortion phenomena in symmetric starlike harmonic functions. Here, the closure property under integral operators lay emphasis on how harmonic functions can retain one-to-one correspondence while experiencing transformations that preserve domain properties like the conformal mapping. In [32] it is further portrayed how this aspect of growth serves as a gateway to exploring the boundaries of analytic function theory in intricate mathematical frameworks.

The work of [18] articulated that estimates for coefficients derived from subfamilies of some bi-univalent mappings disclose essential bounds essential for establishing the growth rates of these functions. The author emphasized that it is mainly beneficial when exploring the inverse relationships between a function and its inverse in maintaining one-to-one correspondence over the unit disk. Related studies by [5] further address partial sums of meromorphic univalent functions, stirring the discourse on distortion behavior while connecting back to the effects of coefficients on overall function behavior in the unit disk domain.

Several studies have ascertain that growth conditions for univalent functions can be well characterized through a combination of distortion theorems and coefficient bounds. The study by [39] present a subclass of analytic univalent functions that explore growth theorem, distortion theorems and coefficient bounds which in turn reveals the geometric properties related to these functions in the unit disk. The study by [14] further asserts how these properties often concern the preservation of convexity, as they discusses relationships between harmonic univalent functions and their coefficients thereby providing sharp inequalities that illustrate the growth behavior of these functions. Likewise, the authors of [3] extended this investigation by focusing on hyperbolic univalent functions as they interpret distortion properties that inform the geometric mapping onto hyperbolic regions.

The research by [33] discovered that a foundational result regarding the growth of UF is captured in the Koebe's theorem, which state that for any univalent function $\varphi : \mathcal{D} \to \mathcal{C}$ it's image contains a disk of radius $\frac{1}{4}$. They assert that this property guarantees that univalent functions can be inverted and their inverses enjoy clear-cut properties inside the unit disk $\mathcal{D}$. The studies by [29] and [17] further validated that the mathematical form of such mappings permits one to come up with the conditions under which the growth of these functions can be quantified thereby offering the root for numerous algorithms intended to estimate coefficients of univalent functions characterized by specific classes satisfying the conditions for the class of maps $\mathcal{S}$.

In conjunction with the growth criteria, several studies have addressed distortion theorems describing how the geometric properties of univalent functions can be quantified. The study by [9] incorporated theorems for these functions where specific bounds are expressed and substantiated for various subclasses for instance those that are starlike or convex. Additionally, the authors of [49] utilized norms of pre-Schwarzian and Schwarzian derivatives to effectively come up with necessary and sufficient conditions for bijection of functions thereby serving as essential tools in comprehending the growth metrics and distortions associated with these mappings.

Again, other studies have shown that distortion theorems provide bounds [35] which are essential for understanding the univalent functions. Precisely for z in a unit disk $\mathcal{D}$, $\frac{1-|z|}{(1+|z|)^3} \leq | \varphi'(z) | \leq \frac{1+|z|}{(1-|z|)^3}$ [12]. This theorem which is well defined by [12] has been generalized to harmonic mappings and other subclasses of univalent functions [41].

The investigations by [41] shows that distortion properties light up the behavior of UF within the UD. They clarified how these properties help create critical criterion defined within specific subclasses [41]. They investigated the anew well-defined subclasses of harmonic univalent functions by coming up with appropriate distortion theorems that organize essential features of these mathematical structures. They clearly portrayed how broad view of distortion theorems lead to better understandings of functional behavior across various subclasses of univalent functions including those that are starlike and harmonic in nature[41]. Remarkably, the authors of [21]further claim that characteristics regarding extreme points and convolution conditions also add up to a broad framework for examining the growth and distortion of these functions.

The study by [28] demonstrates that the nature of growth and distortion for univalent functions involves dimensional aspects for instance the extension of bijection criteria by the use of integral operators. They embarked upon bijection conditions pertinent to integral operators by proving how these operators preserve the univalent nature while respecting the essential growth conditions of the functions involved. Their study is further lightened up by the findings of [21]whose efforts emphasize the nuanced interdependencies between growth, distortion and the algebraic structures governing the univalent functions in consideration. The authors of [50] derived sharp estimates for distortion and growth rates by exploring harmonic mappings specifically the ones with negative coefficient analytic parts. Their exploration of harmonic mappings, generally on the concept of univalent functions, revealed the relationship between coefficients and how they significantly influence the growth behavior in the unit disk. Their study is very vital for formulating algorithms that account for the exceptional features of different subclasses of harmonic mappings. Several approaches have been utilized by researchers in developing algorithms for estimating these growth and distortion conditions ([24] , [42]). For instance, the study by [24] demonstrates how specific subclasses of univalent functions can be explored through their Maclaurin's series expansions where coefficient estimates are derived by methods such as subordination techniques. Likewise, the study by [2] established a robust computational framework for harmonic

univalent functions where coefficient characterizations lead to the formulation of covering theorems that provide tighter bounds for these functions.
Moreover, the investigation by [40] revealed that algorithms for analyzing UF often involve estimating the coefficients of function obtained through the expansion of the power series. For instance, it is clear in research work of [11] that the coefficients of a UF satisfy the de Branges theorem, which states that $|a_n| \leq n$ for all $n \geq 2$, which forms the basis for many coefficient estimation [11]. Other studies have ascertained that numerical analysis of growth and distortion conditions can be effectively conducted by implementing these algorithms and applying them to illustrative cases of univalent functions. The study by [6] gives an insight on how numerical methods involve iterative procedures or boundary value problems thereby allowing for the visualization of distortion effects as parameters varies. Moreover, [46] further asserts that results from such numerical experiments can tell stability properties and enlighten on the approximations of univalent functions under different mappings in turn deeply enhancing our comprehension of their geometric properties.
Finally, on the same note as [40] clarifies, numerical methods such as the Loewner differential equation, have been developed to study the dynamics of univalent functions. These methods involve solving differential equations that describe the evolution of the function under certain constraints. For instance, [3] further clarifies how the Loewner equation can be utilized to examine the Schwarzian derivatives of UF.

4 Research methodology

The systematic approaches that we employ to successfully achieve our objectives are discussed herein. We outline the analysis techniques and characterization of growth and distortion conditions. The methodology guarantees an efficient approach that integrates mathematical analysis and i is crafted in a manner that it provides logical, comprehensible and analytical sound approach that employ both classical techniques from geometric function theory and modern computational tools in addressing these specific objectives. In short, the combination of theoretical analysis, algorithm development and numerical analysis provides a robust framework for achieving our results. The methodology is rooted out of the established framework of univalent function theory while also embracing modern computational methods to extend its scope and applicability.
This research will adopt a hybrid study design incorporating both qualitative theoretical analysis and computational quantitative methods. The study primarily incorporated a deductive reasoning approach stemming from complex analysis and geometric function theory. The deductive component begins with the normalization of UF in the family of maps $\mathcal{S}$ from the foundational princi-

ples of univalent function theory [34]. The theoretical framework is extended by investigating specific subclasses like starlike which obey particular growth and distortion bounds. The algorithmic design integrated in computational quantitative methods facilitated the construction of procedures capable of approximating or verifying growth and distortion conditions, while the numerical simulations analyzed these phenomena within the unit disk. This structured methodology ensured a broad analysis of univalent functions by linking abstract mathematical reasoning with practical computational techniques.
Theoretical Analysis is very essential for establishing rigorous mathematical bounds for example, Koebe's theorem [5]. Algorithmic approach is necessary for practical computation, to help automate growth and distortion computations [23]. Numerical experiments are intended for validating theoretical results and assessing real-world applicability [6]. The second stage is the algorithmic development used for designing numerical algorithm or computational methods for evaluating growth and distortion conditions, and implementation in Python. The final stage entailed numerical analysis intended for implementing and testing the algorithms on known univalent functions like Koebe function and starlike functions using numerical experiments, and for comparison with theoretical predictions. The research work is theoretical and computational for that reason data is derived from various sources. The first source is standard normalized univalent functions like Koebe function and convex or starlike functions. The second source is numerically generated functions like polynomials with constrained coefficients. The third source is published coefficient bounds from existing literature [21]. Finally, data is derived from computational libraries including pre-existing software like Python's NumPy, SciPy and Matplotlib libraries.

4.1 Fundamental principles

Theoretical foundations where conditions for growth and distortion bounds of univalent functions are derived, and key theorems including Koebe, Bieberbach, and Schwarz Lemma are reviewed. These theorems provides the foundation for analyzing the distortion and growth behavior of univalent functions, then the results from the analysis are used to explore sharper bounds, improved inequalities, and generalized conditions for the subclasses of these functions.

Theorem 4.1 ***(Koebe's distortion theorem)****-The inequality* $\frac{1-|z|}{(1+|z|)^3} \leq \mid \varphi'(z) \mid \leq \frac{1+|z|}{(1-|z|)^3}$ *provides bounds on the modulus of* $\varphi'(z)$ *for* φ *satisfying the standardization conditions* $\varphi(0) = 0, \varphi'(0) = 1$ *in the class of* $\mathcal{S}$ *[34]. The theorem demonstrates how the geometrical properties of a UF are distorted under transformation in a unit disk* $\mathcal{D}$.

For growth analysis, we have the growth theorem stated as:

Theorem 4.2 ***(Growth Theorem)***-*The inequality* $\frac{|z|}{(1+|z|)^2} \leq | \varphi(z) | \leq \frac{|z|}{(1-|z|)^2}$ *relates the maximum modulus* $| \varphi(z) |$ *to the radius* $|z|$ *[14]. The growth of the function* $\varphi(z)$ *is shown by the behavior of the modulus of function* $\varphi(z)$ *towards the boundary of the UD as* $|z| \to 1$.

4.2 Analytical techniques

The characterization of growth and distortion conditions is tackled by rigorous derivation of analytical results involving extremal function theory and conformal mapping methods. This is made possible by employing advanced mathematical tools like conformal mapping, Koebe function, subordination principles, sharp inequalities for$|\varphi(z)|$ and $|\varphi'(z)|$based on various subclasses of $\mathcal{S}$, Schwarz's lemma and, Loewner's theory for the establishment of parametric representation and distortion of functions [41]. The methodology comprises of growth analysis and distortion analysis techniques.

4.3 Growth analysis techniques

The characterization of growth conditions is by establishing sharp bounds for $|\varphi(z)|$ in $\mathcal{D}$. The objective is met by applying the maximum modulus principle to relate $|\varphi(z)|$ to boundary behavior, then, using subordination principles to compare growth rates of different function classes and lastly, deriving inequalities using Loewner's differential equation and Gronwall's area theorem [34].

4.4 Distortion analysis techniques

The characterization of distortion conditions is met by determining optimal bounds for $|\varphi'(z)|$. This specific objective is met by applying Schwarz Lemma and Koebe distortion theorem for initial estimates, then refining bounds using variational methods and extremal function techniques and finally, incorporating Loewner theory for dynamic distortion estimates under parameter variations [4]. We acknowledge that special subclasses like convex and starlike functions are used to analyze growth and distortion under geometric constraints whereas the bounded univalent functions are for studying distortion under additional conditions for boundedness.

5 Main results

Now we embark on characterization of growth and distortion conditions for KF in the UD. The growth behavior of univalent functions within $\mathcal{D}$ is central to understanding their geometric and analytic properties. This section focuses on representing Koebe function as an extremal function for the growth analysis in the unit disk, verifying its existence in the class $\mathcal{S}$. These growth conditions are interpreted through classical theorems including growth theorem and de Branges' theorem.
The KF as an extremal function is defined as:

$$k(z) = \frac{z}{(1-z)^2}, \quad for \;\; z \in \mathcal{D} = \{z \in \mathcal{C} : |z| < 1\}. \tag{1}$$

We start by considering power series representation of Koebe Function.

Proposition 5.1 *The Koebe function is analytic and can be represented as a power series from a standard polynomial function $\phi(z)$ defined as*

$$\phi(z) = a_0 + a_1 z + a_2 z^2 + \ldots, \tag{2}$$

where a_i for $i = 0, 1, 2, \ldots$ are constants in complex plane and $z \in \mathcal{C}$.

Proof. Differentiating the function $\phi(z)$ in Equation 2 we have

$$\phi'(z) = a_1 + 2a_2 z + \ldots \tag{3}$$

$$\vdots$$

$$\phi^{(8)}(z) = 40320 a_8 + \ldots \tag{4}$$

Now at the origin ($z = 0$), we have

$$\phi(0) = a_0 \to a_0 = \phi(0) \;\; \to a_0 = \phi(0) \tag{5}$$

$$\phi'(0) = a_1 \to a_1 = \phi'(0) \;\; \to a_1 = \frac{\phi'(0)}{1!} \tag{6}$$

$$\phi''(0) = 2a_2 \to 2a_2 = \phi''(0) \;\; \to a_2 = \frac{\phi''(0)}{2!} \tag{7}$$

$$\vdots$$

$$\phi^{(8)}(0) = 40320a_8 \to 40320a_8 = \phi^{(8)}(0) \ \to a_8 = \frac{\phi^{(8)}(0)}{8!} \tag{8}$$

Substituting a_i for $i = 0, 1, 2, \ldots$ in the original standard polynomial function $\phi(z)$, we have

$$\phi(z) = \phi(0) + \frac{\phi'(0)}{1!}z + \frac{\phi''(0)}{2!}z^2 + \ldots \tag{9}$$

Suppose the standard polynomial function $\phi(z)$ is a binomial function defined as

$$\phi(z) = (1+z)^n. \tag{10}$$

Differentiating the above binomial function above continuously we have

$$\phi'(z) = n(1+z)^{n-1} \tag{11}$$

$$\phi''(z) = n(n-1)(1+z)^{n-2} \tag{12}$$

$$\vdots$$

$$\phi^{(8)}(z) = n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)(1+z)^{n-8} \tag{13}$$

At the origin ($z = 0$), we have

$$\phi(0) = 1 \tag{14}$$

$$\phi'(0) = n \tag{15}$$

$$\phi''(0) = n(n-1) \tag{16}$$

$$\vdots$$

$$\phi^{(8)}(0) = n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7) \tag{17}$$

Substituting $\phi(0)$, $\phi'(0), \phi''(0), ..., \phi^{(8)}(0), \ldots$ in Equation 17 we have

$$\phi(z) = 1 + \frac{n}{1!}z + \frac{n(n-1)}{2!}z^2 + + \ldots \tag{18}$$

substituting z with $-z$, we have

$$\phi(-z) = 1 + \frac{n}{1!}(-z) + \frac{n(n-1)}{2!}(-z)^2 + \ldots \tag{19}$$

Upon simplification it reduces to

$$\phi(-z) = 1 - \frac{n}{1!}z + \frac{n(n-1)}{2!}z^2 - \dots \tag{20}$$

Koebe function is defined as $k(z) = \frac{z}{(1-z)^2}$ can be expressed in the form

$$k(z) = z(1-z)^{-2} = z(1+(-z))^{-2}, \tag{21}$$

where $\phi(-z) = (1+(-z))^{-2}$ taking n=-2. Therefore, it becomes

$$k(z) = z\Big\{1 - \frac{n}{1!}z + \frac{n(n-1)}{2!}z^2 - \frac{n(n-1)(n-2)}{3!}z^3 + \cdots\Big\} \tag{22}$$

Now substituting n by -2 we have

$$k(z) = \frac{z}{(1-z)^2} = z\Big\{1 - \frac{-2}{1!}z + \frac{-2(-2-1)}{2!}z^2 - \cdots\Big\} = \sum_{n=1}^{\infty} nz^n \tag{23}$$

Thus, the power series representation of Koebe function is given as

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \dots \ = \sum_{n=1}^{\infty} nz^n \tag{24}$$

This series converges absolutely for $|z| < 1$, proving that $k(z)$ is analytic in $\mathcal{D}$.

Next we give a result on the continuity, boundedness, injectivity and univalence of KF in the following theorem.

Theorem 5.2 *The Koebe function is continuous, bounded, injective and extremal univalent.*

Proof. The proof for continuity and boundedness is trivial. To prove that KF is injective, we need to show that if $k(z_1) = k(z_2)$ then $z_1 = z_2$. It follows from [11] that $(z_1 - z_2)(1 - z_1 z_2) = 0$. This gives two possibilities, that is, $z_1 - z_2 = 0 \Longrightarrow z_1 = z_2$ or $1 - z_1 z_2 = 0 \Longrightarrow z_1 z_2 = 1$. However, for $z_1, z_2 \in \mathcal{D}$, we have $|z_1 z_2| = |z_1||z_2| < 1$. Therefore, $z_1 z_2 = 1$ is impossible in $\mathcal{D}$. Thus, the only solution is $z_1 = z_2$ proving that k is injective on $\mathcal{D}$. By Proposition 5.1, KF is analytic in the unit disk, and again having demonstrated its injectivity on $\mathcal{D}$, it is therefore enough that it is one of the extremal univalent function for the analysis of growth and distortion conditions in the unit disk.
Next, we consider normalization of KF in the class $\mathcal{S}$. Koebe function is normalized in the class of $\mathcal{S}$ since it satisfies the conditions:

Theorem 5.3 *The Koebe function is satisfies normalization criterion in the class of* $\mathcal{S}$.

Proof. From the hypothesis of the theorem, we have the following conditions:

$$k(0) = \frac{0}{(1-0)^2} = 0 \tag{25}$$

$$k'(z) = \frac{1+z}{(1-z)^3} \Longrightarrow k'(0) = \frac{1+0}{(1-0)^3} = 1. \tag{26}$$

The above conditions can be verified via power series representation from where we have the constant term $a_0 = 0$ (since the series starts at $n = 1$) and the linear term $a_1 z = 1 \cdot z$ implying $a_1 = 1$. This matches the normalization conditions.
Next, we consider a result for growth.

Proposition 5.4 *For $z \in \mathcal{D}$ with $|z| = r < 1$ we have*

$$\frac{r}{(1+r)^2} \le |\varphi(z)| \le \frac{r}{(1-r)^2}. \tag{27}$$

Proof. Inequality 27 indicates that the growth of the function is constrained between two functions of r, which diverge as $r \to 1^-$. The lower bound limits how slowly the function may grow near the boundary of the unit disk while the upper bound represents the fastest possible growth under the univalence condition. The significance of this theorem is twofold in the sense that it confirms that univalent functions cannot grow arbitrarily fast within D as well as providing benchmarks for comparing specific univalent functions. The KF achieves equality according to the theorem since $a_n = n$ and $a_1 = 1$.

Now, we consider radial growth of the modulus of KF and we analyze how it behaves as z approaches the boundary of $\mathcal{D}$.

Theorem 5.5 *Let $z = re^{i\theta}$, where $0 \le r < 1$ and $\theta \in [0, 2\pi)$ then*

$$|k(z)| = \frac{r}{|1 - re^{i\theta}|^2} \approx re^{i\theta}(1 + 2re^{i\theta} + 3r^2e^{2i\theta} + \cdots) \tag{28}$$

Proof. First, we check the behavior at the key points:

(i). At $r = 0$: $|k(0)| = \frac{0}{|1-0e^{i\theta}|^2} = 0$.

(ii). For small r: $k(z) \approx re^{i\theta}(1 + 2re^{i\theta} + 3r^2e^{2i\theta} + \cdots)$ indicating the function grows almost linearly.

(iii). As $r \to 1^-$: The denominator $(1 - re^{i\theta})^2 \to 0$ causing the modulus of $|k(z)|$ to blow up towards infinity that is $|k(z)| \to \infty$.

Next we consider Extreme Cases:
1. Maximum Growth (on the Boundary $|z| = r$). By the Maximum Modulus Principle, the maximum of $|k(z)|$ for $|z| \leq r$ occurs on $|z| = r$. Let $z = re^{i\theta}$, then: $|k(z)| = \left|\frac{re^{i\theta}}{(1-re^{i\theta})^2}\right| = \frac{r}{|1-re^{i\theta}|^2}$. To maximize $|k(z)|$, minimize $|1 - re^{i\theta}|$. The minimum occurs at $\theta = 0$, that is, where $z = r$). $\min_{\theta=0} |1 - re^{i\theta}| = |1 - re^0| = |1 - r(cos0 + isin0)| = |1 - r(1 + i0)| = |1 - r(1)| = 1 - r$. Therefore, $\max_{|z|=r} |k(z)| = \frac{r}{(1-r)^2} \sim \frac{1}{(1-r)^2}$, as $r \to 1^-$, $\max |k(z)| \to +\infty$.
2. Minimum Growth (on the Boundary $|z| = r$) The minimum modulus occurs when $|1 - re^{i\theta}|$ is maximized, which happens at $\theta = \pi$, that is, $\max_{\theta=\pi} |1 - re^{i\theta}| = |1 - re^{i\pi}| = |1 - r(cos\pi + isin\pi)| = |1 - r(-1 + i0)| = |1 - r(-1)| = 1 + r$. Therefore, $\min_{|z|=r} |k(z)| = \frac{r}{(1+r)^2} \sim \frac{1}{(1+r)^2}$. As $r \to 1^-$, this approaches $\lim_{r\to 1^-} \min_{|z|=r} |k(z)| = \frac{1}{(1+1)^2} = \frac{1}{4} = 0.25$ Hence, KF exhibits extremal growth behavior making it central in the analysis of growth conditions.

Next we consider distortion behavior of Koebe function which is fundamental in understanding the geometric properties of univalent functions within the unit disk. This section focuses on representing the Koebe function as an extremal function for distortion analysis verifying its properties and analyzing the bounds for the modulus of its derivative. We utilize the Koebe function as an Extremal Function for analyzing distortion conditions. The derivative of KF is given by $k'(z) = \sum_{n=0}^{\infty}(n+1)^2 z^n$. The derivative of the growth theorem gives the distortion theorem.

Proposition 5.6 *Growth theorem states that for $z \in \mathcal{D}$ with $|z| = r < 1$ we have*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |\varphi'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}, \quad |z| = r < 1. \tag{29}$$

Proof. These inequalities quantify how the geometry of $\mathcal{D}$ is distorted under φ. We need to to verify that the Koebe function achieves equality in both bounds making it extremal. For the Koebe function $k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + \ldots : z \in \mathcal{D} = \{z \in \mathcal{C} : |z| < 1\}$. For *modulus of the derivative* we have $|k'(z)| = |\frac{1+z}{(1-z)^3}| = \frac{|1+z|}{|1-z|^3}$. For *maximum distortion on the Boundary* $|z| = r$, we have by the Maximum Modulus Principle, the maximum of $|k'(z)|$ for $|z| \leq r$ occurs on $|z| = r$. Let $z = re^{i\theta}$, then $|k'(z)| = \left|\frac{1+re^{i\theta}}{(1-re^{i\theta})^3}\right| = \frac{1+r}{|1-re^{i\theta}|^3}$ To maximize $|k'(z)|$, minimize $|1 - re^{i\theta}|$. The minimum occurs at $\theta = 0$, that is, $z = r$ $\min_{\theta=0} |1 - re^{i\theta}| = |1 - re^0| = |1 - r| = 1 - r$. Therefore, $\max_{|z|=r} |k'(z)| = \frac{1+r}{(1-r)^3} \sim \frac{2}{(1-r)^3}$. This matches the upper bound. As $r \to 1^-$, $\max |k'(z)| \to +\infty$. For *minimum distortion on the boundary,* $|z| = r$. The minimum derivative modulus occurs when $|1 - re^{i\theta}|$ is maximized, which happens at $\theta = \pi$, that is, $\max_{\theta=\pi} |1 - re^{i\theta}| = |1 - re^{i\pi}| = |1 - r(cos\pi + isin\pi)| = |1 - r(-1)| = 1 + r$. Therefore, $\min_{|z|=r} |k'(z)| = \frac{1-r}{(1+r)^3}$. This matches the lower bound of the Distortion Theorem. As $r \to 1^-$, this approaches $\lim_{r\to 1^-} \min_{|z|=r} |k'(z)| = \frac{1-1}{(1+1)^3} = 0$.

Remark 5.7 *For the implications of the Distortion Theorem we consider:*
1. Boundary Behavior*: Near the origin* $(r \approx 0)$ *the distortion is close to 1. The lower bound* $\frac{1-r}{(1+r)^3} \to 0$ *indicating that derivatives may vanish near the boundary. As* $r \to 1^-$*, the upper bound* $\frac{1+r}{(1-r)^3} \to +\infty$ *showing that derivatives of elements of class* $\mathcal{S}$ *can grow rapidly near the boundary.*
2. Geometric Interpretation*: The theorem controls how* φ *stretches or compresses distances in* $\mathcal{D}$*. The Koebe function stretches the disk maximally toward the boundary point* $z = 1$*.*

6 Open Problems

We have analyzed growth and distortion conditions for univalent functions in the unit disk. Several researches have been conducted over decades with interesting findings obtained for various functions like the conformal mappings and slice regular functions with nice and very crucial in applications in various fields. However, complete analysis of these conditions has not been done. Recently, researchers gave an open question on the growth and distortion theorems asking whether the family of regular slice mappings is the largest subfamily of the unit ball. In this paper, we have analyzed, in particular, the Koebe function and established its GDC by establishing its minimal and maximal extremal boundary points. This leaves an open question as stated below.
Problem 1: Can one develop an efficient algorithm for analyzing the growth and distortion conditions given in this work?

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