Int. J. Open Problems Compt. Math., Vol. 18, No. 3, September 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376

A Study on Comb Graph Product in Equitable Coloring

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Abstract

In this paper, we study the equitable chromatic number of comb product graphs formed by combining standard graph classes. Specifically, we determine the comb product of path with path, path with complete, complete with path, complete with complete, cycle with cycle, cycle with path, path with cycle.

Keywords: *Equitable Coloring; Comb Product; Path graph; Cycle Graph; Complete Graph.*

2010 Mathematics Subject Classification: 03F55, 46S40.

1 Introduction

Graph coloring is a fundamental area in graph theory with applications in scheduling, load balancing, and resource allocation. Among its variations, *equitable coloring* is especially significant when fairness or balanced distribution is required. An equitable coloring of a graph M is a proper vertex coloring in which the sizes of any two color classes differ by at most one. Formally, it is a partition of the vertex set $V(M)$ into k independent sets $V_1, V_2, \ldots, V_k$ such that $||V_i| - |V_j|| \leq 1$ for all i, j. The smallest integer k is called the *equitable chromatic number*, denoted $\chi_=(M)$. This concept was introduced by Meyer [8], who proved that for any connected graph M other than a complete graph or an odd cycle, we have $\chi_=(M) \leq \Delta(M)$, where $\Delta(M)$ is the maximum degree. A more general and foundational result is the Hajnal–Szemerédi Theorem [4], which states that every graph M with maximum degree Δ admits an equitable coloring with at most $\Delta + 1$ colors, i.e., $\chi_=(M) \leq \Delta + 1$. Several refinements and extensions have been established since then. For instance, Chen, Lih, and Wu [2] proved that every tree is equitably 2-colorable. Kierstead and Kostochka [7] extended the Hajnal–Szemerédi result to list colorings. Yap [11] showed that for planar graphs with $\Delta \geq 8$, the equitable chromatic number satisfies $\chi_{=}(M) \leq \Delta$. Hanna Furma´nczyk [5] discussed vairous graph products. The comb product or rooted product, originally introduced by Godsil and McKay [3], involves attaching a rooted graph H to every vertex of a base graph M, and serves as a fertile structure for studying equitable coloring and related graph invariants.

2 Preliminaries

A **Comb Product** [3], [1] also called as **Rooted Product** with a graph $T^{'}$ and a rooted graph $E^{'}$ denoted by $T^{'} \circ E^{'}$ is a graph formed by taking $\left|V(T^{'})\right|$ copies of $E^{'}$ and grafting the m-th copy of $E^{'}$ at the vertex i to the m-th vertex of $T^{'}$.

A **Path** [10], [9] is a finite sequence of vertices and edges that have distinct end vertices.

A **Cycle** [10], [9] is a finite sequence of vertices and edges that have common end vertices.

A **Complete Graph** [10], [6] is a graph in which each vertex is adjacent to all other vertices.

Proposition 2.1: For any graph M, $\chi_{=}(M) \geq \chi(M)$ [8]

3 Main results

Theorem 3.1. *The equitable coloring of comb product of path P_q with path graph $P_{q'}$ is given by,*

$$\chi_{=}[P_q \circ P_{q'}] = 2 \ ; \quad \forall q, q' \geq 2$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[P_q \circ P_{q'}] = \{b_d, b_{dd'}; 1 \leq d \leq q, 1 \leq d' \leq q'\}$$

where b_d be the nodes of Path Graph (P_q) and $b_{dd'}$ be the nodes of Path Graph $(P_{q'})$. Explicitly, by definition b_d and b_{d1} have same color since both are interlinked. We consider b_{d1}.

For $1 \leq d \leq q$, $1 \leq d' \leq q'$. In the vertex set $b_{dd'}$, if $q \equiv 1 \bmod 2$, we rerun the color sequence (1, 2). Likewise, if $q \equiv 0(\text{mod}2)$, we rerun the color sequence (2, 1) to the vertices $b_{dd'}$ respectively. An illustration is given in Figure 1.

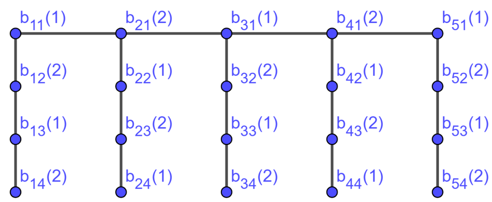

Figure 1: An example of $P_5 \circ P_4$

Hence, if q is even, then each color cropped q times. If q is odd and q' is even, then the colors cropped $\left\lceil \frac{qq'}{2} \right\rceil$ each. And if q is odd and q' is odd, then the color 1 cropped $\left\lfloor \frac{qq'}{2} \right\rfloor + 1$ and color 2 cropped $\left\lfloor \frac{qq'}{2} \right\rfloor$. And the absolute difference between any two color class is either 0 or 1.

Here by, the above coloring strategy gives the upper bound

$$\chi_{=}(P_q \circ P_{q'}) \leq 2, \quad \forall q, q' \geq 2 \tag{1}$$

By Proposition 2.1, we have

$$\chi_{=}(P_q \circ P_{q'}) \geq \chi(P_q \circ P_{q'}) = 2, \quad \forall q, q' \geq 2$$

Therefore the lower bound is given by

$$\chi_{=}(P_q \circ P_{q'}) \geq 2, \quad \forall q, q' \geq 2 \tag{2}$$

From (1)&(2), we get

$$\chi_{=}(P_q \circ P_{q'}) = 2, \quad \forall q, q' \geq 2$$

□

Theorem 3.2. *The equitable coloring of comb product of path P_q with complete graph K_t is given by,*

$$\chi_{=}[P_q \circ K_t] = t \ ; \quad \forall q \geq 2, t \geq 4$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[P_q \circ K_t] = \{b_d, h_{nn'}; 1 \leq d, n \leq q, 1 \leq n' \leq t\}$$

where b_d be the nodes of Path Graph (P_q) and $h_{nn'}$ be the nodes of Complete Graph (K_t).

Explicitly, by definition b_d and h_{n1} have same color since both are interlinked. We consider h_{n1}.

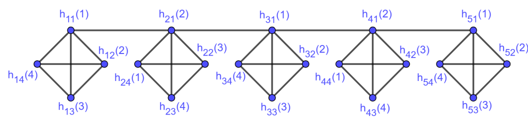

Figure 2: An example of $P_5 \circ K_4$

For $1 \leq n \leq q$, $1 \leq n' \leq t$. In the vertex set $h_{nn'}$, if $n \equiv 1(\text{mod}2)$, we rerun the color sequence (1, 2,..., t) to the vertices. Likewise, if $n \equiv 0(\text{mod}\,2)$, we rerun the color sequence (2, 3, 4,...,t, 1) to the vertices $h_{nn'}$ respectively. An illustration is given in Figure 3.

Hence, each color cropped q times. And the absolute difference between any two color class is either 0 or 1.
Here by, the above coloring strategy gives the upper bound

$$\chi_{=}(P_q \circ K_t) \leq t, \quad \forall q \geq 2, t \geq 4 \tag{3}$$

By Proposition 2.1, we have

$$\chi_{=}(P_q \circ K_t) \geq \chi(P_q \circ K_t) = t, \quad \forall q \geq 2, t \geq 4$$

Therefore, the lower bound is given by

$$\chi_{=}(P_q \circ K_t) \geq t, \quad \forall q \geq 2, t \geq 4 \tag{4}$$

From (3)&(4), we get

$$\chi_{=}(P_q \circ K_t) = t, \quad \forall q \geq 2, t \geq 4$$

□

Theorem 3.3. *The equitable coloring of comb product of complete K_t with path graph P_q is given by,*

$$\chi_{=}[K_t \circ P_q] = t \ ; \quad \forall q \geq 2, t \geq 3$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[K_t \circ P_q] = \{h_n, b_{dd'}; 1 \leq n, d \leq t, 1 \leq d' \leq q\}$$

where h_n be the nodes of Complete Graph (K_t) and $b_{dd'}$ be the nodes of Path Graph (P_q).

Explicitly, by definition h_n and b_{d1} have same color since both are interlinked. We consider b_{d1}.

For $1 \leq d \leq t, 1 \leq d' \leq q$. If $d' \equiv 1(\bmod 2)$ we assign the color sequence (1, 2,..., t) to the vertices $b_{dd'}$; if $d' \equiv 0(\bmod 2)$ we assign the color sequence (2, 3, 4,..., t, 1) to $b_{dd'}$. Each color is cropped q times. And the absolute difference

between any two color class is either 0 or 1.
Here by, the above coloring strategy gives the upper bound

$$\chi_=(K_t \circ P_q) \leq t, \quad \forall q \geq 2, t \geq 3 \tag{5}$$

By Proposition 2.1, we have

$$\chi_=(K_t \circ P_q) \geq \chi(K_t \circ P_q) = t, \quad \forall q \geq 2, t \geq 3$$

Therefore, the lower bound is given by

$$\chi_=(K_t \circ P_q) \geq t, \quad \forall q \geq 2, t \geq 3 \tag{6}$$

From (5)&(6), we get

$$\chi_=(K_t \circ P_q) = t, \quad \forall q \geq 2, t \geq 3$$

□

Theorem 3.4. *The equitable coloring of comb product of complete K_t with complete graph $K_{t'}$ is given by,*

$$\chi_=[K_t \circ K_{t'}] = \begin{cases} t & \text{if } t' < t, \\ t' & \text{otherwise} \end{cases} \quad \text{where } t, t' \geq 3.$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[K_t \circ K_{t'}] = \{h_n, h_{nn'}; 1 \leq n \leq t, 1 \leq n' \leq t'\}$$

where h_n be the nodes of Complete Graph (K_t) and $h_{nn'}$ be the nodes of Complete Graph $(K_{t'})$.

Explicitly, by definition h_n and h_{n1} have same color since both are interlinked. We consider h_{n1}.

Case 1: If $t' < t$

For $1 \leq n \leq t$. We assign the color sequence $(1, 2, \ldots, t)$ to h_{n1}; to h_{n2}, we assign the sequence $(2, 3, \ldots, t, 1)$; to h_{n3}, the sequence $(3, 4, \ldots, t, 1, 2)$; and we continue similarly, so that for $h_{nt'}$, the assigned color sequence is $(t - 1, t, 1, \ldots, t - 2)$.

Case 2: If $t' \geq t$

For $1 \leq n \leq t, 1 \leq n' \leq t'$. We assign the color sequence $(1, 2, \ldots, t')$ to $h_{1n'}$; to $h_{2n'}$, we assign the sequence $(2, 3, \ldots, t', 1)$; to $h_{3n'}$, the sequence $(3, 4, \ldots, t', 1, 2)$; and continuing similarly, for $h_{tn'}$, we assign the sequence $(t, t + 1, \ldots, t', 1, 2, \ldots, t - 1)$ if $(t' > t)$ and if (t' = t) we assign the color sequence $t, 1, 2, ..., t - 1$ to vertices $h_{tn'}$.

The absolute difference between any two color class is either 1 or 0. Here by, the above coloring strategy gives the upper bound

$$\chi_{=}(K_t \circ K_{t'}) \leq \begin{cases} t & \text{if } t' < t, \\ t' & \text{otherwise} \end{cases} \quad \text{where } t, t' \geq 3. \tag{7}$$

By Proposition 2.1, we have

$$\chi_{=}(K_t \circ K_{t'}) \geq \chi(K_t \circ K_{t'}) = \begin{cases} t & \text{if } t' < t, \\ t' & \text{otherwise} \end{cases} \quad \text{where } t, t' \geq 3.$$

Therefore, the lower bound is given by

$$\chi_{=}(K_t \circ K_{t'}) \geq \begin{cases} t & \text{if } t' < t, \\ t' & \text{otherwise} \end{cases} \quad \text{where } t, t' \geq 3. \tag{8}$$

From (7)&(8), we get

$$\chi_{=}(K_t \circ K_{t'}) = \begin{cases} t & \text{if } t' < t, \\ t' & \text{otherwise} \end{cases} \quad \text{where } t, t' \geq 3.$$

□

Theorem 3.5. *The equitable coloring of comb product of cycle C_r with cycle graph $C_{r'}$ is given by,*

$$\chi_{=}[C_r \circ C_{r'}] = \begin{cases} \begin{cases} 2 & \textit{if r' is even} \\ 3 & \textit{if r' is odd} \end{cases} & \textit{if r is even} \\ 3 & \textit{if r is odd} \end{cases} \quad \textit{where } r \geq 3, r' \geq 4.$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[C_r \circ C_{r'}] = \{g_m, g_{mm'}; 1 \leq m \leq r, 1 \leq m' \leq r'\}$$

where g_m be the nodes of Cycle Graph (C_r) and $g_{mm'}$ be the nodes of Cycle Graph ($C_{r'}$).

Explicitly, by definition g_m and g_{m1} have same color since both are interlinked. We consider g_{m1}.

Case 1: If r is odd

For $1 \leq m \leq r, 1 \leq m' \leq r' - 1$. For $m \equiv 1(\text{mod}3)$ and if $1 \leq m \leq r - 1$, we rerun the color sequence (1,2,3) to $g_{mm'}$. And if $(m = r) \wedge (m \equiv 1(\text{mod}3))$ then we give the color sequence (2,3,1) to $g_{mm'}$. For $m \equiv 2(\text{mod}3)$, we assign the color sequence (2,3,1) to $g_{mm'}$. And for $m \equiv 0(\text{mod}3)$, we rerun the color (3,1,2) to $g_{mm'}$.

For $m' = r' \& 1 \leq m \leq r$. For $m' \equiv 1,2(\text{mod}3)$, then the vertex $g_{mr'}$ will have the same color as g_{m2}. For $m' \equiv 0(\text{mod}3)$, then the vertex $g_{mr'}$ will have the same color as g_{m3}.

Case 2: If r is even

Sub case 1: If $r' \equiv 0(\text{mod}2)$

For $1 \leq m \leq r, 1 \leq m' \leq r'$

For $m \equiv 1(\text{mod}2)$, we assign the color sequence (1,2) to $g_{mm'}$ while the other case $m \equiv 0(\text{mod}2)$, we have the color sequence (2,1) to the vertices $g_{mm'}$.

Sub case 2: If $r' \equiv 1(\text{mod}2)$

For $1 \leq m \leq r, 1 \leq m' \leq r'-1$. For $m \equiv 1(\text{mod}3)$ and if $1 \leq m \leq r-1$, we rerun the color sequence (1,2,3) to $g_{mm'}$. And if $(m = r) \wedge (m \equiv 1(\text{mod}3))$ then we give the color sequence (2,3,1) to $g_{mm'}$. For $m \equiv 2(\text{mod}3)$, we assign the color sequence (2,3,1) to $g_{mm'}$. And for $m \equiv 0(\text{mod}3)$, we rerun the color (3,1,2) to $g_{mm'}$.

For $m' = r' \& 1 \leq m \leq r$. For $m' \equiv 1,2(\text{mod}3)$, then the vertex $g_{mr'}$ will have the same color as g_{m2}. For $m' \equiv 0(\text{mod}3)$, then the vertex $g_{mr'}$ will have the same color as g_{m3}.

The absolute difference between any two color class is either 1 or 0. Here by, the above coloring strategy gives the upper bound

$$\chi_{=}(C_r \circ C_{r'}) \leq \begin{cases} \begin{cases} 2 & \text{if r' is even} \\ 3 & \text{if r' is odd} \end{cases} & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad \text{where } r \geq 3, r' \geq 4. \quad (9)$$

By Proposition 2.1, we have

$$\chi_{=}(C_r \circ C_{r'}) \geq \chi(C_r \circ C_{r'}) = \begin{cases} \begin{cases} 2 & \text{if r' is even} \\ 3 & \text{if r' is odd} \end{cases} & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad \text{where } r \geq 3, r' \geq 4.$$

Therefore, the lower bound is given by

$$\chi_{=}(C_r \circ C_{r'}) \geq \begin{cases} \begin{cases} 2 & \text{if r' is even} \\ 3 & \text{if r' is odd} \end{cases} & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad \text{where } r \geq 3, r' \geq 4. \quad (10)$$

From (9)&(10), we get

$$\chi_{=}(C_r \circ C_{r'}) = \begin{cases} \begin{cases} 2 & \text{if r' is even} \\ 3 & \text{if r' is odd} \end{cases} & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad \text{where } r \geq 3, r' \geq 4.$$

□

Theorem 3.6. *The equitable coloring of comb product of cycle C_r with path graph P_q is given by,*

$$\chi_{=}[C_r \circ P_q] = \begin{cases} 2 & \text{if } r \text{ is even} \\ 3 & \text{if } r \text{ is odd} \end{cases} \quad where, r \geq 3, q \geq 2$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[C_r \circ P_q] = \{g_m, b_{dd'}; 1 \leq m, d \leq r, 1 \leq d' \leq q\}$$

where g_m be the nodes of Cycle Graph (C_r) and $b_{dd'}$ be the nodes of Path Graph P_q.

Explicitly, by definition g_m and b_{d1} have same color since both are interlinked. We consider b_{d1}.

Case 1: If r is odd

Sub case 1: If $q < 4$

For $1 \leq d \leq r-1, 1 \leq d' \leq 3$. We assign the color sequence (1,2,3) for the vertices b_{d1}; for b_{d2} we assign (2,3,1); for b_{d3} we assign the color sequence (3,1,2) respectively.

For $d = r, 1 \leq d' \leq 3$. If $d \equiv 1, 2 \bmod 3$ we assign the color sequence (2,3,1) to the vertices $b_{rd'}$. And if $d \equiv 0 \bmod 3$, we assign the color sequence (3,1,2) to $b_{rd'}$. An illustration is given in Figure 3.

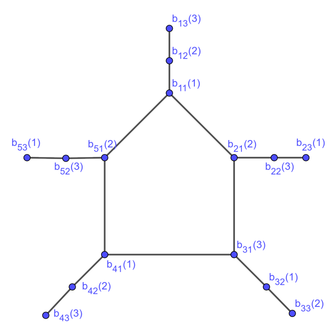

Figure 3: An example of $C_5 \circ P_3$

Sub case 2: If $q \geq 5$ For $1 \leq d \leq r, 1 \leq d' \leq q-1$. For $d \equiv 1(\text{mod}3)$ and $1 \leq d \leq r-1$, we rerun the color sequence (1,2,3) to $b_{dd'}$. And if $(d = r) \wedge (d \equiv 1(\text{mod}3))$ then we give the color sequence (2,3,1) to $b_{dd'}$ respectively. For $d \equiv 2(\text{mod}3)$, we assign the color sequence (2,3,1) to $b_{dd'}$. And for $d \equiv 0(\text{mod}3)$, we rerun the color (3,1,2) to $b_{dd'}$.

For $1 \leq d \leq r, d' = q$. For $d' \equiv 1, 2(\text{mod}3)$, then the vertex $b_{dd'}$ will have the same color as in b_{d2}. And if $d' \equiv 0(\text{mod}3)$, then the vertex $b_{dd'}$ will have the same color as in b_{d3}.

Case 2: If r is even

Sub case 1: If $q \equiv 0(\text{mod}2)$

For $1 \le d \le r \& 1 \le d' \le q$. For $d \equiv 1(\text{mod}2)$, we assign the color sequence (1,2) to $b_{dd'}$ while the other case $d \equiv 0(\text{mod}2)$, we have the color sequence (2,1) to the vertices $b_{dd'}$.

Sub case 2: If $q \equiv 1(\text{mod}2)$

For $1 \le d \le r, 1 \le d' \le q-1$. If $d \equiv 1(\text{mod}3)$ and $1 \le d \le r-1$, we rerun the color sequence (1,2,3) to $b_{dd'}$. If $(d = r) \wedge (d \equiv 1(\bmod 3))$, the vertex $b_{dd'}$ will have the color sequence (2,3,1). If $d \equiv 2(\text{mod}3)$, we rerun the color sequence (2,3,1) to $b_{dd'}$. If $d \equiv 0(\text{mod}3)$, we rerun the color sequence (3,2,1) to $b_{dd'}$.

For $d' = q \& 1 \le d \le r$

If $d' \equiv 1, 2(\text{mod}3)$, then the vertex $b_{dd'}$ will have the same color as vertex b_{d2}. If $d' \equiv 0(\text{mod}3)$, then the vertex $b_{dd'}$ will have the same color as vertex b_{d3}.

The absolute difference between any two color class is either 1 or 0. Here by, the above coloring strategy gives the upper bound

$$\chi_{=}(C_r \circ P_q) \le \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \ge 3, q \ge 2 \tag{11}$$

By Proposition 2.1, we have

$$\chi_{=}(C_r \circ P_q) \ge \chi(C_r \circ P_q) = \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \ge 3, q \ge 2$$

Therefore, the lower bound is given by

$$\chi_{=}(C_r \circ P_q) \ge \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \ge 3, q \ge 2 \tag{12}$$

From (11)&(12), we get

$$\chi_{=}(C_r \circ P_q) = \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases}$$

□

Theorem 3.7. *The equitable coloring of comb product of path P_q with cycle graph C_r is given by,*

$$\chi_{=}[P_q \circ C_r] = \begin{cases} 2 & \textit{if r is even} \\ 3 & \textit{if r is odd} \end{cases} \quad where, r \ge 4, q \ge 2$$

Proof. From the definition of comb product, the vertex is defined as follows

$$V[P_q \circ C_r] = \{b_d, g_{mm'}; 1 \leq m, d \leq q, 1 \leq m' \leq r\}$$

where b_d be the nodes of Path Graph (P_q) and $g_{mm'}$ be the nodes of Cycle Graph (C_r).

Explicitly, by definition b_d and g_{m1} have same color since both are inter-linked. We consider g_{m1}.

Case 1: If q is odd

For $1 \leq m \leq q, 1 \leq m' \leq r-1$. For $m \equiv 1(\text{mod}3)$ and if $1 \leq m \leq q-1$, we rerun the color sequence (1,2,3) to $g_{mm'}$. And if $(m = q) \wedge m \equiv 1(\text{mod}3)$ then we give the color sequence (2,3,1) to $g_{mm'}$. For $m \equiv 2(\text{mod}3)$, we assign the color sequence (2,3,1) to $g_{mm'}$. And for $m \equiv 0(\text{mod}3)$, we rerun the color (3,1,2) to $g_{mm'}$.

For $m' = r\&1 \leq m \leq q$. For $m' \equiv 1, 2(\text{mod}3)$, then the vertex g_{mr} will have the same color as g_{m2}. For $m' \equiv 0(\text{mod}3)$, then the vertex g_{mr} will have the same color as g_{m3}.

Case 2: If q is even

Sub case 1: If $r \equiv 0(\text{mod}2)$

For $1 \leq m \leq q, 1 \leq m' \leq r$. For $m \equiv 1(\text{mod}2)$, we assign the color sequence (1,2) to $g_{mm'}$ while the other case $m \equiv 0(\text{mod}2)$, we have the color sequence (2,1) to the vertices $g_{mm'}$.

Sub case 2: If $r \equiv 1(\text{mod}2)$

For $1 \leq m \leq q, 1 \leq m' \leq r-1$. For $m \equiv 1(\text{mod}3)$ and if $1 \leq m \leq q-1$, we rerun the color sequence (1,2,3) to $g_{mm'}$. And if $m = q \wedge m \equiv 1(\text{mod}3)$ then we give the color sequence (2,3,1) to $g_{mm'}$. For $m \equiv 2(\text{mod}3)$, we assign the color sequence (2,3,1) to $g_{mm'}$. And for $m \equiv 0(\text{mod}3)$, we rerun the color (3,1,2) to $g_{mm'}$.

For $m' = r\ \&1 \leq m \leq q$. For $m' \equiv 1, 2(\text{mod}3)$, then the vertex $g_{mm'}$ will have the same color as g_{m2}. For $m' \equiv 0(\bmod 3)$, then the vertex $g_{mm'}$ will have the same color as g_{m3}.

The absolute difference between any two color class is either 1 or 0. Here by, the above coloring strategy gives the upper bound

$$\chi_=(P_q \circ C_r) \leq \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \geq 4, q \geq 2 \tag{13}$$

By Proposition 2.1, we have

$$\chi_=(P_q \circ C_r) \geq \chi(P_q \circ C_r) = \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \geq 4, q \geq 2$$

Therefore, the lower bound is given by

$$\chi_=(P_q \circ C_r) \geq \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \geq 4, q \geq 2 \tag{14}$$

From (13)&(14), we get

$$\chi_{=}(P_q \circ C_r) = \begin{cases} 2 & \text{if r is even} \\ 3 & \text{if r is odd} \end{cases} \quad where, r \geq 4, q \geq 2$$

□

4 Observation

In general, comb graph product of any two graphs are not commutative. But applying a function like equitable coloring to comb graph product, we procured the results are commutative in nature.

5 Open Problem

This work can be extended to more complex graphs such as wheel, helm, and closed helm graphs. Due to their unique structural characteristics, analyzing the equitable coloring of their comb products poses several open problems. In particular, determining tight bounds or exact values of the equitable chromatic number for these graph products remains an open and challenging area for further research.

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