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Characterization of the M-Projective Curvature Tensor in Sasakian Manifolds with General Connections

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Abstract

This paper aims to investigate specific properties of Sasakian manifolds with respect to a general connection. Furthermore, we provide a characterization of Sasakian manifolds that admit a general connection, using the M-projective curvature tensor. The study concludes with an example focusing on three-dimensional Sasakian manifolds.

Keywords: *Sasakian manifold, $M-$projective curvature tensor, general connection.*

2010 Mathematics Subject Classification: 53C15, 53C25.

1 Introduction

The concept of a Sasakian structure was introduced by S. Sasaki in 1960 [8]. A contact manifold is called a Sasakian manifold if it is normal. In certain aspects, Sasakian manifolds can be regarded as the odd-dimensional counterparts of Kähler manifolds. In 1971, Pokhariyal and Mishra [7] introduced a tensor field M on a Riemannian manifold, defined as follows:

$$\begin{aligned} M(D,E)F &= R(D,E)F - \frac{1}{2(n-1)}[Ric(E,F)D - Ric(D,F)E] \quad (1) \\ &- \frac{1}{2(n-1)}[g(E,F)QD - g(D,F)QE], \end{aligned}$$

for all $D, E, F \in \chi(M)$, such a tensor field M is referred to as the M-projective curvature tensor, Here, $R(D, E)F$ denotes the Riemannian curvature tensor of type $(0, 3)$, Ric represents the Ricci tensor of type $(0, 2)$ and Q is the Ricci operator.

The properties of the M -projective curvature tensor in Sasakian and Kahler manifolds were studied by Ojha [5]. Furthermore, in [6], it was demonstrated that this tensor serves as a unifying framework that connects the conformal, conharmonic, and concircular curvature tensors. Singh [10] established that an M-projectively flat para-Sasakian manifold is necessarily an Einstein manifold. He further showed that if a para-Sasakian Einstein manifold satisfies the condition $R(\xi, D)M = 0$then it is locally isometric to a unit sphere $S^n(1)$.

The symbols ∇^S, ∇^G, ∇^F, ∇^T, ∇^q, and ∇ stands for Schouten-Van Kamper connection , the General connection, the Zamkovoy connection, the Generalized Tanaka-Webster connection, the quarter-symmeteric connection, and the Levi-civita connection respectively.

The general connection ∇^G is defined as

$$\nabla^G_D E = \nabla_D E + K_1[(\nabla_D \eta)(E)\xi - \eta(E)\nabla_D \xi] + K_2 \eta(D)\phi E \tag{2}$$

for all $D, E \in \chi(M)$ and the pair (K_1, K_2) being real constants. The beauty of such connection ∇^G lies in the fact that it has the flavour of

1. quarter symmetric metric connection([3], [1]) for $(K_1, K_2) \equiv (0, -1)$.
2. Schouten-Van Kampen connection [9] for $(K_1, K_2) \equiv (1, 0)$.
3. Tanaka Webster connection [11] for $(K_1, K_2) \equiv (1, -1)$.
4. Zamkovoy connection [11] for $(K_1, K_2) \equiv (1, 1)$.

In [2], Biswas, Das, Baishya and Bakshi studied $\eta-$Ricci solitons on Kenmotsu manifolds admitting a general connection. Additionally, Mandal and Das [4] studied the M-Projective curvatue tensor in Sasakian manifolds admitting Zamkovoy connection.

In a Sasakian manifold M of dimension $(n > 2)$, the $M-$projective curvature tensor M^G with respect to the general connection ∇^G is given by

$$\begin{aligned} M^G(D, E)F &= R^G(D, E)F - \frac{1}{2(n-1)}[Ric^G(E, F)D - Ric^G(D, F)E] \qquad (3) \\ &- \frac{1}{2(n-1)}[g(E, F)Q^G D - G(D, F)Q^G E], \end{aligned}$$

where R^G, Ric^G and Q^G denote the Riemannian curvature tensor, the Ricci tensor, and the Ricci operator, respectively, all defined with respect to the general connection ∇^G.

Definition 1.1 *[2] An $n-$dimensional Sasakian manifold M is said to be an $\eta-$Einstein manifold if its Ricci tensor satisfies $Ric(D,E) = K_1 g(D,E) + K_2\eta(D)\eta(E)$, for all vector fields D, $E \in \chi(M)$ where K_1 and K_2 are scalars on M*

Definition 1.2 *[2] An $n-$dimensional Sasakian manifold M is called M-projectively flat if its M-projective curvature tensor vanishes identically (that is, $M^G = 0$).*

Definition 1.3 *[2] An $n-$dimensional Sasakian manifold M is said to be ξ-M-projectively flat if the $M^G(D,E)\xi = 0$ for all $D, E \in \chi(M)$.*

2 Preliminaries

In this section, we present several definitions and fundamental concepts that will be used throughout the paper. Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta$,g) where ϕ is a $(1,1)$ tensor field, ξ, is a vector field, η is a 1-form η and g is a Riemannian metric, satisfying the following conditions:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta o \phi = 0, \tag{4}$$

where I denotes the identity endomorphism. A smooth manifold M endowed with an almost contact structure is called an almost contact manifold. A Riemannian metric g on M is said to be compatible with an almost contact structure (ϕ, ξ, η, g) it satisfies the condition:

$$g(D,E) = g(\phi D, \phi E) + \eta(D)\eta(E) \tag{5}$$

$$g(\phi D, E) = -g(D, \phi E). \tag{6}$$

for all vector fields D and E on M.
A K-contact manifold is a contact metric manifold Conversely, a contact metric manifold is K-contact if and only if the Lie derivative of ϕin the direction of the characteristic vector field ξ vanishes identically, i.e., identically .
A Sasakian manifold is a always a K-conatct manifold. vanishes identically, K-contact manifold is Sasakian manifold.
It is well known that a contact metric manifold is Sasakian if and only if

$$R(D,E)\xi = \eta(E)D - \eta(D)E. \tag{7}$$

In a Sasakian manifold equipped with the structure (ϕ, ξ, η, g), the following relations also hold [4]:

$$(\nabla_D \eta)E = g(D, \phi E) \tag{8}$$

$$R(\xi, D)E = g(D, E)\xi - \eta(E)D \tag{9}$$

$$Ric(D, \xi) = (n-1)\eta(D) \tag{10}$$

$$R(D, \xi)E = \eta(E)D - g(D, E)\xi \tag{11}$$

$$Q\xi = (n-1)\xi. \tag{12}$$

3 Main results

3.1 Some properties of Sasakian manifold admitting general connection

Proposition 3.1 *Let M be an $n-$dimensional Sasakian manifold equipped with a general connection ∇^G. Then:*

1. *the curvature tensor R^G of ∇^G is given by equation* (13).
2. *the Ricci tensor Ric^G of ∇^G is given by equation* (14).
3. *the scalar curvature r^G of ∇^G is given by equation* (17).
4. *the Ricci tensor Ric^G of ∇^G is symmetric.*

Proof: *Let the curvature tensor R^G corresponding to the general connection ∇^G be defined by $R^G(D, E)F = \nabla^G_D \nabla^G_E F - \nabla^G_E \nabla^G_D F - \nabla^G_{[D,E]} F$ for all $D, E, F \in \chi(M)$. Using equations (2) and (5) through (11)together with the definition above, we obtain the expression for the Riemannian curvature tensor with respect to the general connection ∇^G as follows:*

$$\begin{aligned} R^G(D, E)F &= R(D, E)F + (K_1 + K_2 - K_1K_2)((g(D, F)\eta(E) \\ &- g(E, F)\eta(D))\xi + (\eta(D)E - \eta(E)D)\eta(F)) \\ &+ (2K_1 - K_1^2)g(D, \phi F)\phi E + 2K_2(g(D, \phi E)\phi F). \end{aligned} \tag{13}$$

Consequently, one can easily bring out the following results:

$$\begin{aligned} Ric^G(E, F) &= Ric(E, F) + (K_1(1 - K_1) + K_2(K_1 + 1))g(E, F) \\ &+ (K_1(K_1 - n) + (n-2)K_1K_2 - nK_2)\eta(E)\eta(F), \end{aligned} \tag{14}$$

$$Ric^G(E, \xi) = (n-1)(1 - K_1 - K_2 + K_1K_2)\eta(E), \tag{15}$$

$$\begin{aligned} Q^G E &= QE + (K_1(1-K_1) + K_2(K_1+1))E \\ &+ (K_1(K_1-n) + (n-2)K_1K_2 - nK_2)\eta(E)\xi, \end{aligned} \tag{16}$$

$$r^G = r - (n-1)K_1^2 + 2(n-1)K_1K_2, \tag{17}$$

$$R^G(D,E)\xi = (K_1K_2 - K_1 - K_2 + 1)(\eta(E)D - \eta(D)E), \tag{18}$$

$$R^G(\xi,E)F = (K_1K_2 - K_1 - K_2 + 1)(g(E,F)\xi - \eta(F)E), \tag{19}$$

$$R^G(D,\xi)F = (K_1 + K_2 - K_1K_2 - 1)(g(D,F)\xi - \eta(F)D). \tag{20}$$

Theorem 3.2 *If a Sasakian manifold M is Ricci flat with respect to the general connection ∇^G, then M is an η-Einstein manifold.*

Proof: *Assume that the Sasakian manifold M is Ricci flat with respect to the general connection ∇^G, then from (14), the Ricci tensor takes the form:*

$$\begin{aligned} Ric(E,F) &= (K_1(1-K_1) + K_2(K_1+1))g(E,F) + (K_1(K_1-n) \\ &+ (n-2)K_1K_2 - nK_2)\eta(E)\eta(F), \end{aligned}$$

which shows that M is an η-Einstein manifold.

3.2 M-Projectively flat Sasakian manifold with respect to the general connection

Theorem 3.3 *Let M be an n-dimensional Sasakian manifold with $M(n>2)$ that is M-projectively flat with respect to the general connection ∇^G. Then M is an $\eta-$Einstein manifold.*

Proof: *Let M be an n-dimensional M-projectively flat Sasakian manifold with respect to general connection, i.e., $M^G = 0$. Then from (3), we have*

$$\begin{aligned} R^G(D,E)F &= \frac{1}{2(n-1)}[Ric^G(E,F)D - Ric^G(D,F)E] \\ &+ \frac{1}{2(n-1)}[g(E,F)Q^G D - G(D,F)Q^G E]. \end{aligned} \tag{21}$$

Taking the inner product of equation (21) with a vector field V, and then contracting over the vector fields D and V, we obtain:

$$nRic^G(E,F) = g(E,F)r^G. \tag{22}$$

Using equations (14), (17) in (22), we obtain:

$$\begin{aligned} Ric(E,F) &= \frac{1}{n}(r + (K_1-n)K_1 - nK_2 + (n-2)K_1K_2)g(E,F) \\ &+ (n(K_1+K_2) - K_1^2 + (2-n)K_1K_2)\eta(E)\eta(F). \end{aligned}$$

Hence, M is an η-Einstein manifold.

Theorem 3.4 *Let M be an n-dimensional Sasakian manifold with $(n > 2)$ which is $\xi - M$-projectively flat with respect to the general connection ∇^G Sasakian manifold. Then M is an η-Einstein manifold.*

Proof: *If $M(n > 2)$ is $\xi - M-$projectively flat with respect to the general connection,*
i.e., $M^G(D,E)\xi = 0$, for all vector fields $D, E \in \chi(M)$.
Then, from (3), we have

$$R^G(D,E)\xi = \frac{1}{2(n-1)}[Ric^G(E,\xi)D - Ric^G(D,\xi)E + \eta(E)Q^GD - \eta(D)Q^GE].$$

Taking the inner product of this equation with an arbitrary vector field V, it follows that

$$\begin{aligned}(K_1K_2 - K_1 &- K_2 + 1)(\eta(E)g(D,V) - \eta(D)g(E,V)) \\ &= \frac{1}{2(n-1)}(Ric^G(E,\xi)g(D,V) - Ric^G(D,\xi)g(E,V) \\ &+ \eta(E)g(Q^GD,V) - \eta(D)g(Q^GE,V)).\end{aligned} \tag{23}$$

Setting $E = \xi$ and using equations (15) and (16) in (23), we get

$$\begin{aligned}Ric(D,V) &= ((n-2)K_1K_2 + (n-1) - nK_1 - nK_2 + K_1^2)g(D,V) \\ &+ (n(K_1+K_2) + (2-n)K_1K_2 - K_1^2)\eta(D)\eta(V).\end{aligned} \tag{24}$$

Therefore, M is an η-Einstein manifold.

Corollary 3.5 *If a Sasakian manifold $M(n > 2)$ admits a general connection ∇^G is $\xi - M-$projectively flat, then its scalar curvature r is constant.*

Let $\{e_i\}$ $(1 \le i \le n)$ be an orthonormal basis of the tangent space at an arbitrary point M. Setting $E = F = e_i$ in the equation (24) and summing over i, $1 \le i \le n$, we obtain: $r = (n-1)K_1^2 + (n^2 - 3n + 2)K_1K_2 + n(1-n)(K_1 + K_2 - 1)$,
which is constant.

Theorem 3.6 *An $n-$dimensional Sasakian manifold is $\xi - M-$projectively flat with respect to the general connection if and only if it is so with respect to Levi-civita connection, provided that the vector fields involved are horizontal.*

Proof: *Using equations (14) and (16), we have*

$$\begin{aligned}&M^G(D,E)F = M(D,E)F + (K_1 + K_2 - K_1K_2)(g(D,F)\eta(E)\xi - g(E,F)\eta(D)\xi) \\ &+ (K_1^2 - 2K_1)(g(E,\phi F)\phi D - g(D,\phi F)\phi E) + (K_1K_2 - K_1 - K_2)(\eta(E)\eta(F)D \\ &- \eta(D)\eta(F)E) + 2K_2g(D,\phi E)\phi F - \frac{1}{2(n-1)}\big[2(K_1(1-K_1) + K_2(K_1+1)) \\ &\times g(E,F)D - g(D,F)E\big] + (K_1(K_1-n) + (n-2)K_1K_2 - nK_2)(\eta(E)\eta(F)D \\ &- \eta(D)\eta(F)E + \eta(D)g(E,F)\xi - \eta(E)g(D,F)\xi).\end{aligned} \tag{25}$$

Setting $F=\xi$ in (25), we obtain

$$M^G(D,E)\xi = M(D,E)\xi + (K_1K_2 - K_1 - K_2)(\eta(E)D - \eta(D)E)$$
$$-\tfrac{1}{2(n-1)}((2-n)K_1 + (2-n)K_2 + nK_1K_2 - K_1^2)(\eta(E)D - \eta(D)E).$$

If D and E are horizantal vector fields (i.e., $\eta(D)=\eta(E)=0$), then from above equation above it follows that $M^G(D,E)\xi = M(D,E)\xi$. That is, the $\xi - M$-projective curvature tensors with respect to the general connection and the Levi-Civita connection coincide when evaluated on horizontal vector fields.

Theorem 3.7 *A Sasakian manifold $M(n>2)$ is locally M-projectively ϕ-symmetric with respect to the general connection ∇^G if and only if it is so with respect to Levi-civita connection.*
Proof: *Using (2), we have*

$$\begin{aligned}
(\nabla^G_V M^G)(D,E)F &= (\nabla_V M^G)(D,E)F + K_1 g(V,\phi M^G(D,E)F)\xi \\
&+ K_1\eta(M^G(D,E)F)\phi V + K_2\eta(V)\phi M^G(D,E)F - K_1 g(V,\phi D)M^G(\xi,E)F \\
&- K_1\eta(D)M^G(\phi V,E)F - \eta(V)K_2 M^G(\phi D,E)F - K_1 g(V,\phi E)M^G(D,\xi)F \quad (26)\\
&- K_1\eta(E)M^G(D,\phi V)F - K_2\eta(V)M^G(D,\phi E)F - K_1 g(V,\phi F)M^G(D,E)\xi \\
&- K_1\eta(F)M^G(D,E)\phi V - K_2\eta(V)M^G(D,E)\phi F.
\end{aligned}$$

By differntiating (2), (13), (14), and (16) with respect to V, we obtain:

$$\begin{aligned}
(\nabla_V M^G)(D,E)F &= (\nabla_V R^G)(D,E)F - \frac{1}{2(n-1)}[(\nabla_V Ric^G)(E,F)D \\
&- (\nabla_V Ric^G)(D,F)E + g(E,F)(\nabla_V Q^G)D - g(D,F)(\nabla_V Q^G)E],
\end{aligned} \tag{27}$$

$$\begin{aligned}
(\nabla_V R^G)(D,E)F &= (\nabla_V R)(D,E)F + (K_1 + K_2 - K_1K_2)(g(D,F)g(V,\phi E)\xi \\
&- g(D,F)\eta(E)\phi V - g(E,F)g(V,\phi D)\xi + g(E,F)\eta(D)\phi V + g(V,\phi D)\eta(F)E \\
&+ g(V,\phi F)\eta(D)E - g(V,\phi E)\eta(F)D - \eta(E)g(V,\phi F)D) + 2K_2(g(V,E)\eta(D)\phi F \\
&- \eta(E)g(D,V)\phi F + g(D,\phi E)g(V,F)\xi - g(D,\phi E)\eta(F)V) + (K_1^2 - 2K_1) \\
&\times (g(V,\phi F)\eta(E)\phi D - \eta(F)g(E,V)\phi D + g(E,\phi F)g(V,\phi D)\xi - g(E,\phi F)\eta(D)V \\
&- g(V,\phi F)\eta(D)\phi E + \eta(F)g(D,V)\phi E - g(D,\phi F)g(V,\phi E)\xi + g(D,\phi E)\xi \\
&+ g(D,\phi F)\eta(E)V),
\end{aligned} \tag{28}$$

$$\begin{aligned}
(\nabla_V Ric^G)(E,F) &= (\nabla_V Ric)(E,F) + (K_1(K_1-n) + (n-2)K_1K_2 \\
&- nK_2)(g(V,\phi E)\eta(F) + g(V,\phi F)\eta(E)),
\end{aligned} \tag{29}$$

$$\begin{aligned}
(\nabla_V Q^G)D &= (\nabla_V Q)D + (K_1(K_1-n) + (n-2)K_1K_2 \\
&- nK_2)(g(V,\phi D)\xi - \eta(D)\phi V),
\end{aligned} \tag{30}$$

respectively.
Now combining the equations (27)-(30), we obtain:

$$
\begin{aligned}
&(\nabla_V M^G)(D,E)F = (\nabla_V R)(D,E)F + (K_1 + K_2 - K_1K_2)(g(D,F)(g(V,\phi E)\xi \\
&- \eta(E)\phi V) - g(E,F)(g(V,\phi D)\xi - \eta(D)\phi V) + g(V,\phi D)\eta(F)E + g(V,\phi F) \\
&\times \eta(D)E - g(V,\phi E)\eta(F)D - g(V,\phi F)\eta(E)D) + 2K_2((g(E,V)\eta(D) \\
&- g(D,V)\eta(E))\phi F + g(D,\phi E)(g(V,F)\xi - \eta(F)V)) + (K_1^2 - 2K_1) \\
&\times ((g(V,\phi F)\eta(E) - g(E,V)\eta(F))\phi D + g(E,\phi F)(g(V,\phi D)\xi - \eta(D)V) \\
&+ (g(D,V)\eta(F) - g(V,\phi F)\eta(D))\phi E + g(D,\phi F)(\eta(E)V - g(V,\phi E)\xi)) \qquad (31)\\
&- \frac{1}{2(n-1)}[(\nabla_V Ric)(E,F)D - (\nabla_V Ric)(D,F)E + g(E,F)(\nabla_V Q)D \\
&- g(D,F)(\nabla_V Q)E + (K_1(K_1 - n) + (n-2)K_1K_2 - nK_2)(g(V,\phi E)\eta(F)D \\
&- g(V,\phi D)\eta(F)E) + g(V,\phi F)(\eta(E)D - \eta(D)E) + g(E,F)(g(V,\phi D)\xi \\
&- \eta(D)\phi V) + g(D,F)(\eta(E)\phi V - g(V,\phi E)\xi)].
\end{aligned}
$$

Now differentiating (1) with respect to V, we have

$$
\begin{aligned}
(\nabla_V M)(D,E)F &= (\nabla_V M)(D,E)F - \frac{1}{2(n-1)}[(\nabla_V Ric)(E,F)D \qquad (32)\\
&- (\nabla_V Ric)(D,F)E + g(E,F)(\nabla_V Q)D - g(D,F)(\nabla_V Q)E].
\end{aligned}
$$

By use of (31), (32) takes the form

$$
\begin{aligned}
&(\nabla_V M^G)(D,E)F = (\nabla_V M)(D,E)F + (K_1 + K_2 - K_1K_2)(g(D,F)(g(V,\phi E)\xi \\
&- \eta(E)\phi V) - g(E,F)(g(V,\phi D)\xi - \eta(D)\phi V) + g(V,\phi D)\eta(F)E + g(V,\phi F) \\
&\times \eta(D)E - g(V,\phi E)\eta(F)D - g(V,\phi F)\eta(E)D) + 2K_2((g(E,V)\eta(D) \\
&- \eta(E)g(D,V))\phi F + g(D,\phi E)(g(V,F)\xi - \eta(F)V)) + (K_1^2 - 2K_1) \\
&\times ((g(V,\phi F)\eta(E) - g(E,V)\eta(F))\phi D + g(E,\phi F)(g(V,\phi D)\xi - \eta(D)V) \\
&- (g(V,\phi F)\eta(D) - g(D,V)\eta(F))\phi E) - (g(D,\phi F)(g(V,\phi E)\xi - \eta(E)V)) \qquad (33)\\
&- \frac{1}{2(n-1)}[(K_1(K_1 - n) + (n-2)K_1K_2 - nK_2)((g(V,\phi E)D - g(V,\phi D)E) \\
&\times \eta(F) + g(V,\phi F)(\eta(E)D - \eta(D)E) + g(E,F)(g(V,\phi D)\xi - \eta(D)\phi V) \\
&+ g(D,F)(\eta(E)\phi V - g(V,\phi E)g(D,F)\xi))]
\end{aligned}
$$

Applying ϕ^2 to both sides of equation (26), we obtain:

$$
\begin{aligned}
&\phi^2(\nabla^G_V M^G)(D,E)F = \phi^2(\nabla_V M^G)(D,E)F - K_1\eta(M^G(D,E)F)\phi V + K_2\eta(V) \\
&\times \phi^2(\phi(M^G(D,E)F)) - K_1\eta(D)\phi^2 M^G(\phi V,E)F - K_1\eta(E)\phi^2 M^G(D,\phi V)F \\
&- K_1\eta(F)\phi^2 M^G(D,E)\phi V - K_2\eta(V)\phi^2 M^G(D,E)\phi F + K_1 g(V,\phi D)\eta(F)\phi E \\
&- K_1 g(V,\phi^2 E)\eta(F)\phi^2 D + (K_1+K_2-K_1K_2)(K_1 g(V,\phi D)\eta(F)\phi^2 E \\
&+ K_1 g(V,\phi E)\eta(F)\phi^2 D) - \frac{1}{2(n-1)}\Big[2nK_1 g(V,\phi E)\eta(F)(\phi^2 E + \phi^2 D) \\
&+ (K_1(1-K_1)+K_2(K_1+1))2K_1\eta(F)g(V,\phi E)(\phi^2 E - \phi^2 D) + (K_1(K_1-n) \\
&+ (n-2)K_1K_2 - nK_2)(K_1 g(V,\phi D)\eta(F))(\phi^2 E + \phi^2(QE)) - K_1 g(V,\phi E)\eta(F) \\
&\times (\phi^2 D + \phi^2(QD))\Big] - K_1 g(V,\phi F)((\eta(D)\phi^2 E - \eta(E)\phi^2 D)(K_1+K_2-K_1K_2 \\
&- 1) - \frac{1}{2(n-1)}\Big[2n(\eta(E)\phi^2 D - \eta(D)\phi^2 E)(K_1(1-K_1)+K_2(K_1+1) \\
&+ K_1(K_1-n) + (n-2)K_1K_2 - nK_2 + 2)\Big],
\end{aligned} \tag{34}
$$

Using equation (33) and assuming that the vector fields D, E, F and V are orthogonal to ξ, we obtain:

$$\phi^2(\nabla^G_V M^G)(D,E)F = \phi^2(\nabla_V M)(D,E)F.$$

Hence the theorem.

Theorem 3.8 *A pseudo-M-projectively flat Sasakian manifold with respect to the general connection is an $\eta-$Einstein manifold with respect to the Levi-Civita connection.*

Proof: *Assume that a Sasakian manifold M is pseudo-M-projectively flat with respect to the general connection, i.e, $g(M^G(\phi D,E)F,\phi W) = 0$, for all vector fields $D,E,F,W \in \chi(M)$.*
Then in view of (3), we have:

$$
\begin{aligned}
R^G(\phi D,E,F,\phi W) &= \frac{1}{2(n-1)}[Ric^G(E,F)g(\phi D,\phi W) - Ric^G(\phi D,F)g(E,\phi W) \\
&+ g(E,F)Ric^G(\phi D,\phi W) - g(\phi D,F)Ric^G(E,\phi W)].
\end{aligned} \tag{35}
$$

Let $(e_i,\xi)(1 \le i \le n-1)$ be a local orthonormal basis of the tangent space at an arbitrary point of the manifold M. Since $(\phi e_i,\xi)(1 \le i \le n-1)$ also forms a local orthonormal basis, we set $D = W = e_i$ and taking summation over i

$(1 \leq i \leq n-1)$ *in (35), then we obtain:*

$$\begin{aligned} R^G(\phi e_i, E, F, \phi e_i) &= \frac{1}{2(n-1)}[\sum_{i=1}^{n-1} Ric^G(E,F)g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} Ric^G(\phi e_i, F)g(E, \phi e_i) \\ &+ \sum_{i=1}^{n-1} g(E,F)Ric^G(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi e_i, F)Ric^G(E, \phi e_i)]. \end{aligned}$$

From which we obtain:

$$Ric(E,F) = ag(E,F) + b\eta(D)\eta(E),$$

where $a = \frac{1}{n+2}[(n-4)(K_1(1-K_1)+K_2(K_1+1))+r-(n-1)-2(n-1)(K_1(1-K_1)+K_2(K_1+1))]$ *and* $b = -\frac{2(n-1)}{(n+2)}[(K_1(1-K_1)+K_2(K_1+1))+K_1(K_1-n)+(n-2)K_1K_2-nK_2]$.
Therefore, M *is an* $\eta-$*Einstein manifold.*

4 Example

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}\}$, where (x,y,z) are the standard coordinates on $\mathbb{R}^3$. Let e_1, e_2, e_3 be a linearly independent frame field on M, given by:

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

which is linearly independent at each point of M and hence form a basis of the tangent space T_pM. Let g be the Riemannian metric on M defined by:

$$g(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let η be the 1-form on M defined by $\eta(D) = g(D, e_3)$ for any vector field $D \in \chi(M)$. Define the $(1,1)$-tensor field ϕ by

$$\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0.$$

By the linearity of both ϕ and g we have

$$\phi^2 D = -D + \eta(D)\xi, \quad \eta(D) = g(D,\xi),$$

$$g(\phi D, \phi E) = g(D,E) - \eta(D)\eta(E).$$

Now, by direct computations, we can easily see that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.$$

Furthermore, by ∇, we denote the Levi-civita connection on M, by using Koszul's formula, we can calculate, easily

$$\begin{aligned} \nabla_{e_1} e_1 = e_3, \quad & \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \\ \nabla_{e_1} e_2 = 0, \quad & \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_3} e_2 = 0, \\ \nabla_{e_1} e_3 = -e_1, \quad & \nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

From the above resuts we see that the structure (ϕ, ξ, η, g) satisfies $(\nabla_D \phi)E = g(D, E)\xi - \eta(E)D, \forall D, E \in \chi(M^3)$, where $\eta(\xi) = \eta(e_3) = 1$. Hence (ϕ, ξ, η, g) is a $3-$dimensional Sasakian manifold. The non zero components of Riemannian curvature tensor with respect to Levi-civita connection ∇ are given by:

$$\begin{aligned} R(e_1, e_2)e_1 = e_2, \quad & R(e_1, e_3)e_1 = e_3, \quad R(e_2, e_3)e_1 = 0, \\ R(e_1, e_2)e_2 = -e_1, \quad & R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = e_3, \\ R(e_1, e_2)e_3 = 0, \quad & R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_3)e_3 = -e_2. \end{aligned}$$

Using the general connection ∇^G defined by equation (2), the covariant derivatives are computed as

$$\begin{aligned} \nabla^G_{e_1} e_1 = (1 + K_1)e_3, \quad & \nabla^G_{e_2} e_1 = 0, \quad \nabla^G_{e_3} e_1 = K_2 e_1, \\ \nabla^G_{e_1} e_2 = 0, \quad & \nabla^G_{e_2} e_2 = (1 + K_1)e_3, \quad \nabla^G_{e_3} e_2 = K_2 e_2, \\ \nabla^G_{e_1} e_3 = (K_1 - 1)e_1, \quad & \nabla^G_{e_2} e_3 = (K_1 - 1)e_2, \quad \nabla^G_{e_3} e_3 = 0. \end{aligned}$$

The non zero components of Riemannian curvature tensor with respect to general connection are given by:

$$\begin{aligned} R^G(e_1, e_2)e_1 = (1 - K_1^2)e_2, \quad & R^G(e_1, e_3)e_1 = (K_1 + 1)(1 + K_2)e_3, \quad R^G(e_2, e_3)e_1 = 0, \\ R^G(e_1, e_2)e_2 = (K_1^2 - 1)e_1, \quad & R^G(e_1, e_3)e_2 = 0, \quad R^G(e_2, e_3)e_2 = (1 + K_1)(1 + K_2)e_3, \\ R^G(e_1, e_2)e_3 = 0, \quad & R^G(e_1, e_3)e_3 = (K_1 - 1)(1 - K_2)e_1, \quad R^G(e_2, e_3)e_3 = (K_1 - 1)(1 - K_2)e_2. \end{aligned}$$

Using the above curvature tensor the Ricci tensors with respect to ∇ and ∇^G are: $Ric(e_1, e_1) = Ric(e_2, e_2) = Ric(e_3, e_3) = -2$.

$Ric^G(e_1, e_1) = K_1(K_1 + 1) + K_2(1 - K_1) - 2 = Ric^G(e_2, e_2)$,
$Ric^G(e_3, e_3) = -2(K_1 + K_2 + K_1K_2 + 1)$.
Finally, from these expressions, one verifies that
$M^G(e_1, e_2)e_3 = 0$.
which confirms that the manifold M is ξ-projectively flat with respect to the general connection.

5 Open Problem

An open problem is to identify Sasakian manifolds that naturally arise in contact and CR geometry, extending the analysis to contact metric manifolds or almost contact manifolds with a general connection could bridge results across related geometric structures.

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