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Asymptoticity of Measure Theoretic Operators and Persistent Homology in Artificial Intelligence

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Abstract

Studies in algebraic structures and their applications to computing remain interesting to date. In this work, we study asymptoticity of measure theoretic operators in norm-attainable (NA) algebras. We show that these operators are asymptotically stable in a measure theoretic sense in maximal two-sided proper rings which are subalgebras of NA algebras and that they have unique aspects that are useful in artificial intelligence.

Keywords: *Asymptoticity, Operator, Persistent homology, Measure, AI*

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1 Introduction

Studies regarding theoretical maps [13] in the sense of measure have been done over decades. These maps can be found in a general setting of Banach algebras (BA). Of great concern has been the asymptotic behaviour (ASB) of these maps in measure theoretic sense [1]. Norm-attainable maps can form a BA with a unit or without. Its of interest to study the ASB of these maps when they occur in BAs with persistent homology [2]. Certain characterizations have been done for BA operators in many aspects. The properties that have been considered include: norms, numerical ranges, orthogonality, spectrum, invertibility, nilpotency, idempotency among others [6]. These properties have been identified to be instrumental in applications in various sub-branches of computing like software engineering, hardware engineering, malware studies among others [3]. In [5] the work described how essential the algebraic structures influence the occurrences of persistence homologies due to their intricate natures [4]. This was further advanced by [8] which described the asymptotic behaviors in a topological sense. This occurs in Topological Data Analysis (TDA) where techniques of simplicial complex (SC) is utilized in computational approach in Persistent homology (PH). This leads to applications in Machine learning (ML) and artificial intelligence (AI) which is the trend as of today because a lot of research activities are in these two areas of computing [9]. For the measure theoretic aspects [13], consider $\chi(x)$ as integrable NA operator. Its Laplace transform (LT) can be given as $\chi(\lambda) = \int_0^\infty e^{\lambda x}\chi(x)\,dx$, $Re\lambda = 0$. In this regard, $\Lambda(\chi)(x)$ is analytic and is defined in the restricted region $\{\lambda : Re\lambda = 0\}$, where $\Lambda(0) = 0$. We have that $\Lambda(\chi(\lambda))$ is the LT of $\Lambda(\chi)(x)$ which is an integrable NA operator and $\Lambda(\chi(\lambda)) = \int_0^\infty e^{\lambda x}\Lambda(\chi)(x)\,dx$, where $Re\lambda = 0$. Now, suppose the ASB of $\chi(x)$ as $x \to \infty$ is known then it is interesting to determine the ASB of $\Lambda(\chi)(x)$ as $x \to \infty$. It has been established in [9] that for algebras with persistent homology on $\chi(x)$, the ASB of $\chi(x)$ and $\Lambda(\chi)(x)$ as $x \to \infty$ are similar at a given constant.

This work is very useful in terms of applications in computer science, its subbranches and related areas. Understanding the underlying algebraic structures is useful in developing efficient algorithms which can be used in controlling malwares that enhances cyber-security [11]. This work is also useful in quantum mechanics whereby high speed computers are being developed to replace the standard ones and this study combined wit AI and ML techniques forms a milestone in the development in the computing world [12].

2 Preliminaries

The terminologies given below form the preliminary notes which are useful for this work.

Definition 2.1 *([1]) An operator* Π *in a BA* Ω *is said to be norm-attainable if there exists a unit vector* $\zeta \in \Omega$ *such that* $\|\Pi\zeta\| = \|\Pi\|$. *The algebra of all such* Π *is called the algebra of norm-attainable operators denoted by* Ω_{NA}.

Definition 2.2 *([8]) Let* $\varpi : \mathcal{R} \to (0, \infty)$ *be a positive operator whereby* $\varpi(0) = 1, \quad \varpi(x+y) \leq \varpi(x)\varpi(y), \quad \forall x, y \in \mathcal{R}$. *We define* $\gamma_-(\varpi) := \lim_{x\to-\infty} \frac{\ln \varpi(x)}{x}$ *and* $\gamma_+(\varpi) := \lim_{x\to\infty} \frac{\ln \varpi(x)}{x}$.

Definition 2.3 *([13]) Let $\otimes_{\mathcal{NA}_\varphi}$ be a nonunital BA of all complex-valued absolutely continuous NA operators χ on $\mathcal{R}$ such that $\|\chi\|_\varpi = \int_{\mathcal{R}} |\chi(x)|\varpi(x)\,dx < \infty$. We have a bounded, positive, Borel-measurable map $\tau : \mathcal{R} \to (0, \infty)$ satisfying: $\lim_{|x|\to\infty} \tau(x) = 1$ and $\sup_{x\in\mathcal{R}} \frac{\tau(x)}{\tau(x-y)} < \infty$.*

Definition 2.4 *([2]) For $\otimes_{\mathcal{NA}_\varpi}$, a seminorm is $P_\tau(\chi) = \sup_{x\in\mathcal{R}} |\chi(x)|\frac{\varpi(x)}{\tau(x)}$.*

Definition 2.5 *([4]) Let $\otimes_{\mathcal{NA}_\varpi}(\tau)$ be the BA consisting of all maps $\chi \in \otimes_{\mathcal{NA}_\varpi}$ for which $\|\chi\|_{\varpi,\tau} = C\left(\|\chi\|_\varpi + P_\tau(\chi)\right) < \infty$, where $C > 0$ is an absolute constant. Then $\otimes_{\mathcal{NA}_\varpi}(\tau)$ is a convolutive nonunital BA.*

Definition 2.6 *([12]) Artificial Intelligence (AI) refers to the ability of machines to behave like humans for instance openAI like DeepSeek.*

Definition 2.7 *([10]) There is part of AI called ML that uses algorithms and Big Data in working like humans.*

Definition 2.8 *([7]) A simplicial complex Δ is a collection of simplices that has each face $\Delta \subset \Delta_0$ and any two simplices σ_i and σ_j in Δ coincide in a common face.*

3 Main results

We begin this section by giving auxiliary results on AI, ML, TDA which are useful in the sequel.

Proposition 3.1 *Consider a Hilbert space $\mathcal{H}$ and $\varpi : \mathcal{H} \to \mathcal{R}$ as a continuous operator in $\otimes_{\mathcal{NA}_\varpi}(\tau)$. We have that $PD(\varpi)$ contains an almost countability up to a big data set (BDS) of points, where PD denotes persistence diagram.*

Proof. Fix a scalar for ϖ as $G(\varpi)$. Consider $\ell = \frac{\varepsilon}{G(\varpi)}$ for some ε. We have that $\mathcal{L} \subset \mathcal{H}$ coincides with π which ia a family of homologies in $\mathcal{L}$ with optimality is attained for persistence. Now, π is attains almost countability in $\mathcal{L}$ which show that it is almost countable in $\mathcal{H}$ for some $\varepsilon > 0$. Hence, for some BDS in $\otimes_{\mathcal{NA}_\varpi}(\tau)$. We have that $PD(\varpi)$ contains an almost countability up to a big data set (BDS) of points.

We now consider utilizing 3.1 in the next result to characterize stability in$\otimes_{\mathcal{NA}_\varpi}(\tau)$ for PH.

Proposition 3.2 *For every $\mathcal{H}$ with finite degree of PH, consider $\varpi, \pi : \mathcal{H} \to \mathcal{R}$ to be NA then $L_w(\varpi, \pi) \leq J^{\frac{\ell}{t}} \|\varpi - \pi\|_\alpha^{\ell - \frac{n}{t}}$.*

Proof. To proof this, we show that $\rho : PD(\varpi) \to PD(\pi)$ is one-one, onto and has Haursdorff Distance in a T_2-space. Now, consider η in $DP(\varpi)$. From [10] $\|\eta - \rho\|_\infty \leq \|\varphi - \phi\|_\infty$ which is bounded above. The rest follows from [7].

Remark 3.3 *The result in Proposition 3.3 is useful in determining the speed at which the Open AI softwares operate. The efficiency and stability issues are also determined by the distance related sequences of the algorithms in the AI platforms as established in [1].*

Lemma 3.4 *Let $\varpi, \pi : \mathcal{H} \to \mathcal{R}$ be NA. The inequality $\|PS_t(\varpi) - PS_t(\pi)\|_\infty \leq \frac{1}{4t} J\|\varpi - \pi\|_\infty$, where $t \geq n + 1$ holds for asymptotic stability.*

Proof. Consider the optimal classification criterion for x and y being natural numbers and $t \geq 1$. Since the operators are asymptotically stable by Lemma 3.4 then the NA operators satisfy PH criterion. The rest follows from maximum likelihood principle (MLP).

Remark 3.5 *With these auxiliary results we now embark on the main results on measure theoretic sense for the NA operators in the next theorem.*

Theorem 3.6 *Every two-sided proper ring $\Xi \subset \otimes_{\mathcal{NA}_\varpi}(\tau)$ of the form $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_\varpi}(\tau)$, where $\Xi_\lambda \subset \otimes_{\mathcal{NA}_\varphi}$ is convoluting and asymptotically stable.*

Proof. For a proper two-sided ring $\Xi_\lambda \subset \otimes_{\mathcal{NA}_\varpi}$, we have the set $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_\varpi}(\tau)$ which is also a proper two-sided ring which is maximal in $\otimes_{\mathcal{NA}_\varpi}(\tau)$. By closed ring criterion, we have that the ring has converging sequences that converge to an optimal point. Also by Lemma 3.4 and MLP, asymptotic stability is obtained in $\otimes_{\mathcal{NA}}$.

Remark 3.7 *Consider the weighted shift operators which are NA $\tau(x)$ and which satisfy $\lim_{n\to\infty} P_\tau(f_n) = 0$, for all $\chi \in \mathcal{A}_\varpi(\tau)$. It is known from [13] that the equality holds only if $\chi_n(x) = \chi(x)(1 - \chi_{[-n,n)}(x))$. Moreover, that all two-sided proper ring $\Xi \subset \otimes_{\mathcal{NA}_\varpi}(\tau)$ is canonically represented as $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_\varpi}(\tau)$, for which $\Xi_\lambda \subset \otimes_{\mathcal{NA}_\varpi}$ is proper and two-sided. Again, $\Xi_\lambda \subset \otimes_{\mathcal{NA}_\varpi}$ forms the family $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_\varphi}(\tau)$ which is in $\otimes_{\mathcal{NA}_\varpi}(\tau)$ and also proper two-sided.*

Corollary 3.8 *Every NA operator in the subalgebra defined by the criterion $\otimes_{\mathcal{NA}_\varpi}(\tau, \beta) = \left\{\chi \in \otimes_{\mathcal{NA}_\varpi}(\tau) : \lim_{x\to\infty} \frac{\chi(x)\varpi(x)}{\tau(x)} = \chi(\infty, \beta) \text{ exists } \right\}$ is asymptotically stable.*

Proof. Suppose that $\otimes_{\mathcal{NA}_\varpi}(\tau, \beta)$ is a NA subalgebra of $\otimes_{\mathcal{NA}_\varpi}(\tau)$. We have that for every $\chi, \eta \in \otimes_{\mathcal{NA}_\varpi}(\tau, \beta)$, an asymptotic convolution in a measure theoretic sense $(\chi * \eta)(\infty, \beta) = \chi(\infty, \beta) \cdot \hat{\eta}(\gamma_+(\varpi) + \beta) + \eta(\infty, \beta) \cdot \hat{\chi}(\gamma_+(\varpi) + \beta)$, holds for which $\hat{\chi}(\lambda) = \int_{\mathcal{R}} e^{\lambda x}\chi(x)\,dx$. But every proper two-sided ring $\Xi \subset \otimes_{\mathcal{NA}_{\varpi}}(\tau,\beta)$ is canonically represented as $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_{\varpi}}(\tau,\beta)$, for some proper two-sided ring $\Xi_\lambda \subset \otimes_{\mathcal{NA}_{\varpi}}$ which is maximal. Now for any proper two-sided ring $\Xi_\lambda \subset \otimes_{\mathcal{NA}_{\varpi}}$, we have $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}_{\varpi}}(\tau,\beta)$ is a proper two-sided ring in $\otimes_{\mathcal{NA}_{\varphi}}(\tau,\beta)$ which by [5] contains NA operators which are asymptotically stable in the measure theoretic sense.

Another consequence of Theorem 3.6 is the next corollary is a characterization with regard to commutativity of rings.

Corollary 3.9 *Consider commutative rings in $\otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$ and consider also $\otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$ as a NA subalgebra. Then every map in the subalgebra defined as $\otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta) = \left\{\chi \in \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau) : \lim_{x\to\infty} \frac{f(x)\varpi(x)}{\tau(x)} = f(\infty,\beta) \text{ exists } \right\}$ is asymptotically stable.*

Proof. Consider $\tau(x)$ which holds for $\lim_{n\to\infty} P_\tau(f_n) = 0, \quad \forall \chi \in \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau)$. We have that convolution below holds with respect to asymptoticity, that is, $(\chi * \eta)(\infty,\beta) = \chi(\infty,\beta)\cdot\hat{\eta}(\gamma_+(\varpi)+\beta) + \eta(\infty,\beta)\cdot\hat{\chi}(\gamma_+(\varpi)+\beta)$. This is true for every $\chi,\eta \in \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$. Now, the commutative ring $\Xi \subset \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$ has a canonical representation $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$, for all rings $\Xi_\lambda \subset \otimes_{\mathcal{NA}_{\varpi}}$. Next, for any commutative ring $\Xi_\lambda \subset \otimes_{\mathcal{NA}_{\varpi}}$, we have $\Xi = \Xi_\lambda \cap \otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$ being is a commutative ring in $\otimes_{\mathcal{NA}^{\circ}_{\varpi}}(\tau,\beta)$. Hence, every map in the commutative ring is asymptotically stable.

Remark 3.10 *We note that studies in algebraic structures and their applications to computing remain interesting to date as illustrated in many recent studies. In this work, we have studied asymptoticity of measure theoretic operators in norm-attainable algebras. We have shown that these operators have unique aspects that are useful in AI and ML. The characterizations particularly in the commutative rings for the measure theoretic operators are useful in PH and SC that are instrumental in computer science and its applications.*

4 Open Problems

We have shown that these operators are asymptotically stable in a measure theoretic sense in maximal two-sided proper rings which are subalgebras of NA algebras and that they have unique aspects that are useful in artificial intelligence. So, in view of this study, some open questions arise naturally. **Problem 1:** Firstly, it is interesting to develop algorithms which are fast and very efficient that can ensure stability in a NA set up of BA. Can the algorithm work in Lorentz algebras? **Problem 2:** Can the maximality of the measures of these operators be obtained using Lie symmetry analytic approaches?

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