Int. J. Open Problems Compt. Math., Vol. 18, No. 3, September 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376
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Uniqueness of Meromorphic functions concerning their shift and Linear difference polynomials

HARINA P. WAGHAMORE AND ROOPA M.

Department of Mathematics, Jnanabharathi Campus, Bangalore University,
Bengaluru, Karnataka, India - 560 056.
e-mail:harinapw@gmail.com.

Received 1 March 2025; Accepted 17 June 2025

Abstract

In this paper, we investigate the uniqueness of meromorphic functions by considering their shift and Linear difference polynomial. We obtain some results which extends and generalizes the results given by Harina P. Waghamore and Preetham Raj N.[6].

Keywords: *Uniqueness, Shift, Differential polynomials, Linear Difference Polynomial, multiplicity, Weighted Sharing.*

2010 Mathematics Subject Classification: 30D35.

1 Introduction

The Nevanlinna value distribution theory is a branch of complex analysis mainly concentrate on the study of value distribution of solutions to the equation $f(z) = a$, where $a \in \mathbb{C} \cup \{\infty\}$ and $f(z)$ is an entire or meromorphic function in the Gaussian complex plane $\mathbb{C}$. For the elementary definitions and standard notations of the Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ etc. The reader can refer ([7], [16], [17]). We shall denote by $S(r, f)$, any quantity which satisfies $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure. Throughout the paper a meromorphic function always means a non-constant meromorphic function in the open complex plane and a constant always means a complex valued constant, unless otherwise mentioned.

Let f and g be two non-constant meromorphic functions in the open complex plane. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^{+} \cup \{\infty\}$ the set, $E(a, f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where each a-point of f with multiplicity k is counted k times in the set and the set, $\overline{E}(a, f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where the multiplicities are ignored. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity) and we have $E(a, f) = E(a, g)$; Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity) and we will have $\overline{E}(a, f) = \overline{E}(a, g)$.

Nevanlinna had proved that, if two non-constant entire functions f and g on the complex plane share four distinct finite values (ignoring multiplicity), then $f = g$ and this number four cannot be reduced [14, page 1]. But in 1976, Rubel and Yang [14] considered a special case by taking g as the first derivative of f and obtained the following result.

Theorem 1.1 *[14] If f is a non-constant entire function in the finite complex plane and if f and f' share two distinct values (counting multiplicity), then f'=f.*

Thus Rubel and Yang [14] showed that a derivative is worth two values. Since then the study of uniqueness of meromorphic functions sharing values with their derivatives became a subject of much interest.

In 2018, Qi et al. [12] by considering the shifts, studied the value sharing problem related to $f'(z)$ and $f(z + c)$, where c is a finite complex number and they obtained the following result.

Theorem 1.2 *[12] Let $f(z)$ be a non-constant meromorphic function of finite order and $n \geq 9$ be an integer. If $[f'(z)]^n$ and $f^n(z + c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = tf(z + c)$, for a constant t, that satisfies $t^n = 1$.*

In 2020, Meng and Liu [11] extended the above result by considering the kth derivative of f and obtained the following results.

Theorem 1.3 *[11] Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions is satisfied:*

- *(I) $[f^{(k)}(z)]^n$ and $f^n(z + c)$ share $(1, 2)$, $(\infty, 0)$ and $n \geq 2k + 8$;*
- *(II) $[f^{(k)}(z)]^n$ and $f^n(z + c)$ share $(1, 2)$, (∞, ∞) and $n \geq 2k + 7$;*
- *(III) $[f^{(k)}(z)]^n$ and $f^n(z + c)$ share $(1, 0)$, $(\infty, 0)$ and $n \geq 3k + 14$;*

then $f^{(k)}(z) = tf(z + c)$, for a constant t, that satisfies $t^n = 1$.

Corollary 1.4 *[11] Let f be a non-constant entire function of finite order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(z+c)$ share $(1,2)$, then $f^{(k)}(z) = tf(z+c)$, for a constant t, that satisfies $t^n = 1$.*

Meng and Liu [11] further studied the same problem by replacing the shifts $f(z+c)$ by q-difference, *i.e.*, $f(qz)$ and obtained the following results.

Theorem 1.5 *[11] Let f be a non-constant meromorphic function of zero order and n be a positive integer. If one of the following conditions is satisfied:*

(I) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, $(\infty,0)$ and $n \geq 2k+8$;

(II) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, (∞,∞) and $n \geq 2k+7$;

(III) $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,0)$, $(\infty,0)$ and $n \geq 3k+14$;

then $f^{(k)}(z) = tf(qz)$, for a constant t, that satisfies $t^n = 1$.

Corollary 1.6 *[11] Let f be a non-constant entire function of finite order and $n \geq 5$ be an integer. If $[f^{(k)}(z)]^n$ and $f^n(qz)$ share $(1,2)$, then $f^{(k)}(z) = tf(qz)$, for a constant t, that satisfies $t^n = 1$.*

In 2021 Harina P. Waghamore and Preetham N. Raj [6] has obtained the results for the Meromorphic functions concerning their shift and differential polynomial.

Theorem 1.7 *[6] Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$ and ∞ IM, and $n \geq 2\gamma_p + 2\sigma + 6$;

(II) $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$ and ∞ CM, and $n \geq 2\gamma_p + 2\sigma + 5$;

(III) $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,0)$ and ∞ IM, and $n \geq 3\gamma_p + 3\sigma + 11$;

then $P[f] = tf(z+c)$, for a constant t, such that $t^n = 1$.

Corollary 1.8 *[6] Let f be a non-constant entire function of finite order and $n \geq 2\gamma + 3$ be an integer. If $\{P[f]\}^n$ and $f^n(z+c)$ share $(1,2)$, then $f^{(k)}(z) = tf(z+c)$, for a constant t, that satisfies $t^n = 1$.*

Theorem 1.9 *[6] Let f be a non-constant meromorphic function of finite order and n be a positive integer. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ and $f^n(qz)$ share $(1,2)$ and ∞ IM, and $n \geq 2\gamma_p + 2\sigma + 6$;

(II) $\{P[f]\}^n$ and $f^n(qz)$ share $(1,2)$ and ∞ CM, and $n \geq 2\gamma_p + 2\sigma + 5$;

(III) $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,0)$ *and* ∞ *IM, and* $n \geq 3\gamma_p + 3\sigma + 11$*;*

then $P[f] = tf(z+c)$*, for a constant* t*, such that* $t^n = 1$*.*

Corollary 1.10 *[6] Let* f *be a non-constant entire function of finite order and* $n \geq 2\gamma + 3$ *be an integer. If* $\{P[f]\}^n$ *and* $f^n(qz)$ *share* $(1,2)$*, then* $f^{(k)}(z) = tf(qz)$*, for a constant* t*, that satisfies* $t^n = 1$*.*

Question : Since $(f^{(k)})^n$ is nothing, but a differential monomial generated by f, it is natural to ask whether $(f^{(k)})^n$ can be extended to a differential polynomial $P[f]$ and their shift $f(z+c)$ as $[L(z,f)]^n$ a linear difference polynomial of a meromorphic (or entire) function generated by $f(z)$.

In this paper we give an affirmative answer to the above question and extend and improve the above mentioned theorems (1.7 - 1.10) and their corollaries to more generalized form. Thus, by investigating the uniqueness of meromorphic functions of the form $[L(z,f)]^n$, $[L_q(z,f)]^n$, and $\{P[f]\}^n$ we obtain the following results.

2 Preliminaries

The proof of the previous theorems proceeds through a number of steps, stated below as lemmas.

We need the following definitions and notations.

Definition 2.1 *[8] Let* k *be a non-negative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$*, we denote by* $E_k(a,f)$ *the set of all* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$*. If* $E_k(a,f) = E_k(a,g)$*, then we say that* f *and* g *share the value* a *with the weight* k*.*

The definition implies that, if f and g share a value a with the weight k, then z_0 is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g-a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f-a$ with multiplicity $m(>k)$ if and only if z_0 is a zero of $g-a$ with multiplicity $n(>k)$, where m is not necessarily equal to n. We write f, g share (a,k) to mean that, f, g share the value a with the weight k. Clearly, if f, g share (a,k), then f, g share (a,p) for any integer p, such that, $0 \leq p < k$. Also we note that, f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 2.2 *[10] For two meromorphic functions* f*,* g *and for* a*,* $b \in \mathbb{C} \cup \{\infty\}$ *and for a positive integer* k*,*

(i) $N_{(k}(r,a;f)$ $(\overline{N}_{(k}(r,a;f))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not less than* k*,*

(ii) $N(r,a;f|g=b)$ ($\overline{N}(r,a;f|g=b)$) *denotes the counting function (reduced counting function) of those a-points of f which are the b-points of g,*

(iii) $N(r,a,f|g\neq b)$ ($\overline{N}(r,a;f|g\neq b)$) *denotes the counting function (reduced counting function) of those a-points of f which are not the b-points of g,*

(iv) $N_p(r,a;f)=\overline{N}(r,a;f)+\sum_{k=2}^{p}\overline{N}_{(k}(r,a;f)$,

(v) $N_2(r,a;f|g=b)$ ($N_2(r,a;f|g\neq b)$) *denotes the counting function of those a-points of f which are (are not) the b-points of g, where an a-point of f with multiplicity m is counted m times if* $m\le 2$ *and twice if* $m>2$,

(vi) $N_{k)}(r,a;f)$ ($\overline{N}_{k)}(r,a;f)$) *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k.*

Definition 2.3 *[1] Let f and g be two meromorphic functions such that f and g share the value 1 IM. Let* z_0 *be a 1-point of f of order p, and a 1-point of g of order q. We denote, by* $\overline{N}_L\left(r,\frac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $q<p$, *by* $\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $2\le q=p$, *by* $\overline{N}_E^{1)}\left(r,\frac{1}{f-1}\right)$ *the counting function of those 1-points of f and g such that* $p=q=1$, *and by* $\overline{N}_{f>2}\left(r,\frac{1}{g-1}\right)$ *the counting functions of those 1-points of f and g such that* $p>q=2$, *each point in these counting functions is counted only once. In the same way, we can define* $\overline{N}_L\left(r,\frac{1}{g-1}\right)$, $\overline{N}_E^{(2}\left(r,\frac{1}{g-1}\right)$, $\overline{N}_{g>2}\left(r,\frac{1}{f-1}\right)$.

Definition 2.4 *[9] Let* $n_{0j},n_{1j},...,n_{kj}$ *be non-negative integers. The expression,*

$$M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}},$$

is called a differential monomial generated by f of degree $\gamma_{M_j}=\sum_{i=0}^{k}n_{ij}$ *and weight* $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$. *The sum,*

$$P[f]=\sum_{i=1}^{l}b_jM_j[f],$$

is called a differential polynomial generated by f of degree γ_p=*max*$\{\gamma_{M_j}:1\le j\le l\}$ *and weight* Γ_p=*max*$\{\Gamma_{M_j}:1\le j\le l\}$, *where* $T(r,b_j)=S(r,f)$ *for* $j=1,2,...,l$.

The numbers, $\underline{\gamma}_p$=min$\{\gamma_{M_j}:1\le j\le l\}$ and k (the highest order of the derivarive of f in $P[f]$) are called, respectively, the lower degree and order of

$P[f]$. $P[f]$ is said to be homogeneous if $\gamma_p = \underline{\gamma}_p$. Also $P[f]$ is called a quasi differential polynomial generated by f if instead of assuming $T(r, b_j) = S(r, f)$ we just assume that, $m(r, b_j) = S(r, f)$ for the co-efficients $b_j (j = 1, 2, ..., l)$. We denote, by σ=max$\{\Gamma_{M_j} - \gamma_{M_j} : 1 \leq j \leq l\}$=max$\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq l\}$.

Definition 2.5 *[5] If $f(z)$ is a non-constant meromorphic function of finite order, then*

$$L(z, f) = b_k f(z + c_k) + ... + b_1 f(z + c_1) + b_0 f(z) = \sum_{j=0}^{k} b_j f(z + c_j), \qquad (1)$$

is called linear difference polynomial of $f(z)$, where $k \in N \cup \{0\}$ and a_j and c_j's are complex constants with atleast one a_j's are non-zero.
For a non-zero complex constant q and a meromorphic function f, the $q-$shift and $q-$difference operators are defined, respectively by $f(qz)$ and $\Delta_q = f(qz) - f(z)$. Here also we generalize these operators as follows.

$$L_q(z, f) = b_r f(q_j z + c_r) + ... + b_1 f(q_1 z + c_1) + b_0 f(qz) = \sum_{j=0}^{r} b_j f(q_j z + c_j), \qquad (2)$$

where r is a non-negative integer, and q_j, b_j, d_j's are complex constants with atleast one of b_j is non-zero.

Let F and G be two non-constant meromorphic functions, we shall denote by Ψ the following function.

$$\Psi = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \qquad (3)$$

Lemma 2.6 *[2] Let F, G be two non-constant meromorphic functions. If F, G share $(1, 2)$ and (∞, k), where $0 \leq k \leq \infty$ and $\Psi \not\equiv 0$, then*

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G),$$

where $\overline{N}_*(r, \infty; F, G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.7 *[15] Let f be a non-constant meromorphic function and let $a_1, a_2, ..., a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + ... + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.8 *[4] Let f be a meromorphic function of finite order $\rho(f)$ and let c be a non-zero complex constant. Then*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + O(logr),$$

for an arbitrary $\epsilon > 0$.

Lemma 2.9 *[18] Suppose that two non-constant meromorphic functions F and G share 1 and ∞ IM. Let Ψ be given as in (2.1). If $\Psi \neq 0$, then*

$$\begin{aligned} T(r,F) + T(r,G) \leq 3\overline{N}(r,F) &+ N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_E^{1)}\left(r, \frac{1}{F-1}\right) \\ &+ 2N_E^{(2}\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) \\ &+ S(r,F) + S(r,G). \end{aligned}$$

Lemma 2.10 *[19] Let f be a zero order meromorphic function and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$\begin{aligned} T(r, f(qz)) &= (1+O(1))T(r, f(z)), \\ N(r, f(qz)) &= (1+O(1))T(r, f(z)), \end{aligned}$$

on a set of lower logarithmic density 1.

Lemma 2.11 *[3] Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial in f. Then*

$$\begin{aligned} m\left(r, \frac{P[f]}{f^{\gamma_p}}\right) &\leq (\gamma_p - \underline{\gamma}_p) m\left(r, \frac{1}{f}\right) + S(r,f), \\ m\left(r, \frac{P[f]}{f^{\underline{\gamma}_p}}\right) &\leq (\gamma_p - \underline{\gamma}_p) m\left(r, f\right) + S(r,f), \\ N\left(r, \frac{P[f]}{f^{\gamma_p}}\right) &\leq (\gamma_p - \underline{\gamma}_p) N\left(r, \frac{1}{f}\right) + \sigma\left[\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r,f), \\ N\left(r, P[f]\right) &\leq \gamma_p N(r,f) + \sigma\overline{N}(r,f) + S(r,f), \\ T(r, P[f]) &\leq \gamma_p T(r,f) + \sigma\overline{N}(r,f) + S(r,f), \end{aligned}$$

where σ=max $\{n_{1j} + 2n_{2j} + 3n_{3j} + ... + kn_{kj}\,; 1 \leq j \leq l\}$.

Lemma 2.12 *Let f be a non-constant meromorphic function having zeros of multiplicity at least s, where s, n be a positive integers, if $F = \{L(z,f)\}^n$, where c is a finite complex number and $G = \{P[f]\}^n$, where $P[f]$ is a differential polynomial, then $F \cdot G \neq 1$.*

Proof 2.12. On the contrary, suppose $F \cdot G = 1$,

$$i.e., \{L(z,f)\}^n\{P[f]\}^n = 1.$$

From the above, it is clear that the function f can't have any zeros and poles. Therefore

$$\overline{N}\left(r,\frac{1}{f}\right) = S(r,f) = \overline{N}(r,f).$$

So by the first fundamental theorem of Nevanlinna and Lemma 2.6, we have,

$$\begin{aligned}
(n+n\gamma_p)T(r,f) &= T\left(r,\frac{1}{[L(z,f)]^n f^{n\gamma_p}}\right) + S(r,f)\\
&\le T\left(r,\frac{\{P[f]\}^n}{f^{n\gamma_p}}\right) + S(r,f)\\
&\le nT\left(r,\frac{\{P[f]\}}{f^{\gamma_p}}\right) + S(r,f)\\
&\le n\left[m\left(r,\frac{P[f]}{f^{\gamma_p}}\right) + N\left(r,\frac{P[f]}{f^{\gamma_p}}\right)\right] + S(r,f)\\
&\le n\left[(\gamma_p - \underline{\gamma}_p)T(r,f)\right] + n\sigma\left[\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right] + S(r,f)\\
(n+n\gamma_p)T(r,f) &\le n\left[(\gamma_p - \underline{\gamma}_p)T(r,f)\right] + S(r,f)\\
(n+n\underline{\gamma}_p)T(r,f) &\le S(r,f),
\end{aligned}$$

which is a contradiction. Thus $F \cdot G \neq 1$. This completes the proof of the lemma 2.12.

Lemma 2.13 *Let f be a non-constant meromorphic function having zeros of multiplicity at least s, where s, n be a positive integers. If $F = [L_q(z,f)]^n$, where c is a finite complex number and $G = \{P[f]\}^n$, where $P[f]$ is a differential polynomial, then $F \cdot G \neq 1$.*

Proof 2.13. In a similar manner we can prove the following Lemma 2.13.

3 Main results

Theorem 3.1 *Let f be a non-constant meromorphic function having zeros of multiplicity at least s with finite order, where s, n are positive integers. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ and $[L(z,f)]^n$ share $(1,2)$ and ∞ IM, and
$n \ge \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + +4t + 6)$;

(II) $\{P[f]\}^n$ *and* $[L(z,f)]^n$ *share* $(1,2)$ *and* ∞ *CM, and* $n \geq \frac{1}{s(t+1)}\left(2\gamma_p + 2\sigma + +4t + 5\right)$*;*

(III) $\{P[f]\}^n$ *and* $[L(z,f)]^n$ *share* $(1,0)$ *and* ∞ *IM, and* $n \geq \frac{1}{s(t+1)}\left(3\gamma_p + 3\sigma + 9t + 11\right)$*;*

then $P[f] = tL(z,f)$*, for a constant t, such that* $t^n = 1$.

Proof of Theorem 3.1. Let us consider,

$$F = [L(z,f)]^n \quad \text{and} \quad G = \{P[f]\}^n. \tag{4}$$

(I). Suppose $\{P[f]\}^n$ and $[L(z,f)]^n$ share $(1,2)$ & ∞ IM, and $n \geq \frac{1}{s(t+1)}\left(2\gamma_p + 2\sigma + 4t + 6\right)$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$ & ∞ IM. Let Ψ be defined as in (3). It can be easily seen that the possible poles of Ψ occur at,

- (i) multiple zeros of F and G
- (ii) those 1-points of F and G whose multiplicities are different
- (iii) those poles of F and G whose multiplicities are different
- (iv) zeros of F' and G' which are not the zeros of $F(F-1)$ and $G(G-1)$ respectively.

We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then it follows from Lemma 2.6 that,

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G). \tag{5}$$

Using Lemma 2.7 and Lemma 2.8, we can write

$$\begin{aligned} T(r,F) &= nT(r,L(z,f)) + S(r,f) \\ &= n(t+1)T(r,f(z)) + O(r^{\rho(f)-1+\epsilon}) + S(r,f). \end{aligned} \tag{6}$$

We obviously have the following,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{L(z,f)}\right) &\leq 2\frac{(t+1)}{s}T(r,L(z,f)) + S(r,f) \\ &\leq 2\frac{(t+1)}{s}T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{7}$$

$$\overline{N}(r,F)=\overline{N}(r,L\left(z,f\right))\leq\frac{1}{s}T(r,L\left(z,f\right))\leq\frac{(t+1)}{s}T(r,f)+O(r^{\rho(f)-1+\epsilon}),\tag{8}$$

$$\overline{N}_{*}(r,\infty;F,G)\leq\overline{N}(r,F)\leq\frac{1}{s}T(r,L\left(z,f\right))\leq\frac{(t+1)}{s}T(r,f)+O(r^{\rho(f)-1+\epsilon})+S(r,f).\tag{9}$$

Since $\overline{E}(\infty,f^{(k)})=\overline{E}(\infty,f)$, we have

$$\overline{N}(r,G)=\overline{N}(r,P[f])=\overline{N}(r,f).\tag{10}$$

From Lemma 2.11, we have

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{P[f]}\right)\leq\frac{2}{s}T(r,P[f])+S(r,f)\leq2\frac{(\gamma_p+\sigma)}{s}T(r,f)+S(r,f).\tag{11}$$

By combining (5) to (12), we deduce that

$$(n(t+1)-\left(\frac{2\gamma_p+2\sigma+5}{s}\right)T(r,f)\leq S(r,f),\tag{12}$$

which contradicts that $n\geq\frac{1}{s(t+1)}\left(2\gamma_p+2\sigma+4t+6\right)$. Thus, we have $\Psi=0$ and hence

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}.\tag{13}$$

By integrating the above equation twice on both sides, we get

$$\frac{1}{F-1}=\frac{C}{G-1}+D,\tag{14}$$

where $C\neq0$ and D are constants. From (14), we have

$$G=\frac{(D-C)F+(C-D-1)}{DF-(D+1)}.\tag{15}$$

Now, we have the following three cases:

Case 1. Suppose that $D\neq0,-1$. Then, from (15), we have

$$\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)=\overline{N}(r,G).\tag{16}$$

From the Second fundamental theorem, Lemma 2.8 and (11), we have

$$\begin{aligned}n(t+1)T(r,f)=T(r,F)+S(r,f)&\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f)\\&\leq\frac{(2t+3)}{s}T(r,f)+O(r^{\rho(f)-1+\epsilon})+S(r,f),\end{aligned}\tag{17}$$

which contradicts $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$.

Case 2. Suppose $D = -1$. From (15), we have

$$G = \frac{(C+1)F - C}{F}. \tag{18}$$

(i) If $C \neq -1$, from (18), we get

$$\overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{19}$$

From Second fundamental theorem, Lemma 2.8 and Lemma 2.11, we get

$$\begin{aligned} n(t+1)T(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) + S(r,f) \\ &\leq \overline{N}(r, L(z,f)) + \overline{N}\left(r, \frac{1}{L(z,f)}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f) \\ &\leq \left(\frac{2(t+1)}{s} + \frac{\gamma_p + \sigma}{s}\right) T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{20}$$

which contradicts with $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$.

(ii) If $C = -1$, then from (18), we get $F \cdot G = 1$,
i.e.,

$$[L(z,f)]^n \{P[f]\}^n = 1, \tag{21}$$

which is a contradiction from Lemma 2.12.

Case 3. Suppose that $D = 0$. From (15), we have

$$G = CF - (C-1). \tag{22}$$

If $C \neq 1$, from (22), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{23}$$

Then from the Second fundamental theorem, Lemma 2.8 and Lemma 2.11, we have

$$\begin{aligned} n(t+1)T(r,f) &= T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}\left(r,\frac{1}{F-\frac{C-1}{C}}\right)+S(r,f)\\ &\le \overline{N}(r,L\left(z,f\right))+\overline{N}\left(r,\frac{1}{L\left(z,f\right)}\right)+\overline{N}\left(r,\frac{1}{P[f]}\right)+S(r,f)\\ &\le \left(\frac{2(t+1)}{s}+\frac{\gamma_p+\sigma}{s}\right)T(r,f)+O(r^{\rho(f)-1+\epsilon})+S(r,f), \end{aligned} \tag{24}$$

which again contradicts $n \ge \frac{1}{s(t+1)}(2\gamma_p+2\sigma+4t+6)$. Hence, $C=1$. Thus From (22), we have $F=G$, *i.e.*,

$$L^n(z,f)=\{P[f]\}^n.$$

Hence $P[f]=tL\left(z,f\right)$, for a constant t, such that $t^n=1$. This proves (I) of Theorem 3.1.

(II). Suppose $\{P[f]\}^n$ and $[L(z,f)]^n$ share $(1,2)$ & ∞ CM and $n \ge \frac{1}{s(t+1)}(2\gamma_p+2\sigma+3t+5)$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$ & ∞ CM. Let Ψ be defined as in (3). We claim $\Psi=0$, on the contrary if $\Psi\ne 0$, then from Lemma 2.6, we have

$$\begin{aligned} T(r,F) \le{}& N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)+\\ &S(r,F)+S(r,G). \end{aligned} \tag{25}$$

It is obvious that,

$$\overline{N}_*(r,\infty;F,G)=0. \tag{26}$$

By combining (25), (7), (8), (9), (11), (12), and (26), we have

$$(n(t+1)-\left(\frac{2\gamma_p+2\sigma+3t+4}{s}\right)T(r,f)\le O(r^{\rho(f)-1+\epsilon})+S(r,f), \tag{27}$$

which contradicts with, $n \ge \frac{1}{s(t+1)}(2\gamma_p+2\sigma+3t+5)$. Hence, we get $\Psi=0$. Now, by following the steps of the proof of (I) of the Theorem 3.1, we can get the proof of (II) of Theorem 3.1.

(III). Suppose $\{P[f]\}^n$ and $[L(z,f)]^n$ share 1, ∞ CM and

$n \geq \dfrac{1}{s(t+1)}\left(3\gamma_p + 3\sigma + 9t + 11\right)$. Then it follows directly from the assumptions of the theorem, that F and G share 1, ∞ CM. Let Ψ be defined as in (3). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.9, we have

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ & + 2N_E^{(2}\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{G-1}\right) \\ & + S(r,F) + S(r,G). \end{aligned} \tag{28}$$

Since,

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{F-1}\right) & + 2N_E^{(2}\left(r,\frac{1}{F-1}\right) + N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) \\ & \leq N\left(r,\frac{1}{G-1}\right) \leq T(r,G) + O(1). \end{aligned} \tag{29}$$

Therefore, from (28) and (29), we get

$$\begin{aligned} T(r,F) \leq\ & 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2N_L\left(r,\frac{1}{F-1}\right) + \\ & N_L\left(r,\frac{1}{G-1}\right) \quad + S(r,F) + S(r,G). \end{aligned} \tag{30}$$

From Lemma 2.7 and Lemma 2.8, we have,

$$T(r,F) = nT(r,L(z,f)) + S(r,f) = n(t+1)T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f). \tag{31}$$

We obviously have,

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{F}{F'}\right) & \leq N\left(r,\frac{F'}{F}\right) + S(r,f) \\ & \leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\ & \leq \overline{N}(r,L(z,f)) + \overline{N}\left(r,\frac{1}{L(z,f)}\right) + S(r,f) \\ & \leq \frac{2(t+1)}{s}T(r,f) + O(r^{\rho(f)-1+\epsilon}) + S(r,f), \end{aligned} \tag{32}$$

$$\begin{aligned}
N_L\left(r,\frac{1}{G-1}\right) \le N\left(r,\frac{G}{G'}\right) &\le N\left(r,\frac{G'}{G}\right)+S(r,f)\\
&\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\
&\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{P[f]}\right)+S(r,f)\\
&\le \left(\frac{1}{s}+\frac{\gamma_p+\sigma}{s}\right)T(r,f)+S(r,f). \qquad (33)
\end{aligned}$$

By combining (30), (8), (9), (12), (31), (32), and (33), we get

$$(n(t+1)-\left(\frac{3\gamma_p+3\sigma+9t+10}{s}\right)T(r,f)\le O(r^{\rho(f)-1+\epsilon})+S(r,f), \qquad (34)$$

which contradicts with $n \ge \frac{1}{s(t+1)}(3\gamma_p+3\sigma+9t+11)$. Now, by following the steps of the proof of (I) of Theorem 3.1, we can get the proof of (III) of Theorem 3.1. This completes the proof of Theorem 3.1.

Corollary 3.2 *Let f be a non-constant entire function having zeros of multiplicity atleast s of finite order and $n \ge \frac{1}{s(t+1)}(2\gamma_p+2t+3)$ be an integer. If $\{P[f]\}^n$ and $[L(z,f)]^n$ share $(1,2)$, then $\{P[f]\}=tL(z,f)$, for a constant t, such that $t^n=1$.*

Proof of Corollary 3.2

Suppose $\{P[f]\}^n$ and $L^n(z,f)$ share $(1,2)$ and $n \ge \frac{1}{s(t+1)}(2\gamma_p+2\sigma+2t+3)$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$. Let Ψ be defined as in (3). We claim $\Psi=0$, on the contrary if $\Psi\ne 0$, then it follows from Lemma 2.6 that

$$T(r,F)\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N_*}(r,\infty;F,G)+S(r,F)+ S(r,G).$$

Since f is a non-constant entire function, we have $\overline{N}(r,f)=S(r,f)$. Hence, the above equation reduces to

$$n(t+1)T(r,f)\le\frac{2(t+1)}{s}T(r,f)+\frac{2\gamma_p}{s}T(r,f)+O(r^{\rho(f)-1+\epsilon})+S(r,f).$$

Thus,

$$(n(t+1)-\left(\frac{2\gamma_p+2t+2}{s}\right)T(r,f)\le S(r,f), \qquad (35)$$

which contradicts with $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2t + 3)$. Now, by following the steps of the proof of (I) of Theorem 3.1, we can easily get the proof of Corollary 3.2.

Theorem 3.3 *Let f be a non-constant meromorphic function having zeros of multiplicity s with zero order and s, n are positive integers. If one of the following conditions holds:*

(I) $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$ and ∞ IM, and $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$;

(II) $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$ and ∞ CM, and $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 5)$;

(III) $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,0)$ and ∞ IM, and $n \geq \frac{1}{s(t+1)}(3\gamma_p + 3\sigma + 9t + 11)$;

then $P[f] = tL_q(z,f)$, for a constant t, such that $t^n = 1$.

Proof of theorem 3.3
Let us consider,

$$F = [L_q(z,f)]^n \text{ and } G = \{P[f]\}^n \tag{36}$$

(I). Suppose $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$ & ∞ IM and $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$. Then it follows directly from the assumptions of the theorem that F and G share $(1,2)$ & ∞ IM. We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then it follows from Lemma 2.6 that,

$$T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G). \tag{37}$$

Using Lemma 2.7 and Lemma 2.10, we can write

$$T(r,F) = n(t+1)T(r,L_q(z,f)) + S(r,f) = n(t+1)T(r,f) + S(r,f), \tag{38}$$

$$\overline{N}(r,F) = \overline{N}(r,L_q(z,f)) = (t+1)\overline{N}(r,f(z)) + S(r,f) \leq \frac{(t+1)}{s}T(r,f) + S(r,f). \tag{39}$$

We obviously have the following,

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{L_q(z,f)}\right) \le 2\frac{(t+1)}{s}T(r,L_q(z,f))+S(r,f) \\ &\le 2\frac{(t+1)}{s}T(r,f)+S(r,f), \end{aligned} \tag{40}$$

$$\overline{N}_*(r,\infty;F,G) \le \overline{N}(r,F) \le \frac{(t+1)}{s}T(r,L_q(z,f)) \le \frac{(t+1)}{s}T(r,f)+S(r,f). \tag{41}$$

Since $\overline{E}(\infty,f^{(k)})=\overline{E}(\infty,f)$, we have

$$\overline{N}(r,G)=\overline{N}(r,P[f])=\overline{N}(r,f). \tag{42}$$

From Lemma 2.11, we have

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) &= 2\overline{N}\left(r,\frac{1}{P[f]}\right) \le \frac{2}{s}T(r,P[f])+S(r,f) \\ &\le 2\frac{(\gamma_p+\sigma)}{s}T(r,f)+S(r,f). \end{aligned} \tag{43}$$

By combining (37) to (43), we deduce,

$$(n(t+1)-\left(\frac{2\gamma_p+2\sigma+4t+5}{s}\right)T(r,f) \le S(r,f), \tag{44}$$

which contradicts that $n \ge \frac{1}{s(t+1)}(2\gamma_p+2\sigma+4t+6)$. Thus, we have $\Psi=0$ and hence,

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}.$$

By integrating the above equation twice on both sides, we get

$$\frac{1}{F-1}=\frac{C}{G-1}+D, \tag{45}$$

where $C\neq 0$ and D are constants. From (45), we have

$$G=\frac{(D-C)F+(C-D-1)}{DF-(D+1)}. \tag{46}$$

Now, we have the following three cases:

Case 1. Suppose that $D\neq 0,-1$. Then, from (46), we have

$$\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)=\overline{N}(r,G). \tag{47}$$

From the second fundamental theorem, Lemma 2.10 and (11), we have

$$n(t+1)T(r,f) = T(r,F) + S(r,f) \leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{D+1}{D}}\right) + S(r,f)$$
$$\leq \frac{(2t+3)}{s}T(r,f) + S(r,f), \tag{48}$$

which contradicts with $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$.

Case 2. Suppose $D = -1$. From (46), we have

$$G = \frac{(C+1)F - C}{F}. \tag{49}$$

(i) If $C \neq -1$, then from (49), we get

$$\overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{50}$$

From Second fundamental theorem, Lemma 2.10 and Lemma 2.11, we get

$$n(t+1)T(r,f) = T(r,F) + S(r,f) \leq \overline{N}(r,F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) + S(r,f)$$
$$\leq \overline{N}(r, [L_q(z,f)]^n) + \overline{N}\left(r, \frac{1}{[L_q(z,f)]^n}\right) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f)$$
$$n(t+1)T(r,f) \leq \frac{(\gamma_p + \sigma + 2t + 3)}{s}T(r,f) + S(r,f), \tag{51}$$

which contradicts with $n \geq \frac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$.

(ii) If $C = -1$ and from (49) we get $F \cdot G = 1$,
i.e.,

$$[L_q(z,f)]^n \{P[f]\}^n = 1, \tag{52}$$

which is a contradiction from Lemma 2.13.

Case 3. Suppose that $D = 0$. From (46), we have

$$G = CF - (C-1). \tag{53}$$

If $C \neq 1$, then from (49), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \tag{54}$$

Then from the Second fundamental theorem, Lemma 2.10 and Lemma 2.11, we have

$$\begin{aligned} n(t+1)T(r,f) = (t+1)T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) \\ &+ S(r,f) \\ &\leq \overline{N}(r,[L_q(z,f)]^n) + \overline{N}\left(r, \frac{1}{[L_q(z,f)]^n}\right) \\ &+ \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f), \end{aligned}$$

i.e.,

$$n(t+1)T(r,f) \leq \frac{(2t + \gamma_p + \sigma + 2)}{s} T(r,f) + S(r,f), \tag{55}$$

which again contradicts $n \geq \dfrac{1}{s(t+1)}(2\gamma_p + 2\sigma + 4t + 6)$. Hence, $C = 1$. Thus From (22), we have $F = G$,*i.e.*,

$$[L_q(z,f)]^n = \{P[f]\}^n.$$

Hence $P[f] = tL_q(z,f)$, for a constant t, such that $t^n = 1$. This proves (I) of Theorem 3.3.

(II). Suppose $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$ & ∞ CM and $n \geq \dfrac{1}{t+1}(2\gamma_p + 2\sigma + 3t + 5)$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$ & ∞ CM. Let Ψ be defined as in (3). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.6, we have

$$\begin{aligned} T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) + \\ S(r,F) + S(r,G). \end{aligned} \tag{56}$$

It is obvious that,

$$\overline{N}_*(r,\infty;F,G) = 0. \tag{57}$$

By combining (56), (58), (39), (40), (42), (43) and (57), we have

$$\left(n(t+1) - \left(\frac{2\gamma_p + 2\sigma + 3t + 4}{s}\right)T(r,f) \leq S(r,f), \tag{58}$$

which contradicts with $n \geq \dfrac{1}{s(t+1)}\left(2\gamma_p+2\sigma+3t+5\right)$. Hence, we get, $\Psi = 0$. Now, by following the steps of the proof of (I) of the Theorem 3.3, we can get the proof of (II) of Theorem 3.3.

(III). Suppose $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $1, \infty$ CM and $n \geq \dfrac{1}{s(t+1)}\left(3\gamma_p+3\sigma+9t+11\right)$. Then it follows directly from the assumptions of the theorem, that F and G share $1, \infty$ CM. Let Ψ be defined as in (3). We claim $\Psi = 0$, on the contrary if $\Psi \neq 0$, then from Lemma 2.9, we have

$$\begin{aligned}T(r,F)+T(r,G) \leq{}& 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)\\ &+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)\\ &+S(r,F)+S(r,G). \qquad (59)\end{aligned}$$

Since,

$$\begin{aligned}N_E^{1)}\left(r,\frac{1}{F-1}\right)&+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)\leq T(r,G)+O(1). \qquad (60)\end{aligned}$$

Therefore, from (59) and (60), we get

$$\begin{aligned}T(r,F)\leq{}& 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)+\\ &N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G). \qquad (61)\end{aligned}$$

From Lemma 2.7 and Lemma 2.10, we have

$$T(r,F)=n\,(t+1)\,T(r,L_q(z,f))+S(r,f)=n\,(t+1)\,T(r,f)+S(r,f). \qquad (62)$$

We obviously have,

$$\begin{aligned}N_L\left(r,\frac{1}{F-1}\right)\leq N\left(r,\frac{F}{F'}\right)&\leq N\left(r,\frac{F'}{F}\right)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f)\\ &\leq \overline{N}(r,[L_q(z,f)]^n)+\overline{N}\left(r,\frac{1}{[L_q(z,f)]^n}\right)\\ &+S(r,f)\\ &\leq 2\left(\frac{t+1}{s}\right)T(r,f)+S(r,f), \qquad (63)\end{aligned}$$

$$\begin{aligned} N_L\left(r,\frac{1}{G-1}\right) \le N\left(r,\frac{G}{G'}\right) &\le N\left(r,\frac{G'}{G}\right)+S(r,f)\\ &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{P[f]}\right)+S(r,f)\\ &\le \left(\frac{\gamma_p+\sigma+1}{s}\right)T(r,f)+S(r,f). \qquad (64) \end{aligned}$$

By combining (61),(38), (39), (40), (43), (63) and (64), we get

$$(n\,(t+1)-\left(\frac{3\gamma_p+3\sigma+9t+10}{s}\right)T(r,f)\le S(r,f), \qquad (65)$$

which contradicts with $n \ge \dfrac{1}{s(t+1)}\left(3\gamma_p+3\sigma+9t+11\right)$. Now, by following the steps of the proof of (I) of Theorem 3.3., we can get the proof of (III) of Theorem 3.3. This completes the proof of Theorem 3.3.

Corollary 3.4 *Let f be a non-constant entire function having zeros of multiplicity s with zero order and $n \ge \dfrac{1}{s(t+1)}\left(2\gamma_p+2t+3\right)$ be an integer. If $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$, then $\{P[f]\}=tL_q(z,f)$, for a constant t, such that $t^n=1$.*

Proof of Corollary 3.4 Suppose $\{P[f]\}^n$ and $[L_q(z,f)]^n$ share $(1,2)$ and $n \ge \dfrac{1}{s(t+1)}\left(2\gamma_p+2t+3\right)$. Then it follows directly from the assumptions of the theorem, that F and G share $(1,2)$. Let Ψ be defined as in (3). We claim $\Psi=0$, on the contrary if $\Psi\ne 0$, then it follows from Lemma 2.6 that

$$\begin{aligned} T(r,F) \le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)+\overline{N_*}(r,\infty;F,G)+\\ S(r,F)+S(r,G). \end{aligned}$$

Since f is a non-constant entire function, we have $\overline{N}(r,f)=S(r,f)$. Hence, the above equation reduces to,

$$n\,(t+1)\,T(r,f)\le 2\frac{(t+1)}{s}T(r,f)+2\gamma_p T(r,f)+S(r,f),$$

Thus,

$$(n\,(t+1)-2\left(\frac{\gamma_p+2t+2}{s}\right)T(r,f)\le S(r,f), \qquad (66)$$

which contradicts with $n \geq \dfrac{1}{s(t+1)}(2\gamma_p + 2t + 3)$. Now, by following the steps of the proof of (I) of Theorem 3.3, we will easily get the proof of Corollary 3.4.

Remarks. If suppose $P[f] = f^{(k)}$, then we get, $(\gamma_p = \gamma_{M_1} = 1)$, $(\Gamma_p = \Gamma_{M_1} = (k+1)), (t = 0, s = 1)$, $(\sigma = \Gamma_{M_1} - \gamma_{M_1} = k)$ and thus,

(i). Theorem 3.1 reduces to Theorem 1.7,

(ii). Corollary 3.2 reduces to Corollary 1.8,

(iii). Theorem 3.3 reduces to Theorem 1.9,

(iv). Corollary 3.4 reduces to Corollary 1.10.

4 Open Problem

1. What happens if we replace $F = f(z + c)$ by $q-$ difference operator $\Delta^u_{q,c} f(z)$ in Theorem 3.1 and Theorem 3.3, where $\Delta^u_{q,c} f = \Delta^{n-1}_{q,c} (\Delta_{q,c} f(z)) = \sum\limits_{r=0}^{u} (-1)^r \binom{u}{r} f(q^{u-r} z + (u-r)c)$, where $q(\neq 0), c \in \mathbb{C}, u(> 1) \in \mathbb{N}, 0 \leq r \leq u$?
2. What happens to the condition n if we use relaxed sharing which is weaker than weakly weighted sharing?.

References

[1] T. C. Alzahary and H. X. Yi, *Weighted value sharing and a question of I. Lahiri*, Complex Variables, Theory and Application, 49 (2004), 1063–1078.

[2] A. Banerjee, *Uniqueness of meromorphic functions that share two sets*, Southeast Asian Bulletin of Mathematics, 31 (2007), 7–17.

[3] S. S. Bhoosnurmath and A. J. Patil, *On the growth and value distribution of meromorphic functions and their differential polynomials*, Journal of Indian Mathematical Society, 74 (2007), 167–184.

[4] Y.M. Chiang and S.-J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, The Ramanujan Journal, 16 (2008), 105–129.

[5] G. Haldar, *Uniqueness of Meromorphic functions concerning k-th derivatives and difference operators*, arXiv preprint arXiv, 2107 (2021), 12345.

[6] Harina P. Waghamore and Preetham N. Raj, *Uniqueness results on meromorphic functions concerning their shift and differential polynomial*, Serdica Mathematical Journal, 47 (2021), 191–212.

[7] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, (1964).

[8] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Mathematical Journal, 161 (2001), 193–206.

[9] I. Lahiri, *Value distribution of certain differential polynomials*, International Journal Mathematics and Mathematical Sciences, 28 (2001), 83–91.

[10] I. Lahiri and A. Sarkar, *Uniqueness of a meromorphic function and its derivative*, Journal of Inequalities Pure and Applied Mathematics, 5 (2004), 9.

[11] C. Meng and G. Liu, *On unicity of meromorphic functions concerning the shifts and derivatives*, Journal of Mathematical Inequalities, 14 (2020), 1095–1112.

[12] X. Qi, N. Li and L. Yang, *Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations*, Computational Methods and Function Theory, 18 (2018), 567–582.

[13] S. Renuka Dyavanal and N. Deepa Angadi, *Uniqueness concerning derivatives of a meromorphic function and its difference polynomial*, Journal of the Indian Mathematical Society, 19 (2024), 229–236.

[14] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, In Complex Analysis: Proceedings of the Conference held at the University of Kentucky, (2006), 101–103.

[15] C. C. Yang, *On deficiencies of differential polynomials,* Math. Z., 125 (1972), 107–112.

[16] C.C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Springer Science and Business Media, Vol. 557 (2004).

[17] L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin, 1993.

[18] J. Zhang, *Meromorphic functions sharing a small function with their differential polynomials*, Kyungpook Mathematical Journal, 50 (2010), 345–355.

[19] J. Zhang and R. Korhonen, *On the Nevanlinna characteristic of* $f(qz)$ *and its applications*, Journal of Mathematical Analysis Applications, 369 (2010), 537–544.