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Existence of solutions to higher order semilinear hyperbolic equations with damping term

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Abstract

In this paper, we study the existence and uniqueness of the initial and boundary value problem for a class of higher order semilinear hyperbolic partial differential equations with damping term. Based on priori estimates of solution we proved the existence of the weak solution in the form of Fourier series under suitable conditions. For this purpose Picard's successive approximation method was used. Furthermore we proved the uniqueness of the weak solution.

Keywords: *Higher Order Partial Differential Equation, Existence, Uniqueness, Picard's Method, Fourier Series Method.*
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1 Introduction

In this paper, we examine the existence of a solution to the following initial and boundary value problem

$$\frac{\partial^2 u}{\partial t^2} + (-1)^k a^2 \frac{\partial^{2k} u}{\partial x^{2k}} + (-1)^m 2\varepsilon \frac{\partial^{2m+1} u}{\partial x^{2m} \partial t} = f(x,t,u),\ (x,t) \in \Omega = \{0 < x < \pi,\ 0 < t < T\}, \tag{1}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0,\ 0 \le x \le \pi, \tag{2}$$

$$\frac{\partial^{2l} u(0,t)}{\partial x^{2l}} = \frac{\partial^{2l} u(\pi,t)}{\partial x^{2l}} = 0,\ l = 0,1,2,...,k-1,\ 0 \le t \le T, \tag{3}$$

where $k \geq 1$ and $m \geq 0$ $(k \geq m)$ are natural numbers and $T > 0$ is a real number, $\varepsilon \geq 0$ is a parameter to be determined later, $f(x,t,u)$ is a given function defined in $\overline{\Omega} \times (-\infty, \infty)$, and $u = u(x,t)$ is a solution to the problem.

It is known that in the case of $f(x,t,u) = F(x,t)$, the problem with the non-homogeneous equation with homogeneous initial or boundary conditions will turn into a problem with homogeneous equations and non-homogeneous initial or boundary conditions, and also if the non-homogeneous equation is given with non-homogeneous initial or boundary conditions, the problem will turn into these two cases. The method of separation of variables is widely used together with the principle of linear combination to solve these problems. This method is also known as the Fourier series method or the eigenfunction expansion method [1].

Baouendi and Grisvard showed that the boundary value problem for the differential equation $x\frac{\partial u}{\partial t} + (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}} = F(x,t)$ has a unique solution [2].

Amanov and Ashyralyev showed the solvability of the initial and boundary value problems and the boundary value problem for the differential equation $\frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^2 u}{\partial t^2} = F(x,t)$ [3]. They established the well-posedness of the problem depends on the evenness and oddness of the number k.

Amanov showed that the initial and boundary value problem for the differential equation $t^m \frac{\partial^{2k} u}{\partial x^{2k}} + (-1)^k \frac{\partial u}{\partial t} = F(x,t)$ has a unique solution [4].

In the references [5] and [6] it is showed that the initial and periodic boundary value problem for the differential equations $\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t,u)$ and $\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial x^2 \partial t} = f(x,t,u)$ have unique solutions respectively.

Yuldasheva showed the unique solvability of the problem with boundary conditions with respect to t and periodic boundary conditions with respect to x for the differential equation $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^{2k} u}{\partial x^{2k}} = f(x,t,u)$ [8].

Tantas and Polat showed that the initial and boundary value problem for the differential equation $\frac{\partial u}{\partial t} + (-1)^k t^m \frac{\partial^{2k} u}{\partial x^{2k}} = f(x,t,u)$ has a unique solution [7].

Since the case of $\dfrac{\partial^{2m+1} u}{\partial x^{2m} \partial t}$ and $f(x,t,u)$ is considered in our current equation, it is clear that it generalizes some of the studies given above. After giving the weak solution in the form of a Fourier series containing the eigenfunctions obtained from the eigenvalue problem related to the current problem, the uniform convergence of the series related to the solution generated by Picard successive approximations is shown. In addition, the uniqueness of the weak solution is proven.

Definition 1.1 *A function $v(x,t) \in C(\overline{\Omega})$ is called a test function if it has continuous partial derivatives involved in equation (1) and it satisfies the boundary conditions in (3) and* $v(x,T) = \dfrac{\partial v(x,T)}{\partial t} = \dfrac{\partial^{2m} v(x,T)}{\partial x^{2m}} = 0.$

Definition 1.2 *The function $u(x,t) \in C(\overline{\Omega})$ that satisfies the following integral equation for an arbitrary test function $v(x,t)$ is called a weak solution of the problem (1)-(3):*

$$\,_{0}^{T}\,_{0}^{\pi}\left[\left(\frac{\partial^2 v}{\partial t^2} + (-1)^k a^2 \frac{\partial^{2k} v}{\partial x^{2k}} - (-1)^m 2\varepsilon \frac{\partial^{2m+1} v}{\partial x^{2m}\partial t}\right) u - f(x,t,u)v\right] dxdt = 0. \quad (4)$$

Using the weak solution in the form of Fourier series, we obtain an infinite number of nonlinear integral equations for the Fourier series coefficients from problems (1)-(3). The space in which the Fourier series coefficients are solutions is defined and the appropriate norm is given.

Definition 1.3 *Let B_T denote the set of continuous functions which are Fourier coefficients*

$$\{\overline{u}(t)\} = \{u_1, u_2, ..., u_n, ...\}$$

in the interval $[0,T]$ that satisfy the condition

$$\sum_{n=1}^{\infty} \max_{0 \le t \le T} |u_n(t)| < \infty.$$

Let the norm in B_T be defined as follows:

$$\|\overline{u}(t)\| = \sum_{n=1}^{\infty} \max_{0 \le t \le T} |u_n(t)|.$$

Clearly, B_T is a Banach space.

2 Solution to the Problem

Let's look for the weak solution of the problem (1)-(3) in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \quad (5)$$

where $u_n(t)$, $(n = \overline{1,\infty})$ is the unknown function. To find it, the following integral equation is obtained under the condition $\Delta = 4\left(\varepsilon^2 n^{4m} - a^2 n^{2k}\right) < 0$ using equation (4):

$$u_n(t) = \frac{2}{\beta\pi}\,_{0}^{t}\,_{0}^{\pi} e^{\alpha(t-\tau)} f\left(\xi, \tau, \sum_{n=1}^{\infty} u_n(\tau) \sin n\xi\right) \sin\beta\,(t-\tau) \sin n\xi d\xi d\tau, \quad (6)$$

where $\alpha = -\varepsilon n^{2m}$ and $\beta = \sqrt{a^2 n^{2k} - \varepsilon^2 n^{4m}}$.

Theorem 2.1 *Under the following conditions, equation (6) admits a unique solution in* B_T*:*

1) $\varepsilon \leq \frac{a}{\sqrt{2}} n^{k-2m}$,
2) $f(x,t,u)$ is continuous with respect to all variables in $\overline{\Omega} \times R$,
3) $|f(x,t,u) - f(x,t,v)| \leq b(x,t)\,|u-v|$, $b(x,t) > 0$ ve $b(x,t) \in L_2(\Omega)$,
4) $f(x,t,0) \in L_2(\Omega)$.

Lemma 2.2 *Under the conditions of Theorem 1, equation (6) has at least one solution in* B_T*.*

Proof. If we apply the method of successive approximations, for equation (6) where $N = \overline{1,\infty}$, we get the following sequence

$$u_n^{(N+1)}(t) = \frac{2}{\beta\pi} \int_0^t \int_0^\pi e^{\alpha(t-\tau)} f\left(\xi, \tau, \sum_{n=1}^{\infty} u_n^{(N)}(\tau)\sin n\xi\right) \sin\beta\,(t-\tau)\sin n\xi\, d\xi d\tau. \tag{7}$$

For simplicity let $Au^{(N)}(\xi,\tau) = \sum\limits_{n=1}^{\infty} u_n^{(N)}(\tau)\sin n\xi$ and $\left\{\overline{u}^{(N)}(t)\right\} = \left\{u_1^{(N)}(t), u_2^{(N)}(t), ..., u_n^{(N)}(t), ...\right\}$. Clearly we have

$$\max_{0\leq\tau\leq T}\left|Au^{(N)}(\xi,\tau)\right| \leq \sum_{n=1}^{\infty}\max_{0\leq\tau\leq T}\left|u_n^{(N)}(\tau)\right| = \left\|\overline{u}^{(N)}(\tau)\right\|_{B_T}. \tag{8}$$

Now we want to show that $\overline{u}^{(N)}(t) \in B_T$ for all N, i.e. $\sum\limits_{n=1}^{\infty}\max\limits_{0\leq t\leq T}\left|u_n^{(N)}(t)\right| < \infty$.

According to the conditions in Theorem 1, it is clear that

$$\left\|\overline{u}^{(0)}(t)\right\| = \sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(0)}(t)\right| = 0 < \infty.$$

For $N = 0$ in (7), we have

$$u_n^{(1)}(t) = \frac{2}{\beta\pi}\int_0^t\int_0^\pi e^{\alpha(t-\tau)} f\left(\xi,\tau,Au^{(0)}(\xi,\tau)\right)\sin\beta\,(t-\tau)\sin n\xi\, d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(1)}(t)\right| \leq \frac{2}{\beta\pi}\left(\int_0^t e^{2\alpha(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left[\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{-2\alpha\beta^2}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(1)}(t)\right| \leq \left(\sum_{n=1}^{\infty}\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\int_0^t\left[\frac{2}{\pi}\int_0^{\pi} f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

is obtained. By applying Bessel's inequality we get

$$\sum_{n=1}^{\infty}\left|u_n^{(1)}(t)\right| \leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t\frac{2}{\pi}\int_0^{\pi} f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}.$$

Thus, $\left\|\overline{u}^{(1)}(t)\right\|_{B_T} = \sum\limits_{n=1}^{\infty}\max\limits_{0\leq t\leq T}\left|u_n^{(1)}(t)\right| \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|f(\xi,\tau,0)\right\|_{L_2(\Omega)}$
$= \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} M_1 < \infty.$

For $N = 1$ in (7), we have

$$u_n^{(2)}(t) = \frac{2}{\beta\pi}\int_0^t\int_0^{\pi} e^{\alpha(t-\tau)} f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)\sin\beta(t-\tau)\sin n\xi d\xi d\tau.$$

$$= \frac{2}{\beta\pi}\int_0^t e^{\alpha(t-\tau)}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right) - f(\xi,\tau,0) + f(\xi,\tau,0)\right)\sin\beta(t-\tau)\sin n\xi d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(2)}(t)\right| \leq \frac{2}{\beta\pi}\left(\int_0^t e^{2\alpha(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left[\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right) - f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$+ \frac{2}{\beta\pi}\left(\int_0^t e^{2\alpha(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left[\int_0^{\pi} f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right) - f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$+ \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^{\pi} f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(2)}(t)\right| \leq \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right) - f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$+ \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|u_n^{(2)}(t)\right| &\leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)-f(\xi,\tau,0)\right)^2 d\xi d\tau\right)^{\frac{1}{2}} \\
&\quad+\frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t\frac{2}{\pi}\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}} \\
&\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|Au^{(1)}(\xi,\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}} \\
&\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, $\left\|\overline{u}^{(2)}(t)\right\|_{B_T}=\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(2)}(t)\right|\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}\|b(\xi,\tau)\|_{L_2(\Omega)}$
$+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\|f(\xi,\tau,0)\|_{L_2(\Omega)}\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}M_2+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}M_1<\infty.$

For $N=2$ in (7), we have

$$\begin{aligned}
u_n^{(3)}(t)&=\frac{2}{\beta\pi}\int_0^t\int_0^\pi e^{\alpha(t-\tau)}f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)\sin\beta(t-\tau)\sin n\xi d\xi d\tau. \\
&=\frac{2}{\beta\pi}\int_0^t e^{\alpha(t-\tau)}\int_0^\pi\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f(\xi,\tau,0)+f(\xi,\tau,0)\right)\sin\beta(t-\tau)\sin n\xi d\xi d\tau.
\end{aligned}$$

If Cauchy's inequality is applied with respect to τ, we get

$$\begin{aligned}
\left|u_n^{(3)}(t)\right|&\leq\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}} \\
&\quad+\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

By summing over n and applying Hölder's inequality

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|u_n^{(3)}(t)\right|&\leq\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}} \\
&\quad+\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\sum_{n=1}^{\infty}\left|u_n^{(3)}(t)\right| \leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au^{(2)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)\left|Au^{(2)}\left(\xi,\tau\right)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(2)}(t)\right\|_{B_T}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}.$$

Thus, $\left\|\overline{u}^{(3)}(t)\right\|_{B_T} = \sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(3)}(t)\right| \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(2)}(t)\right\|_{B_T}\left\|b(\xi,\tau)\right\|_{L_2(\Omega)} + \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|f(\xi,\tau,0)\right\|_{L_2(\Omega)}$

$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(2)}(t)\right\|_{B_T} M_2 + \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} M_1 < \infty.$

Let's show its truth for each N by induction:

For $N=k-1$ in (7), $\left\|\overline{u}^{(k)}(t)\right\|_{B_T} = \sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(k)}(t)\right| \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(k-1)}(t)\right\|_{B_T} M_2\left(\tau\right)$ $+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} M_1\left(\tau\right) < \infty$ be correct.

For $N=k$ in (7), we have

$$u_n^{(k+1)}(t) = \frac{2}{\beta\pi}\int_0^t\int_0^\pi e^{\alpha(t-\tau)}f\left(\xi,\tau,Au^{(k)}\left(\xi,\tau\right)\right)\sin\beta\left(t-\tau\right)\sin n\xi d\xi d\tau.$$

$$=\frac{2}{\beta\pi}\int_0^t e^{\alpha(t-\tau)}\int_0^\pi\left(f\left(\xi,\tau,Au^{(k)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)+f\left(\xi,\tau,0\right)\right)\sin\beta\left(t-\tau\right)\sin n\xi d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(k+1)}(t)\right| \leq \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(k)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$+\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(k+1)}(t)\right| \leq \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(k)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\sum_{n=1}^{\infty}\left|u_n^{(k+1)}(t)\right| \leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au^{(k)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)\left|Au^{(k)}\left(\xi,\tau\right)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(k)}(t)\right\|_{B_T}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)d\xi d\tau\right)^{\frac{1}{2}}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)d\xi d\tau\right)^{\frac{1}{2}}.$$

Thus, $\left\|\overline{u}^{(k+1)}(t)\right\|_{B_T}=\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(k+1)}(t)\right| \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(k)}(t)\right\|_{B_T}\left\|b(\xi,\tau)\right\|_{L_2(\Omega)}$ $+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|f(\xi,\tau,0)\right\|_{L_2(\Omega)} \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(k)}(t)\right\|_{B_T}M_2+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}M_1 < \infty$. Then $\overline{u}^{(N)}(t)\in B_T$.

Now, let us show that the sequence $\left\{\overline{u}^{(N)}(t)\right\}$ is uniformly convergent in B_T as $N\longrightarrow\infty$. For this, it is sufficient to show that the series

$$\overline{u}^{(0)}(t)+\sum_{N=0}^{\infty}\left(\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right)$$

is uniformly convergent. First, we want to obtain estimates for the differences $\left|\overline{u}_n^{(N+1)}(t)-\overline{u}_n^{(N)}(t)\right|$.

It is clear that

$$\left\|\overline{u}^{(1)}(t)-\overline{u}^{(0)}(t)\right\|_{B_T}=\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(1)}(t)-u_n^{(0)}(t)\right|=\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(1)}(t)\right|\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}M_1=A_T<\infty.$$

We have

$$\left|u_n^{(2)}(t)-u_n^{(1)}(t)\right|\leq\frac{2}{\beta\pi}\int_0^t e^{\alpha(t-\tau)}\int_0^\pi\left|f\left(\xi,\tau,Au^{(1)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right|\left|\sin n\xi\right|d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(2)}(t)-u_n^{(1)}(t)\right|\leq\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(1)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,0\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(2)}(t)-u_n^{(1)}(t)\right|\leq\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\times$$
$$\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)-f(\xi,\tau,0)\right)\sin n\xi d\xi\right]^2d\tau\right)^{\frac{1}{2}}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\sum_{n=1}^{\infty}\left|u_n^{(2)}(t)-u_n^{(1)}(t)\right|\leq\frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)-f(\xi,\tau,0)\right)^2d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^{\pi}b^2(\xi,\tau)\left|Au^{(1)}(\xi,\tau)\right|^2d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}\left(\int_0^t\int_0^{\pi}b^2(\xi,\tau)\,d\xi d\tau\right)^{\frac{1}{2}}.$$

Thus, $\left\|\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t)\right\|_{B_T}=\sum\limits_{n=1}^{\infty}\max\limits_{0\leq t\leq T}\left|u_n^{(2)}(t)-u_n^{(1)}(t)\right|\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}\|b(\xi,\tau)\|_{L_2(\Omega)}$
$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}A_TM_2<\infty.$

We have

$$\left|u_n^{(3)}(t)-u_n^{(2)}(t)\right|\leq\frac{2}{\beta\pi}\int_0^t\int_0^{\pi}e^{\alpha(t-\tau)}\left|f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)\right||\sin n\xi|\,d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(3)}(t)-u_n^{(2)}(t)\right|$$
$$\leq\left(\frac{1}{\varepsilon a^2n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)\right)\sin n\xi d\xi\right]^2d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(3)}(t)-u_n^{(2}(t)\right|$$
$$\leq\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)\right)\sin n\xi d\xi\right]^2d\tau\right)^{\frac{1}{2}}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\sum_{n=1}^{\infty}\left|u_n^{(3)}(t)-u_n^{(2)}(t)\right|$$
$$\leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(2)}(\xi,\tau)\right)-f\left(\xi,\tau,Au^{(1)}(\xi,\tau)\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|Au^{(2)}(\xi,\tau)-Au^{(1)}(\xi,\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|\overline{u}^{(2)}(t)-\overline{u}^{(1)}(t\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left\{\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(1)}(t)\right\|_{B_T}\left(\int_0^\tau\int_0^\pi b^2(\xi_1,\tau_1)\,d\xi_1 d\tau_1\right)^{\frac{1}{2}}\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^2 A_T\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left(\int_0^\tau\int_0^\pi b^2(\xi_1,\tau_1)\,d\xi_1 d\tau_1\right)d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^2 A_T\frac{1}{\sqrt{2!}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi d\tau\right)^2\right]^{\frac{1}{2}}$$
$$\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^2 A_T\frac{1}{\sqrt{2!}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi d\tau\right)^{\frac{1}{2}}\right]^2.$$

Thus, $\left\|\overline{u}^{(3)}(t)-\overline{u}^{(2)}(t)\right\|_{B_T}=\sum\limits_{n=1}^{\infty}\max\limits_{0\leq t\leq T}\left|u_n^{(3)}(t)-u_n^{(2)}(t)\right|\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^2 A_T\frac{1}{\sqrt{2!}}\left\|b(\xi,\tau)\right\|^2_{L_2(\Omega)}$ $\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^2 A_T\frac{1}{\sqrt{2!}}M_2^2<\infty.$

Let's show its truth for each N by induction:

For $N=k-1$, $\left|\overline{u}^{(k)}(t)-\overline{u}^{(k-1)}(t)\right|=\sum\limits_{n=1}^{\infty}\left|u_n^{(k)}(t)-u_n^{(k-1)}(t)\right|\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^{k-1}A_T\frac{1}{\sqrt{(k-1)!}}\times$ $\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi d\tau\right)^{\frac{k-1}{2}}$ be correct. From here we get

$$\left\|\overline{u}^{(k)}(t)-\overline{u}^{(k-1)}(t)\right\|_{B_T}=\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\left|u_n^{(k)}(t)-u_n^{(k-1)}(t)\right|$$
$$\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^{k-1}A_T\frac{1}{\sqrt{(k-1)!}}\left\|b(\xi,\tau)\right\|^{(k-1)}_{L_2(\Omega)}$$
$$\leq\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^{k-1}A_T\frac{1}{\sqrt{(k-1)!}}M_2^{(k-1)}<\infty.$$

For $N = k$, we have

$$\left|u_n^{(k+1)}(t) - u_n^{(k)}(t)\right| \leq \frac{2}{\beta\pi}\int_0^t e^{\alpha(t-\tau)}\int_0^\pi \left|f\left(\xi,\tau,Au^{(k)}(\xi,\tau)\right) - f\left(\xi,\tau,Au^{(k-1)}(\xi,\tau)\right)\right| \left|\sin n\xi\right| d\xi d\tau.$$

If Cauchy's inequality is applied with respect to τ, we get

$$\left|u_n^{(k+1)}(t) - u_n^{(k)}(t)\right|$$

$$\leq \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}} \left(\int_0^t \left[\frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au^{(k)}(\xi,\tau)\right) - f\left(\xi,\tau,Au^{(k-1)}(\xi,\tau)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.$$

By summing over n and applying Hölder's inequality

$$\sum_{n=1}^{\infty}\left|u_n^{(k+1)}(t) - u_n^{(k)}(t)\right|$$

$$\leq \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} \left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au^{(k)}(\xi,\tau)\right) - f\left(\xi,\tau,Au^{(k-1)}(\xi,\tau)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\sum_{n=1}^{\infty}\left|u_n^{(k+1)}(t) - u_n^{(k)}(t)\right|$$

$$\leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}} \left(\int_0^t\frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au^{(k)}(\xi,\tau)\right) - f\left(\xi,\tau,Au^{(k-1)}(\xi,\tau)\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|Au^{(k)}(\xi,\tau) - Au^{(k-1)}(\xi,\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|\overline{u}^{(k)}(t) - \overline{u}^{(k-1)}(t\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left\{\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^{k-1} A_T\frac{1}{\sqrt{(k-1)!}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^{\frac{k-1}{2}}\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{(k-1)!}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left(\int_0^\tau\int_0^\pi b^2(\xi_1,\tau_1)d\xi_1 d\tau_1\right)^{k-1} d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{(k-1)!}}\frac{1}{\sqrt{k}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^k\right]^{\frac{1}{2}}$$

$$\leq \left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{k!}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}\right]^k.$$

Thus, $\left\|\overline{u}^{(k+1)}(t)-\overline{u}^{(k)}(t)\right\|_{B_T}=\sum\limits_{n=1}^{\infty}\max\limits_{0\le t\le T}\left|u_n^{(k+1)}(t)-u_n^{(k)}(t)\right|\le\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{k!}}\left\|b(\xi,\tau)\right\|_{L_2(\Omega)}^k$ $\le\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{k!}}M_2^k<\infty$. From here it is obvious that

$$\overline{u}^{(N+1)}(t)=\overline{u}^{(0)}(t)+\sum_{k=0}^{N}\left(\overline{u}^{(k+1)}(t)-\overline{u}^{(k)}(t)\right)\le\sum_{k=0}^{\infty}\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{k!}}M_2^k.$$

The uniform convergence of the sequence $\left\{\overline{u}^{(N)}(t)\right\}$ in B_T is obtained from the convergence of the series $\sum\limits_{k=0}^{\infty}\left[\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\right]^k A_T\frac{1}{\sqrt{k!}}M_2^k$. As a result, the series $\overline{u}^{(0)}(t)+\sum\limits_{N=0}^{\infty}\left(\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right)$ is uniformly convergent.

Let $\lim\limits_{N\to\infty}\overline{u}^{(N+1)}(t)=\overline{u}(t)$. Since the sequence $\left\{\overline{u}^{(N)}(t)\right\}$ is uniformly convergent, the function $\overline{u}(t)$ is continuous in B_T. Let us show that the function $\overline{u}(t)$ satisfies the integral equation (6):

We have

$$\begin{aligned}\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right|&=\sum_{n=1}^{\infty}\left|u_n(t)-u_n^{(N+1)}(t)\right|\\&\le\sum_{n=1}^{\infty}\frac{2}{\beta\pi}\int_0^t\int_0^\pi e^{\alpha(t-\tau)\pi}\left|f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right|\left|\sin n\xi\right|d\xi d\tau.\end{aligned}$$

If Cauchy's inequality is applied with respect to τ, we get

$$\begin{aligned}&\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right|\\&\le\sum_{n=1}^{\infty}\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

By applying Hölder's inequality

$$\begin{aligned}&\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right|\\&\le\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}\end{aligned}$$

is obtained. If Bessel inequality and the Lipschitz condition is applied, we get

$$\begin{aligned}
\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right| &\leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left(\int_0^t \frac{2}{\pi}\int_0^\pi \left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)\left|Au\left(\xi,\tau\right)-Au^{(N)}\left(\xi,\tau\right)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)\left|\overline{u}(t)-\overline{u}^{(N)}(t)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}(t)-\overline{u}^{(N)}(t)\right\|_{B_T}\left(\int_0^t\int_0^\pi b^2\left(\xi,\tau\right)d\xi d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

If we show that $\lim \lim\limits_{N\to\infty}\left\|\overline{u}(t)-\overline{u}^{(N)}(t)\right\|_{B_T}=0$, it follows that the function $\overline{u}(t)$ satisfies the integral equation (6). If Cauchy's inequality, Hölder's inequality, Bessel's inequality and Lipschitz's condition are applied, then

$$\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right| = \sum_{n=1}^{\infty}\left|u_n(t)-u_n^{(N+1)}(t)\right|$$

$$\begin{aligned}
&\leq \sum_{n=1}^{\infty}\frac{2}{\beta\pi}\int_0^t\int_0^\pi e^{\alpha(t-\tau)\pi}\left|f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right|\left|\sin n\xi\right| d\xi d\tau \\
&\leq \sum_{n=1}^{\infty}\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}} \\
&+\sum_{n=1}^{\infty}\left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(N+1)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N+1)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}} \\
&+\frac{1}{\sqrt{\varepsilon}a}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^\pi\left(f\left(\xi,\tau,Au^{(N+1)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)\sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left({}_0^t\frac{2}{\pi}{}_0^\pi\left(f\left(\xi,\tau,Au\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N+1)}\left(\xi,\tau\right)\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\frac{1}{\sqrt{\varepsilon}a}\left(\frac{\pi^2}{6}\right)^{\frac{1}{2}}\left({}_0^t\frac{2}{\pi}{}_0^\pi\left(f\left(\xi,\tau,Au^{(N+1)}\left(\xi,\tau\right)\right)-f\left(\xi,\tau,Au^{(N)}\left(\xi,\tau\right)\right)\right)^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left({}_{00}^{t\,\pi}b^2\left(\xi,\tau\right)\left|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left({}_{00}^{t\,\pi}b^2\left(\xi,\tau\right)\left|\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_{B_T}\left({}_{00}^{t\,\pi}b^2\left(\xi,\tau\right)d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)\right\|_{B_T}\left({}_{00}^{t\,\pi}b^2\left(\xi,\tau\right)d\xi d\tau\right)^{\frac{1}{2}}$$

is obtained. From here we get

$$\begin{aligned}
\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_{B_T} &\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_{B_T}\left\|b(\xi,\tau)\right\|_{L_2(\Omega)} \\
&+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left[\left(\frac{\pi}{6}\right)^{\frac{1}{2}}\right]^N A_T\frac{1}{\sqrt{N!}}\left\|b(\xi,\tau)\right\|_{L_2(\Omega)}^N\left\|b(\xi,\tau)\right\|_{L_2(\Omega)} \\
&\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_{B_T}M_2+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}A_T\frac{1}{\sqrt{N!}}M_2^{N+1} \\
&\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}M_2\left\|\overline{u}(t)-\overline{u}^{(N+1)}(t)\right\|_{B_T}+\left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}}M_2A_T\frac{\left[\left(\frac{\pi}{6}\right)^{\frac{1}{2}}M_2\right]^N}{\sqrt{N!}}.
\end{aligned}$$

It is clear that $\lim_{N\to\infty}\left\|\overline{u}(t)-\overline{u}^{(N)}(t)\right\|_{B_T}=0$. Thus, it is shown that the function $\overline{u}(t)$ satisfies the integral equation (6).

Lemma 2.3 *Under the conditions of Theorem 1, equation (6) has at most one solution in B_T.*

Proof. To show the uniqueness of the solution, let us assume that $\overline{v}(t)$ is another solution. We want to obtain an estimate for $|\overline{u}(t)-\overline{v}(t)|$:

If Cauchy's inequality, Hölder's inequality, Bessel's inequality and Lipschitz's condition are applied, then

$$|\overline{u}(t)-\overline{v}(t)|=\sum_{n=1}^{\infty}|u_n(t)-v_n(t)|$$

$$\leq \sum_{n=1}^{\infty} \frac{2}{\beta\pi} \int_0^t e^{\alpha(t-\tau)} \int_0^\pi |f(\xi,\tau,Au(\xi,\tau)) - f(\xi,\tau,Av(\xi,\tau))| \, |\sin n\xi| \, d\xi d\tau$$

$$\leq \sum_{n=1}^{\infty} \left(\frac{1}{\varepsilon a^2 n^{2k+2m}}\right)^{\frac{1}{2}} \left(\int_0^t \left[\frac{2}{\pi}\int_0^\pi (f(\xi,\tau,Au(\xi,\tau)) - f(\xi,\tau,Av(\xi,\tau))) \sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\varepsilon}a} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \int_0^t \left[\frac{2}{\pi}\int_0^\pi (f(\xi,\tau,Au(\xi,\tau)) - f(\xi,\tau,Av(\xi,\tau))) \sin n\xi d\xi\right]^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\varepsilon}a} \left(\frac{\pi^2}{6}\right)^{\frac{1}{2}} \left(\int_0^t \frac{2}{\pi}\int_0^\pi (f(\xi,\tau,Au(\xi,\tau)) - f(\xi,\tau,Av(\xi,\tau)))^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \left(\int_0^t\int_0^\pi b^2(\xi,\tau) \, |\overline{u}(t) - \overline{v}(t)|^2 \, d\xi d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \|\overline{u}(t) - \overline{v}(t)\|_{B_T} \left(\int_0^t\int_0^\pi b^2(\xi,\tau) \, d\xi d\tau\right)^{\frac{1}{2}}$$

is obtained. From here we get

$$\begin{aligned} \|\overline{u}(t) - \overline{v}(t)\|_{B_T} &\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \|\overline{u}(t) - \overline{v}(t)\|_{B_T} \|b(\xi,\tau)\|_{L_2(\Omega)} \\ &\leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \|\overline{u}(t) - \overline{v}(t)\|_{B_T} M_2 \\ &= \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} M_2 \|\overline{u}(t) - \overline{v}(t)\|_{B_T}. \end{aligned}$$

It is clear that $\|\overline{u}(t) - \overline{v}(t)\|_{B_T} = 0$. Thus, $\overline{u}(t) = \overline{v}(t)$ and $u_n(t) = v_n(t)$, $\left(n = \overline{1,\infty}\right)$. This result can also be obtained by applying Gronwall's inequality to the inequality $|\overline{u}(t) - \overline{v}(t)| \leq \left(\frac{\pi}{3\varepsilon a^2}\right)^{\frac{1}{2}} \left(\int_0^t\int_0^\pi b^2(\xi,\tau) \, |\overline{u}(t) - \overline{v}(t)|^2 \, d\xi d\tau\right)^{\frac{1}{2}}$. In other words, it was shown that the solution of the integral equation (6) is unique.

Proof of Theorem 1. From Lemma 1 and Lemma 2, equation (6) has a unique solution. Thus, the theorem is proved.

Theorem 2.4 *Under the conditions of Theorem 1, the problem (1)-(3) has a unique weak solution represented by the uniformly convergent series of (5).*

Proof. The series (5) constructed using the solution of equation (6) is continuous since it is uniformly convergent. Let the sequence of partial sums of the series (5) be defined as follows:

$$u_{(l)}(x,t) = \sum_{n=1}^{l} u_n(t) \sin nx.$$

From Teorem 1 and $\lim_{l\to\infty} u_{(l)}(x,t) = u(x,t),\ \lim_{l\to\infty} f\left(x,t,u_{(l)}(x,t)\right) = f\left(x,t,u(x,t)\right)$. Let

$$S_l =_0^T {}_0^\pi \left[\left(\frac{\partial^2 v}{\partial t^2} + (-1)^k a^2 \frac{\partial^{2k} v}{\partial x^{2k}} - (-1)^m 2\varepsilon \frac{\partial^{2m+1} v}{\partial x^{2m}\partial t}\right) u_{(l)} - f(x,t,u_{(l)})v\right] dxdt$$

be defined. We want to show that $\lim_{l\to\infty} S_l = 0$. By using partial integration repeatedly,

$$\begin{aligned}
S_l =&_0^T {}_0^\pi \left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=1}^{l} u_n(t)\sin nx\right) + (-1)^k a^2 \left(\sum_{n=1}^{l} u_n(t)(-1)^k n^{2k}\sin nx\right)\right. \\
&\left. +(-1)^m 2\varepsilon \frac{\partial}{\partial t}\left(\sum_{n=1}^{l} u_n(t)(-1)^m n^{2m}\sin nx\right) - f\left(x,t,u_{(l)}\right)\right] vdxdt \\
=&_0^T {}_0^\pi \left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=1}^{l} u_n(t)\sin nx\right) + (-1)^k a^2 \frac{\partial^{2k}}{\partial x^{2k}}\left(\sum_{n=1}^{l} u_n(t)\sin nx\right)\right. \\
&\left. +(-1)^m 2\varepsilon \frac{\partial^{2m+1}}{\partial x^{2m}\partial t}\left(\sum_{n=1}^{l} u_n(t)\sin nx\right) - f\left(x,t,u_{(l)}\right)\right] vdxdt \\
=&_0^T {}_0^\pi \left(\frac{\partial^2}{\partial t^2} u_{(l)} + (-1)^k a^2 \frac{\partial^{2k}}{\partial x^{2k}} u_{(l)} + (-1)^m 2\varepsilon \frac{\partial^{2m+1}}{\partial x^{2m}\partial t} u_{(l)} - f\left(x,t,u_{(l)}\right)\right) vdxdt
\end{aligned}$$

is obtained. From here we get

$$\lim_{l\to\infty} S_l =_0^T {}_0^\pi \left(\frac{\partial^2}{\partial t^2} u + (-1)^k a^2 \frac{\partial^{2k}}{\partial x^{2k}} u + (-1)^m 2\varepsilon \frac{\partial^{2m+1}}{\partial x^{2m}\partial t} u - f\left(x,t,u\right)\right) vdxdt.$$

From equation (1), we have

$$\lim_{l\to\infty} S_l = 0.$$

Thus, the function $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$ is a weak solution of the problem (1)-(3). The theorem is proved.

3 Open Problem

We examined the existence and uniqueness of the initial and boundary value problem (1)-(3). Under the conditions in the theorem, existence and uniqueness are proven to be valid for every bounded T, that is, global existence and uniqueness. Similar studies can be done for the boundary value problem to the equation considered. Moreover stability of the problem can be studied.

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