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Some new integral formulas with applications

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Abstract

In this article, we present new explicit integral formulas derived by a combination of classical identities and advanced techniques. Applications to integral inequalities, including a logarithmic Hardy-Hilbert-type integral inequality, are also discussed. In addition, three open problems are proposed, which suggest new avenues for further investigation.

Keywords: *Integral, Pi, Fresnel Integral Formula, Arctangent Function, Logarithmic Function, Laplace Transform*
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1 Introduction

The concept of integration is fundamental to many areas of mathematics. It occurs naturally in a wide range of problems, from pure theoretical questions to applied models in physics, engineering and finance. Despite much progress, modern research still regularly encounters new integrals. These integrals often arise from complex systems with non-elementary functions, oscillatory behavior or singularities. Classical techniques, such as primitives, change of variables, integration by parts and contour integration, sometimes fail to provide closed-form formulas. Numerical methods provide approximations but may lack precision, especially in sensitive applications.

Of course, there are already large collections of integral formulas, most of which can be found in the reference book [3]. However, they do not always cover all practical needs. Special cases involving complicated functional forms often require individual treatment. Exact expressions of integrals also contribute to more stable computations in applied sciences. Such work deepens our understanding of advanced mathematical models. For these reasons, active research into the evaluation of integrals remains essential. Several recent articles illustrate this continuing need and progress in the field, including [6, 7, 8, 9, 1, 2].

In this study, we develop new integral formulas by combining classical results with advanced integration techniques. Some of these formulas may exist implicitly within more general expressions, but to our knowledge, they have not been published in explicit form with direct and detailed proofs. We aim to fill this gap. As an illustration of the kind of results we establish, we highlight the following two key examples:

Example 1:

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \arctan\left(\frac{x}{2}\right) dx = \pi.$$

Example 2:

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{1}{x^2}\right) dx = 2\pi\sqrt{2}.$$

The constant π plays a central role in many of our derived formulas. Detailed and rigorous proofs are given for each result. To complement these formulas, we also explore their applications to various types of integral inequalities. These include a special logarithmic integral inequality of the Hardy-Hilbert type, which does not appear in the reference book [11]. We also draw a parallel between some of our results and three open problems.

The rest of the article is as follows: Section 2 is devoted to our results, i.e., integral formulas and integral inequalities. The proofs are given in Section 3. The open problems are discussed in Section 4. A conclusion is proposed in Section 5.

2 Results

2.1 Integral formulas

A ratio-trigonometric-power integral formula is presented in the result below. The proof combines a change of variables, an integration by parts and the Fresnel integral formula, i.e.,

$$\int_0^{+\infty} \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

See [11, 8.257].

Proposition 2.1 *For any $\alpha > 0$, we have*

$$\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}} dx = \sqrt{\alpha\pi}.$$

In particular, if we take $\alpha = 1$, we obtain

$$\int_0^{+\infty} \frac{\sin^2(x)}{x\sqrt{x}} dx = \sqrt{\pi} \approx 1.77245.$$

From this proposition, we can also immediately derive the following integral formula using the trigonometric formula $\sin^2(x) = [1 - \cos(2x)]/2$ for any $x \in \mathbb{R}$:

$$\int_0^{+\infty} \frac{1 - \cos(2\alpha x)}{x\sqrt{x}} dx = 2\sqrt{\alpha\pi}.$$

The proposition below suggests a ratio-power integral formula. The proof is derived from Proposition 2.1 and the use of a special formula for the Laplace transform.

Proposition 2.2 *For any $\alpha > 0$, we have*

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2 + 4\alpha^2)} dx = \frac{\pi}{4\alpha\sqrt{\alpha}}.$$

In particular, if we take $\alpha = 1$, we obtain

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2 + 4)} dx = \frac{\pi}{4} \approx 0.78539.$$

Another ratio-power integral formula is presented in the result below. The proof is derived from Proposition 2.2 and the Leibnitz integral theorem.

Proposition 2.3 *For any $\alpha > 0$, we have*

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2 + 4\alpha^2)^2} dx = \frac{3\pi}{64\alpha^3\sqrt{\alpha}}.$$

In particular, if we take $\alpha = 1$, we obtain

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2 + 4)^2} dx = \frac{3\pi}{64} \approx 0.1472621.$$

The proposition below suggests another ratio-power integral formula. The proof is derived from Proposition 2.3 and the Leibnitz integral theorem.

Proposition 2.4 *For any $\alpha > 0$, we have*

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3} dx = \frac{21\pi}{2048\alpha^5\sqrt{\alpha}}.$$

In particular, if we take $\alpha = 1$, we obtain

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4)^3} dx = \frac{21\pi}{2048} \approx 0.032214.$$

An original arctangent-power integral formula is given in the result below. The proof is derived from Proposition 2.2 and the Fubini-Tonelli integral theorem.

Proposition 2.5 *For any $\beta > 0$, we have*

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx = \pi\left[1 - \frac{1}{\sqrt{\beta}}\right].$$

In particular, if we take $\beta = 1/2$, we get

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx = \pi\left[1 - \sqrt{2}\right] \approx -1.30129.$$

Another interesting consequence of this proposition is the result below.

Proposition 2.6 *We have*

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \arctan\left(\frac{x}{2}\right) dx = \pi.$$

As far as we know, this is a new, or at least little known, integral representation for π.

A parametric version of this proposition is presented in the result below. The proof is based on Proposition 2.6 and a change of variables.

Proposition 2.7 *For any $\eta > 0$, we have*

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \arctan(\eta x)\, dx = \pi\sqrt{2\eta}.$$

The proposition below suggests an original logarithmic-power integral formula. The proof combines a reparameterization of Proposition 2.2 and the Fubini-Tonelli integral theorem.

Proposition 2.8 *For any $\beta > 0$, we have*

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{4\beta}{x^2}\right) dx = 4\pi\beta^{1/4}.$$

In particular, if we take $\beta = 1/4$, we get the following elegant formula:

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{1}{x^2}\right) dx = 2\pi\sqrt{2} \approx 8.8857.$$

The simple reparameterization $\epsilon = 4\beta$ allows to reformulate Proposition 2.8 as follows: For any $\epsilon > 0$, we have

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{\epsilon}{x^2}\right) dx = 2\pi\sqrt{2}\epsilon^{1/4}. \quad (1)$$

An original arctangent-power integral formula is provided in the result below. The proof is derived from Proposition 2.3 and the Fubini-Tonelli integral theorem.

Proposition 2.9 *For any $\theta > 0$, we have*

$$\int_0^{+\infty} \frac{1}{x^3\sqrt{x}} \left[\frac{2\theta x}{x^2 + 4\theta^2} - \frac{2x}{x^2 + 4} + \arctan\left(\frac{2\theta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx$$
$$= \frac{3\pi}{40}\left[1 - \frac{1}{\theta^2\sqrt{\theta}}\right].$$

To the best of our knowledge, it is new to literature.

The formulas obtained can be determinant tools in various mathematical scenarios. In the rest of this section, we propose to use them to obtain valuable upper bounds for sophisticated integrals, including some that depend on one or two auxiliary functions.

2.2 Some applications to integral inequalities

The proposition below suggests a simple integral inequality. The proof is based on the Cauchy-Schwarz integral inequality, and Propositions 2.1 and 2.7.

Proposition 2.10 *For any $\alpha, \eta > 0$, we have*

$$\int_0^{+\infty} \frac{|\sin(\alpha x)|\sqrt{\arctan(\eta x)}}{x\sqrt{x}} dx \leq (2\alpha\eta\pi)^{1/4}\sqrt{\pi}.$$

In this way, we have identified a simple candidate upper bound for a very sophisticated integral, which is not easy to evaluate even with numerical software.

Another simple integral inequality based on our findings is given in the result below. The proof combines the Cauchy-Schwarz integral inequality, and Propositions 2.4 and 2.5.

Proposition 2.11 *For any $\alpha > 0$ and $\beta > 1$, we have*

$$\int_0^{+\infty} \frac{1}{x(x^2+4\alpha^2)^{3/2}} \sqrt{\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)} dx \\ \leq \frac{\pi}{32\alpha^{11/4}} \sqrt{\frac{21}{2}} \sqrt{1 - \frac{1}{\sqrt{\beta}}}.$$

To illustrate the precision of this upper bound, we arbitrarily consider the values $\alpha = 1$ and $\beta = 2$. Then we have

$$\int_0^{+\infty} \frac{1}{x(x^2+4\alpha^2)^{3/2}} \sqrt{\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)} dx \\ = \int_0^{+\infty} \frac{1}{x(x^2+4)^{3/2}} \sqrt{\arctan\left(\frac{4}{x}\right) - \arctan\left(\frac{2}{x}\right)} dx \approx 0.152112$$

and

$$\frac{\pi}{32\alpha^{11/4}} \sqrt{\frac{21}{2}} \sqrt{1 - \frac{1}{\sqrt{\beta}}} \frac{\pi}{32} \sqrt{\frac{21}{2}} \sqrt{1 - \frac{1}{\sqrt{2}}} \approx 0.172166.$$

It is obvious that $0.152112 < 0.172166$, illustrating the sharpness of our result.

The proposition below suggests an integral inequality of the Hölder type. The proof combines the Hölder integral inequality and Proposition 2.1.

Proposition 2.12 *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that*

$$\int_0^{+\infty} x^{3(q-1)/2} f^q(x) dx < +\infty.$$

Then we have

$$\int_0^{+\infty} |\sin(\alpha x)|^{2/p} f(x) dx \leq (\alpha\pi)^{1/(2p)} \left[\int_0^{+\infty} x^{3(q-1)/2} f^q(x) dx\right]^{1/q}.$$

The upper bound obtained is thus much simpler than the integral on the left-hand side; the presence of $|\sin(\alpha x)|$ can complicate the exact evaluation considerably.

An integral inequality of the Hardy-Hilbert type is shown in the result below. It has the feature to involve a logarithmic function, following the spirit of the articles [10, 5], completed by the books [4, 11].

The proof is based on the Hölder integral inequality, the Fubini-Tonelli integral theorem and Equation (1).

Proposition 2.13 *Let $p > 1$, $q = p/(p-1)$, $\omega > 0$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} x^{p/2-1} f^p(x) dx < +\infty, \quad \int_0^{+\infty} y^{q/2-1} g^q(y) dy < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{\omega}{x^2y^2}\right) f(x)g(y) dx dy$$
$$\leq 2\pi\sqrt{2}\omega^{1/4} \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}.$$

The proposition below completes this result. It deals with only one function.

Proposition 2.14 *Let $p > 1$, $q = p/(p-1)$, $\omega > 0$, and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that*

$$\int_0^{+\infty} x^{p/2-1} f^p(x) dx < +\infty.$$

Then we have

$$\int_0^{+\infty} y^{-(q/2-1)(p-1)} \left[\int_0^{+\infty} \log\left(1 + \frac{\omega}{x^2y^2}\right) f(x) dx\right]^p dy$$
$$\leq \left[2\pi\sqrt{2}\omega^{1/4}\right]^p \int_0^{+\infty} x^{p/2-1} f^p(x) dx.$$

Such Hardy-Hilbert-type integral inequalities find important applications in operator theory and the study of various function spaces. We refer again to [11], which contains all the relevant details.

The next section contains the rigorous proofs of the above propositions.

3 Proofs

Proof of Proposition 2.1. First, making the change of variables $y = \alpha x$, we have

$$\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}} dx = \int_0^{+\infty} \frac{\sin^2(y)}{(y/\alpha)\sqrt{y/\alpha}} \left(\frac{1}{\alpha} dy\right) = \sqrt{\alpha} \int_0^{+\infty} \frac{\sin^2(y)}{y\sqrt{y}} dy. \quad (2)$$

Let us work on the last integral term. Doing an integration by parts, using the immediate limit results $\lim_{x \to 0} \sin^2(x)/\sqrt{x} = 0$ and $\lim_{x \to +\infty} \sin^2(x)/\sqrt{x} = 0$, and the trigonometric formula $\sin(2x) = 2\sin(x)\cos(x)$ for any $x \in \mathbb{R}$, we get

$$\int_0^{+\infty} \frac{\sin^2(y)}{y\sqrt{y}} dy = \left[-\frac{2}{\sqrt{y}} \sin^2(y) \right]_{y\to 0}^{y\to+\infty} + 2\int_0^{+\infty} \frac{2\sin(y)\cos(y)}{\sqrt{y}} dy$$
$$= 0 + 2\int_0^{+\infty} \frac{\sin(2y)}{\sqrt{y}} dy = 2\int_0^{+\infty} \frac{\sin(2y)}{\sqrt{y}} dy. \tag{3}$$

Let us work on the last integral term. Making the change of variables $y = z^2/2$ and recognizing the Fresnel integral formula, i.e., $\int_0^{+\infty} \sin(x^2)dx = (1/2)\sqrt{\pi/2}$, we obtain

$$\int_0^{+\infty} \frac{\sin(2y)}{\sqrt{y}} dy = \int_0^{+\infty} \frac{\sin(z^2)}{\sqrt{z^2/2}} (zdz) = \sqrt{2}\int_0^{+\infty} \sin(z^2)dz$$
$$= \sqrt{2} \times \frac{1}{2}\sqrt{\frac{\pi}{2}} = \frac{1}{2}\sqrt{\pi}. \tag{4}$$

Combining Equations (2), (3) and (4), we have

$$\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}} dx = \sqrt{\alpha} \times 2 \times \frac{1}{2}\sqrt{\pi} = \sqrt{\alpha\pi}.$$

This concludes the proof of Proposition 2.1. □

Proof of Proposition 2.2. First, we recall a useful formula involving the Laplace transform and its inverse. Let $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions, let $\mathcal{L}[f(\cdot)](s) = \int_0^{+\infty} e^{-sx} f(x)dx$ be the Laplace transform of f at $s > 0$, and let $\mathcal{L}^{-1}[g(\cdot)](s)$ be the inverse Laplace transform of g at $s > 0$, i.e., satisfying $g(x) = \mathcal{L}[\mathcal{L}^{-1}[g(\cdot)](\cdot)](x)$ for any $x \in \mathbb{R}$, assuming that there exist. Then we have

$$\int_0^{+\infty} f(x)g(x)dx = \int_0^{+\infty} \mathcal{L}[f(\cdot)](s)\mathcal{L}^{-1}[g(\cdot)](s)ds. \tag{5}$$

Aiming to use Proposition 2.1, which involves functions well mastered in terms of the Laplace transform and its inverse, we write

$$\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}} dx = \int_0^{+\infty} f(x)g(x)dx, \tag{6}$$

where

$$f(x) = \sin^2(\alpha x), \quad g(x) = \frac{1}{x\sqrt{x}}.$$

It is known that

$$\mathcal{L}[\sin^2(\cdot)](s) = \frac{2}{s(s^2+4)},$$

from which we derive

$$\mathcal{L}[f(\cdot)](s) = \mathcal{L}[\sin^2(\alpha\cdot)](s) = \frac{1}{\alpha}\mathcal{L}[\sin^2(\cdot)]\left(\frac{s}{\alpha}\right) = \frac{2\alpha^2}{s(s^2+4\alpha^2)}. \tag{7}$$

The inverse Laplace transform of g is also known. It is given by

$$\mathcal{L}^{-1}[g(\cdot)](s) = \mathcal{L}^{-1}\left(\frac{1}{\cdot^{3/2}}\right)(s) = 2\sqrt{\frac{s}{\pi}}. \tag{8}$$

It follows from Equations (5), (6), (7) and (8) that

$$\begin{aligned}
&\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}}dx = \int_0^{+\infty} f(x)g(x)dx = \int_0^{+\infty} \mathcal{L}[f(\cdot)](s)\mathcal{L}^{-1}[g(\cdot)](s)ds \\
&= \int_0^{+\infty} \frac{2\alpha^2}{s(s^2+4\alpha^2)} \times 2\sqrt{\frac{s}{\pi}}ds = \frac{4\alpha^2}{\sqrt{\pi}}\int_0^{+\infty} \frac{\sqrt{s}}{s(s^2+4\alpha^2)}ds \\
&= \frac{4\alpha^2}{\sqrt{\pi}}\int_0^{+\infty} \frac{1}{\sqrt{s}(s^2+4\alpha^2)}ds.
\end{aligned}$$

The last integral is the one of interest, and we can write

$$\int_0^{+\infty} \frac{1}{\sqrt{s}(s^2+4\alpha^2)}ds = \frac{\sqrt{\pi}}{4\alpha^2}\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}}dx.$$

Using Proposition 2.1 and standardizing the notation in x, we immediately obtain

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)}dx = \frac{\sqrt{\pi}}{4\alpha^2} \times \sqrt{\alpha\pi} = \frac{\pi}{4\alpha\sqrt{\alpha}}.$$

This concluses the proof of Proposition 2.2. □

Proof of Proposition 2.3. Proposition 2.2 ensures that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)}dx = \frac{\pi}{4\alpha\sqrt{\alpha}},$$

with $\alpha > 0$. Considering α as a variable, differentiating both sides with respect to it, and developing only the right-hand side term, we get

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)}dx\right] = \frac{\pi}{4}\frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha\sqrt{\alpha}}\right) = \frac{\pi}{4} \times \left(-\frac{3}{2}\right)\frac{1}{\alpha^2\sqrt{\alpha}} \\
&= -\frac{3\pi}{8\alpha^2\sqrt{\alpha}}.
\end{aligned} \tag{9}$$

For the left-hand side term, exchanging the partial derivative and integral symbols by the Leibniz integral theorem, we obtain

$$\frac{\partial}{\partial \alpha}\left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)}dx\right] = \int_0^{+\infty} \frac{1}{\sqrt{x}}\frac{\partial}{\partial \alpha}\left(\frac{1}{x^2+4\alpha^2}\right)dx$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{x}}\left[-\frac{8\alpha}{(x^2+4\alpha^2)^2}\right]dx = -8\alpha\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx. \quad (10)$$

It follows from Equations (9) and (10) that

$$-8\alpha\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx = -\frac{3\pi}{8\alpha^2\sqrt{\alpha}},$$

so that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx = \frac{3\pi}{64\alpha^3\sqrt{\alpha}}.$$

This concludes the proof of Proposition 2.3. □

Proof of Proposition 2.4. Proposition 2.3 ensures that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx = \frac{3\pi}{64\alpha^3\sqrt{\alpha}},$$

with $\alpha > 0$. Considering α as a variable, differentiating both sides with respect to it, and developing only the right-hand side term, we get

$$\frac{\partial}{\partial \alpha}\left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx\right] = \frac{3\pi}{64}\frac{\partial}{\partial \alpha}\left(\frac{1}{\alpha^3\sqrt{\alpha}}\right) = \frac{3\pi}{64}\times\left(-\frac{7}{2}\right)\frac{1}{\alpha^4\sqrt{\alpha}}$$
$$= -\frac{21\pi}{128\alpha^4\sqrt{\alpha}}. \quad (11)$$

For the left-hand side term, exchanging the partial derivative and integral symbols by the Leibniz integral theorem, we obtain

$$\frac{\partial}{\partial \alpha}\left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx\right] = \int_0^{+\infty} \frac{1}{\sqrt{x}}\frac{\partial}{\partial \alpha}\left[\frac{1}{(x^2+4\alpha^2)^2}\right]dx$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{x}}\left[-\frac{16\alpha}{(x^2+4\alpha^2)^3}\right]dx = -16\alpha\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3}dx. \quad (12)$$

It follows from Equations (11) and (12) that

$$-16\alpha\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3}dx = -\frac{21\pi}{128\alpha^4\sqrt{\alpha}},$$

so that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3} dx = \frac{21\pi}{2048\alpha^5\sqrt{\alpha}}.$$

This concludes the proof of Proposition 2.4. □

Proof of Proposition 2.5. It follows from Proposition 2.2 that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)} dx = \frac{\pi}{4\alpha\sqrt{\alpha}},$$

with $\alpha > 0$. Considering α as a variable, integrating both sides with respect to it from 1 to β (including the case $\beta \in (0,1)$), and developing only the right-hand side term, we get

$$\int_1^{\beta} \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)} dx\right] d\alpha = \frac{\pi}{4} \int_1^{\beta} \frac{1}{\alpha\sqrt{\alpha}} d\alpha = \frac{\pi}{4} \left[-\frac{2}{\sqrt{\alpha}}\right]_{\alpha=1}^{\alpha=\beta}$$
$$= \frac{\pi}{2}\left[1 - \frac{1}{\sqrt{\beta}}\right]. \tag{13}$$

For the left-hand side term, exchanging the integral symbols by the Fubini-Tonelli integral theorem, we obtain

$$\int_1^{\beta} \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)} dx\right] d\alpha = \int_0^{+\infty} \frac{1}{\sqrt{x}} \left[\int_1^{\beta} \frac{1}{x^2+4\alpha^2} d\alpha\right] dx$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{x}} \left[\frac{1}{2x} \arctan\left(\frac{2\alpha}{x}\right)\right]_{\alpha=1}^{\alpha=\beta} dx$$
$$= \frac{1}{2}\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx. \tag{14}$$

It follows from Equations (13) and (14) that

$$\frac{1}{2}\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx = \frac{\pi}{2}\left[1 - \frac{1}{\sqrt{\beta}}\right],$$

so that

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx = \pi\left[1 - \frac{1}{\sqrt{\beta}}\right].$$

This ends the proof of Proposition 2.5. □

Proof of Proposition 2.6. Proposition 2.5 ensures that

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right] dx = \pi\left[1 - \frac{1}{\sqrt{\beta}}\right].$$

Applying $\beta \to +\infty$, and developing only the right-hand side term, we get

$$\lim_{\beta\to+\infty}\int_0^{+\infty}\frac{1}{x\sqrt{x}}\left[\arctan\left(\frac{2\beta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right]dx=\lim_{\beta\to+\infty}\pi\left[1-\frac{1}{\sqrt{\beta}}\right]$$
$$=\pi. \tag{15}$$

For the left-hand side term, applying the dominated convergence theorem, and using the limit result $\lim_{x\to+\infty}\arctan(x)=\pi/2$ and the formula $\arctan(x)+\arctan(1/x)=\pi/2$ for $x>0$, we obtain

$$\begin{aligned}&\lim_{\beta\to+\infty}\int_0^{+\infty}\frac{1}{x\sqrt{x}}\left[\arctan\left(\frac{2\beta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right]dx\\&=\int_0^{+\infty}\frac{1}{x\sqrt{x}}\left[\lim_{\beta\to+\infty}\arctan\left(\frac{2\beta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right]dx\\&=\int_0^{+\infty}\frac{1}{x\sqrt{x}}\left[\frac{\pi}{2}-\arctan\left(\frac{2}{x}\right)\right]dx\\&=\int_0^{+\infty}\frac{1}{x\sqrt{x}}\arctan\left(\frac{x}{2}\right)dx.\end{aligned}\tag{16}$$

It follows from Equations (15) and (16) that

$$\int_0^{+\infty}\frac{1}{x\sqrt{x}}\arctan\left(\frac{x}{2}\right)dx=\pi.$$

This ends the proof of Proposition 2.6. □

Proof of Proposition 2.7. Making the change of variables $y=2\eta x$, we get

$$\int_0^{+\infty}\frac{1}{x\sqrt{x}}\arctan(\eta x)\,dx=\int_0^{+\infty}\frac{1}{[y/(2\eta)]\sqrt{y/(2\eta)}}\arctan\left(\frac{y}{2}\right)\left(\frac{1}{2\eta}dy\right)$$
$$=\sqrt{2\eta}\int_0^{+\infty}\frac{1}{y\sqrt{y}}\arctan\left(\frac{y}{2}\right)dx.$$

On the other hand, Proposition 2.6 gives

$$\int_0^{+\infty}\frac{1}{y\sqrt{y}}\arctan\left(\frac{y}{2}\right)dx=\pi.$$

So we have

$$\int_0^{+\infty}\frac{1}{x\sqrt{x}}\arctan(\eta x)\,dx=\pi\sqrt{2\eta}.$$

This ends the proof of Proposition 2.7. □

Proof of Proposition 2.8. It follows from Proposition 2.2 that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)} dx = \frac{\pi}{4\alpha\sqrt{\alpha}},$$

with $\alpha > 0$. By the reparameterization $\gamma = \alpha^2$, we obtain

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\gamma)} dx = \frac{\pi}{4\gamma^{3/4}}.$$

Considering γ as a variable, integrating both sides with respect to it from 0 to β, and developing only the right-hand side term, we get

$$\int_0^{\beta} \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\gamma)} dx\right] d\gamma = \frac{\pi}{4}\int_0^{\beta} \frac{1}{\gamma^{3/4}} d\gamma = \frac{\pi}{4}\left[4\gamma^{1/4}\right]_{\gamma=0}^{\gamma=\beta}$$
$$= \pi\beta^{1/4}. \tag{17}$$

For the left-hand side term, exchanging the integral symbols by the Fubini-Tonelli integral theorem, we get

$$\int_0^{\beta} \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\gamma)} dx\right] d\gamma = \int_0^{+\infty} \frac{1}{\sqrt{x}} \left[\int_0^{\beta} \frac{1}{x^2+4\gamma} d\gamma\right] dx$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{x}} \left[\frac{1}{4}\log\left(x^2+4\gamma\right)\right]_{\gamma=0}^{\gamma=\beta} dx$$
$$= \frac{1}{4}\int_0^{+\infty} \frac{1}{\sqrt{x}} \left[\log\left(x^2+4\beta\right) - \log\left(x^2\right)\right] dx$$
$$= \frac{1}{4}\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1+\frac{4\beta}{x^2}\right) dx. \tag{18}$$

It follows from Equations (17) and (18) that

$$\frac{1}{4}\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1+\frac{4\beta}{x^2}\right) dx = \pi\beta^{1/4},$$

so that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1+\frac{4\beta}{x^2}\right) dx = 4\pi\beta^{1/4}.$$

This concludes the proof of Proposition 2.8. □

Proof of Proposition 2.9. Proposition 2.3 ensures that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2} dx = \frac{3\pi}{64\alpha^3\sqrt{\alpha}},$$

with $\alpha > 0$. Considering α as a variable, integrating both sides with respect to it from 1 to θ (including the case $\theta \in (0,1)$), and developing only the right-hand side term, we obtain

$$\int_1^\theta \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx\right] d\alpha = \frac{3\pi}{64}\int_1^\theta \frac{1}{\alpha^3\sqrt{\alpha}}d\alpha = \frac{3\pi}{64}\left[-\frac{2}{5\alpha^2\sqrt{\alpha}}\right]_{\alpha=1}^{\alpha=\theta}$$
$$= \frac{3\pi}{160}\left[1-\frac{1}{\theta^2\sqrt{\theta}}\right]. \tag{19}$$

For the left-hand side term, exchanging the integral symbols by the Fubini-Tonelli integral theorem, we get

$$\int_1^\theta \left[\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^2}dx\right] d\alpha = \int_0^{+\infty} \frac{1}{\sqrt{x}}\left[\int_1^\theta \frac{1}{(x^2+4\alpha^2)^2}d\alpha\right] dx$$
$$= \int_0^{+\infty} \frac{1}{\sqrt{x}}\left[\frac{1}{4x^3}\left[\frac{2\alpha/x}{1+(2\alpha/x)^2}+\arctan\left(\frac{2\alpha}{x}\right)\right]\right]_{\alpha=1}^{\alpha=\theta} dx$$
$$= \frac{1}{4}\int_0^{+\infty} \frac{1}{x^3\sqrt{x}}\left[\frac{2\theta/x}{1+(2\theta/x)^2}-\frac{2/x}{1+(2/x)^2}+\arctan\left(\frac{2\theta}{x}\right)\right.$$
$$\left.-\arctan\left(\frac{2}{x}\right)\right] dx$$
$$= \frac{1}{4}\int_0^{+\infty} \frac{1}{x^3\sqrt{x}}\left[\frac{2\theta x}{x^2+4\theta^2}-\frac{2x}{x^2+4}+\arctan\left(\frac{2\theta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right] dx. \tag{20}$$

It follows from Equations (19) and (20) that

$$\frac{1}{4}\int_0^{+\infty} \frac{1}{x^3\sqrt{x}}\left[\frac{2\theta x}{x^2+4\theta^2}-\frac{2x}{x^2+4}+\arctan\left(\frac{2\theta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right] dx$$
$$= \frac{3\pi}{160}\left[1-\frac{1}{\theta^2\sqrt{\theta}}\right],$$

so that

$$\int_0^{+\infty} \frac{1}{x^3\sqrt{x}}\left[\frac{2\theta x}{x^2+4\theta^2}-\frac{2x}{x^2+4}+\arctan\left(\frac{2\theta}{x}\right)-\arctan\left(\frac{2}{x}\right)\right] dx$$
$$= \frac{3\pi}{40}\left[1-\frac{1}{\theta^2\sqrt{\theta}}\right].$$

This ends the proof of Proposition 2.9. □

Proof of Proposition 2.10. Applying the Cauchy-Schwarz integral inequality, we obtain

$$\int_0^{+\infty} \frac{|\sin(\alpha x)|\sqrt{\arctan(\eta x)}}{x\sqrt{x}}dx = \int_0^{+\infty} \sqrt{\frac{\sin^2(\alpha x)}{x\sqrt{x}}}\sqrt{\frac{\arctan(\eta x)}{x\sqrt{x}}}dx$$
$$\leq \sqrt{\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}}dx}\sqrt{\int_0^{+\infty} \frac{\arctan(\eta x)}{x\sqrt{x}}dx}. \tag{21}$$

Propositions 2.1 and 2.7 give

$$\int_0^{+\infty} \frac{\sin^2(\alpha x)}{x\sqrt{x}}dx = \sqrt{\alpha\pi}, \quad \int_0^{+\infty} \frac{\arctan(\eta x)}{x\sqrt{x}}dx = \pi\sqrt{2\eta}. \tag{22}$$

It follows from Equations (21) and (22) that

$$\int_0^{+\infty} \frac{|\sin(\alpha x)|\sqrt{\arctan(\eta x)}}{x\sqrt{x}}dx \leq \sqrt{\sqrt{\alpha\pi}\pi\sqrt{2\eta}} = (2\alpha\eta\pi)^{1/4}\sqrt{\pi}.$$

This concludes the proof of Proposition 2.10. □

Proof of Proposition 2.11. The Cauchy-Schwarz integral inequality gives

$$\int_0^{+\infty} \frac{1}{x(x^2+4\alpha^2)^{3/2}}\sqrt{\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)}dx$$
$$= \int_0^{+\infty} \sqrt{\frac{1}{\sqrt{x}(x^2+4\alpha^2)^3}}\sqrt{\frac{1}{x\sqrt{x}}\left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right]}dx$$
$$\leq \sqrt{\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3}dx} \times$$
$$\sqrt{\int_0^{+\infty} \frac{1}{x\sqrt{x}}\left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right]dx}. \tag{23}$$

Propositions 2.4 and 2.5 imply that

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^3}dx = \frac{21\pi}{2048\alpha^5\sqrt{\alpha}} \tag{24}$$

and

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}}\left[\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)\right]dx = \pi\left[1 - \frac{1}{\sqrt{\beta}}\right]. \tag{25}$$

It follows from Equations (23), (24) and (25) that

$$\int_0^{+\infty} \frac{1}{x(x^2+4\alpha^2)^{3/2}} \sqrt{\arctan\left(\frac{2\beta}{x}\right) - \arctan\left(\frac{2}{x}\right)} dx$$
$$\leq \sqrt{\frac{21\pi}{2048\alpha^5\sqrt{\alpha}}\pi\left[1-\frac{1}{\sqrt{\beta}}\right]} = \frac{\pi}{32\alpha^{11/4}}\sqrt{\frac{21}{2}}\sqrt{1-\frac{1}{\sqrt{\beta}}}.$$

This ends the proof of Proposition 2.11. □

Proof of Proposition 2.12. Making a suitable decomposition of the integrand and using the Hölder integral inequality at the parameters p and q, we obtain

$$\int_0^{+\infty} |\sin(\alpha x)|^{2/p} f(x) dx = \int_0^{+\infty} \frac{|\sin(\alpha x)|^{2/p}}{x^{3/(2p)}} x^{3/(2p)} f(x) dx$$
$$\leq \left[\int_0^{+\infty} \frac{\sin(\alpha x)}{x\sqrt{x}} dx\right]^{1/p} \left[\int_0^{+\infty} x^{3q/(2p)} f^q(x) dx\right]^{1/q}. \tag{26}$$

Using Proposition 2.1 and the identity $q/p = q-1$, we find that

$$\left[\int_0^{+\infty} \frac{\sin(\alpha x)}{x\sqrt{x}} dx\right]^{1/p} \left[\int_0^{+\infty} x^{3q/(2p)} f^q(x) dx\right]^{1/q}$$
$$= \left[\sqrt{\alpha\pi}\right]^{1/p} \left[\int_0^{+\infty} x^{3(q-1)/2} f^q(x) dx\right]^{1/q}. \tag{27}$$

It follows from Equations (26) and (27) that

$$\int_0^{+\infty} |\sin(\alpha x)|^{2/p} f(x) dx \leq (\alpha\pi)^{1/(2p)} \left[\int_0^{+\infty} x^{3(q-1)/2} f^q(x) dx\right]^{1/q}.$$

This concludes the proof of Proposition 2.12. □

Proof of Proposition 2.13. Making a suitable decomposition of the integrand via the identity $1/p + 1/q = 1$, and using the Hölder integral inequality, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)g(y) dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \left[x^{1/(2q)} y^{-1/(2p)} \log^{1/p}\left(1+\frac{\omega}{x^2y^2}\right) f(x)\right] \times$$
$$\left[x^{-1/(2q)} y^{1/(2p)} \log^{1/q}\left(1+\frac{\omega}{x^2y^2}\right) g(y)\right] dxdy$$
$$\leq \mathfrak{A}^{1/p}\mathfrak{B}^{1/q}, \tag{28}$$

where

$$\mathfrak{A} = \int_0^{+\infty} \int_0^{+\infty} x^{p/(2q)} \frac{1}{\sqrt{y}} \log\left(1 + \frac{\omega}{x^2y^2}\right) f^p(x) dx dy$$

and

$$\mathfrak{B} = \int_0^{+\infty} \int_0^{+\infty} y^{q/(2p)} \frac{1}{\sqrt{x}} \log\left(1 + \frac{\omega}{x^2y^2}\right) g^q(y) dx dy.$$

Let us study $\mathfrak{A}$ and $\mathfrak{B}$, one after the other.

For $\mathfrak{A}$, exchanging the integral symbols by the Fubini-Tonelli integral theorem, applying Equation (1) with $\epsilon = \omega/x^2$, and using the identity $p/(2q) = (p-1)/2$, we obtain

$$\begin{aligned}
\mathfrak{A} &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{1}{\sqrt{y}} \log\left(1 + \frac{\omega}{x^2y^2}\right) dy\right] dx \\
&= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[2\pi\sqrt{2}\left(\frac{\omega}{x^2}\right)^{1/4}\right] dx \\
&= 2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) dx \\
&= 2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \qquad (29)
\end{aligned}$$

For $\mathfrak{B}$, we perform the same developments but applying Equation (1) with $\epsilon = \omega/y^2$, which gives

$$\begin{aligned}
\mathfrak{B} &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{\omega}{x^2y^2}\right) dx\right] dy \\
&= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[2\pi\sqrt{2}\left(\frac{\omega}{y^2}\right)^{1/4}\right] dy \\
&= 2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) dy \\
&= 2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (30)
\end{aligned}$$

It follows from Equations (28), (29), and (30) that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{\omega}{x^2y^2}\right) f(x)g(y) dx dy \\
&\leq \left[2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[2\pi\sqrt{2}\omega^{1/4} \int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q} \\
&= 2\pi\sqrt{2}\omega^{1/4} \left[\int_0^{+\infty} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 2.13. □

Proof of Proposition 2.14. Let us set

$$\mathfrak{C} = \int_0^{+\infty} y^{-(q/2-1)(p-1)} \left[\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^p dy.$$

Making a suitable decomposition of the integrand and using the Fubini-Tonelli integral theorem, we obtain

$$\begin{aligned}\mathfrak{C} &= \int_0^{+\infty} y^{-(q/2-1)(p-1)} \left[\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^{p-1} \times \\ &\left[\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right] dy \\ &= \int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)g_\diamond(y)dxdy, \end{aligned} \tag{31}$$

where

$$g_\diamond(y) = \left[y^{-(q/2-1)} \int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^{p-1}.$$

It follows from Proposition 2.13 applied to the functions f and $g_\diamond$ that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)g_\diamond(y)dxdy \\ &\leq 2\pi\sqrt{2}\omega^{1/4} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx \right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g_\diamond^q(y)dy \right]^{1/q}. \end{aligned} \tag{32}$$

Let us now work on the last integral term. Using the identity $q(p-1) = p$, we get

$$\begin{aligned}&\int_0^{+\infty} y^{q/2-1} g_\diamond^q(y)dy \\ &= \int_0^{+\infty} y^{q/2-1} \left[y^{-(q/2-1)} \int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^{q(p-1)} dy \\ &= \int_0^{+\infty} y^{q/2-1} \left[y^{-(q/2-1)} \int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^{p} dy \\ &= \int_0^{+\infty} y^{-(q/2-1)(p-1)} \left[\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx \right]^{p} dy \\ &= \mathfrak{C}. \end{aligned} \tag{33}$$

It follow from Equations (31), (32) and (33) that

$$\mathfrak{C} \leq 2\pi\sqrt{2}\omega^{1/4} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx \right]^{1/p} \mathfrak{C}^{1/q}.$$

The identity $1/p = 1 - 1/q$ yields

$$\mathfrak{C}^{1/p} = \mathfrak{C}^{1-1/q} \leq 2\pi\sqrt{2}\omega^{1/4}\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p},$$

which is equivalent to

$$\int_0^{+\infty} y^{-(q/2-1)(p-1)}\left[\int_0^{+\infty} \log\left(1+\frac{\omega}{x^2y^2}\right) f(x)dx\right]^p dy$$
$$= \mathfrak{C} \leq \left[2\pi\sqrt{2}\omega^{1/4}\right]^p \int_0^{+\infty} x^{p/2-1} f^p(x)dx.$$

This ends the proof of Proposition 2.14. □

4 Open problems

Three open problems are formulated below.

Open problem 1: Based on Proposition 2.2 and using the Leibnitz integral theorem, we have obtained a formula for the integral

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(x^2+4\alpha^2)^n}dx,$$

with $\alpha > 0$ and $n = 2$ in Proposition 2.3, then with $n = 3$ in Proposition 2.4. On the same basis, we can derive the individual formula of this integral for $n = 4$, then $n = 5$,... However, a possible general formula depending on n remains to be determined, leading to an open problem.

Open problem 2: The Frullani integral is a classical result in real analysis. It provides an elegant formula for a certain class of improper integrals involving functions evaluated at scaled arguments.

The basic form of the Frullani integral is stated below. Let $f : (0, +\infty) \mapsto \mathbb{R}$ be a continuous function such that the limit

$$\lim_{x\to 0^+} f(x), \quad \lim_{x\to +\infty} f(x)$$

exist and are finite. Then, for any $a, b > 0$, we have

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = \left[\lim_{x\to 0^+} f(x) - \lim_{x\to +\infty} f(x)\right] \log\left(\frac{b}{a}\right).$$

This result thus transforms an improper integral into a simple expression involving the logarithm of the ratio b/a and the boundary values of f.

On the other hand, the formula in Proposition 2.5 ensures that, for any $\beta > 0$, we have

$$\int_0^{+\infty} \frac{1}{x\sqrt{x}} \left[f\left(\frac{2\beta}{x}\right) - f\left(\frac{2}{x}\right) \right] dx = \pi \left[1 - \frac{1}{\sqrt{\beta}} \right],$$

with $f(x) = \arctan(x)$. We can draw a parallel between this formula involving a difference of functions and that of the Frullani integral, although the setting is slightly different and there is no logarithmic term on the right-hand side. We can assume that a variant of the Frullani integral can be obtained of which Proposition 2.5 is a special case, but the details require further investigation.

Open problem 3: From a convergence point of view, the following integral is well defined:

$$\mathfrak{D}(\alpha) = \int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{1}{x^{\alpha}}\right) dx,$$

with $\alpha > 1/2$. We have found in Proposition 2.8 that $\mathfrak{D}(2) = 2\pi\sqrt{2} \approx 8.8857$. With a numerical software, we can calculate approximate values, such as $\mathfrak{D}(1) \approx 6.28319$, $\mathfrak{D}(1.5) \approx 7.2552$, or $\mathfrak{D}(2.5) \approx 10.6896$, among others. However, if it exists, the exact formula behind $\mathfrak{D}(\alpha)$ is, to our knowledge, an open problem.

5 Conclusion

Finally, this article presents new, explicit integral formulas that have been derived using a combination of classical and advanced techniques. Applications are given for integral inequalities, including an innovative logarithmic Hardy-Hilbert-type integral inequality. Additionally, three open problems are proposed to encourage further exploration in this area. Our findings emphasize the importance of ongoing research into integrals and inequalities, thereby enriching the field of mathematical analysis.

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