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Lie Symmetry Analysis of Wave Equation Emanating from Collapse of Shafts in Power Transmission System

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Abstract

The problematic phenomena of an apparently unintentional beating and the potential collapse of shafts in power transmission systems was discovered by motor ship constructors. In this study, we examine a fourth order Ordinary Differential Equation (ODE), which shows how the collapse of shafts in power transmission networks occurs dynamically. The main focus in this work is to examine and find the solution to the wave equation arising due to collapse of shafts in power transmission systems using Lie symmetry.

Keywords: *Lie Symmetry, Wave equation, Power transmission, ODE.*

2010 Mathematics Subject Classification: Primary 22E20; secondary 34C10.

1 Introduction

Differential equations are prevalent in various natural phenomena and technological challenges [5]. These equations pertain to the behaviour of specific unidentified dependent variables at a particular point, such as time or place. These mathematical differential equations, which have been defined, possess solvable derivatives. Multiple endeavors have been undertaken to resolve these differential equations by employing numerical methods, namely utilizing the finite difference approach. Nevertheless, numerical solutions are merely estimations that rely on specific initial boundary conditions [1]. Consequently, they are not appropriate for resolving problems related to mechanical vibrations that require precise values. Therefore, this work offers an alternate analytical approach for solving equations related to the failure of shafts in power transmission systems and other mechanical issues, utilizing Lie symmetry. This has the potential to enhance understanding in the field of applied mathematics and serve as a pathway for additional research.

To study nonlinear differential equations and produce their exact and implicit solutions in a fully algorithmic manner, symmetry analysis of differential equations was created. Lie found that studying the associated vector fields, or infinitesimal generators, was the most effective method for comprehending Lie groups because an infinitesimal generator contains all of the information about transformations. Figuring out the small changes from the symmetry group gives us the group generator, which is the same as creating the group [6]. The Lie's algorithm used to analyze the symmetry of differential equations was further advanced through the efforts of [10] in the late 1950's.

According to [11], a differential equation can be made to reveal its symmetries which are then used to construct exact solutions [4]. Therefore, it can be claimed that symmetry approaches connect to a number of ODE-related subjects, such as the usage of laplace transformation and methods for varying parameters and indeterminate coefficients [2]. In-terms of applications, this work is useful in computer science in generating faster algorithms that could assist engineers in comprehending the oscillation of a vehicle caused by road fissures. This could facilitate engineers in enhancing the design of automobiles and other mobile engines.

The conventional integration methods for ODEs, according to Norwegian mathematician Lie, derive their solutions using the symmetries of the equations [8] which can be utilized in solving cyber security issues. Therefore, it is possible to make any differential equation reveal its symmetries, which are then utilized to build the exact solutions. Consequently, he came to the realization that both techniques might be combined and expanded upon to create a generic integration process predicated on the differential equation's invariance under a continuous set of symmetries[7].

2 Preliminaries and literature review

According to [9], Canonical variables is a strategy that relies on point symmetries to build transformations that simplify the equation before solving it. The first integrals and first order partial differential equation methods are strongly associated with the canonical variables method. Canonical transformations can be computed given a symmetry group. enabling the integration of first-order ODEs or lowering the order of ODEs of higher order. Furthermore, if a two-dimensional symmetry group is admitted by a second-order equation, we can directly modify the variables such that the equation becomes integrable rather than successively reducing the order. This modification of variables was dubbed the canonical variables approach by Lie [5].

A collection of variables is designated as canonical by Lie if the equation can be simplified and allows for integration by quadrature. Let's look at a set of transformations that comprise two generators, V_1 and V_2, in more detail. If the following relations do not exist: $V_1 = cV_2$ where c is a constant; and $(V_1, V_2) = c_1V_1 + c_2V_2$ where c_1 and c_2 are also constants, then two infinitesimal transformations, V_1 and V_2, are independent of one another. Assuming that the product may be represented differently by each of the two independent transformations, the second relation can be made simpler.

Very few equations admit enough point symmetries to allow for reduction to quadratures in an effort to get around this restriction. One such extension results from the observation of what are known as hidden symmetries, which are point symmetries that unintentionally develop. It has been demonstrated that solutions of certain equations lacking sufficient Lie point symmetries with the corresponding Lie algebras can be reached through hidden symmetries. Type I hidden symmetries can arise when an equation's order is increased, while Type II hidden symmetries can arise when an equation's order is decreased [8]. We provide a comprehensive summary of the research conducted by different academics about the derivation of equations for fourth-order differential equations in dynamic motion. Our focus is mostly be on the elucidation of the general equation and its correlation with the failure of shafts in power transmission systems.

Regarding prolongations, a systematic programme of applying the Lie continuous group of transformations methods of up to the third order has been considered. In our work we have looked up to the fourth order prolongation. We have applied the fourth extension to expand the wave equation. The work of [7] studied an ODE with a scalar coefficient which had a given number of Lie symmetries contained in a total of seven equivalent classes. However in our work we solve a fourth order ODE with a fourth degree which admits a second order symmetry [10].

According to [2], Lie developed the idea of continuous groups of transformations which has been termed as Lie groups; named after him to consolidate and extend a number of specialized approaches to solving ODEs [1]. Lie's work has systematically connected numerous topics and methods in ordinary differential equations. We now focus on the symmetry group which is one of the most significant groups in relation to DEs.

The authors in [9] introduced the idea of generalized conditional symmetry and further expanded on this approach. Group theory is used in all of these techniques for determining symmetries and related similarity reduction of a given PDE. The innovative characteristics direct algorithmic approach for identifying similarity reduction of PDEs are completely self-explanatory without the need for group analysis. A geometrical object's symmetry is defined by [3], as a transformation whose action appears to leave the object untouched. He also used symmetries to classify objects.

Moreover, the work of [8] extended the idea of functions to non-local symmetries of DEs. They applied a systematic approach to identify a particular class of non-local symmetries, known as potential symmetries. Contributors to this discussion on rotating shafts pointed out that the mapping techniques established significantly increased roles to DEs. Many scholars have studied the equations of shaft collapse as explained in different articles. The solution of a third order first degree non-linear ODE that arises was considered in the study of Lie symmetry. In contrast, we examined a fourth order ODE in our study that is a fourth degree polynomial in y_3 and admits second order symmetry.

On the general solution [11] examined novel single generators of Lie point symmetries that allow an ODE's order to be reduced once. He used two-point symmetries to describe a double reduction of order. The work of [8] discovered first integrals for higher order ODEs and demonstrated how to lower an ODE's order by utilizing higher order symmetries. Additionally, he contrasted several ODE integration and supplementary techniques. He handled dimensional analysis in great detail. In order to identify solutions of differential equations, different aspects of symmetries were described. The author focused on developing solutions and the first integral that resulted from these symmetries and integrating factors by using an explicit approach. The author included a full discussion of dimensional analysis in his work and used examples from physical and engineering difficulties, such as those involving heat conditions and wave propagation. He presents the reader with the Backingham pi-theorem, which presents the idea of invariance. He was able to show how this results in generalizations by showing how boundary value problems are invariant under changing scaling. This gets the reader ready to think about differential equations' more general invariance under transformation groups.

Many authors were able to establish fundamental ideas about functions and Lie algebras, which are required in the development of infinitesimal generators of various orders. He illustrated how, as stated in [1], a Lie group makes contact and higher order transformations easily regarded. This makes it possible to take differential equation integrating factors into account. He examined a reduction procedure for ODEs that lowers their n^{th} order to r quadrative and a $(n-r)^{th}$ symmetries [7]. He demonstrated the process of locating higher order, acknowledged point, and contact symmetries. Additionally, he demonstrated how to expand the reduction process to include symmetries and use corresponding to identify admitted first integrals. utilizing initial integrals to

get order reductions and integrating factors.
This greatly broadens and streamlines the traditional theorems for determining conservation laws, to include any ODE not just those that admit a variable principle [6]. Specifically, he demonstrated how to compute integrating factors using a variety of computational techniques that are similar to those used to compute variable that changes [5]. He made a clear comparison between the unique ways that admitted local symmetries and acknowledged integrating factors reduce order. He gave an example of how to solve boundary value problems using invariance under point symmetries. By examining their topological characteristics, he demonstrated that invariant solutions comprise separatrices and singular envelop solutions. He also developed an approach to create exceptional solutions, or invariant solutions, that arise from accepted.

3 Research methodology

We mainly focus on methods for finding solutions that stay the same under certain group changes, called Lie groups. The space that contains the system's independent and dependent variables is influenced by the Lie groups. Building generators for infinitesimal transformations, prolongations, determining equations and integrating factors.

3.1 Invariant transformation

When all of the group's transformations result in a point on curve C mapping into another point on the curve, that curve is said to be invariant. This means that the solutions to a certain differential equation stay the same when certain transformations are applied, specifically under a smaller group of transformations that the system allows [1]. In order for the infinitesimal co-efficients to simultaneously disappear, the curve must form an orbit [7]. The equation below can be used to parametrically express a family of curves with one parameter.

$$\phi(x, y) = C \tag{1}$$

where the function $\phi(x, y)$ defines the family and C defines the parameter that labels various curves of the family. We therefore say the family is invariant if the incase of each curve of it is another curve of the family. Any specific value of λ has to be true for the image points (x', y'). The solutions to these ODE systems are part of a smaller group within the larger group that the system allows as shown in the equation below.

$$\phi(x'.y') = \phi(X(x, y; \lambda), Y(x, y; \lambda)) = C', \tag{2}$$

where C' is a parameter different from C whose value depends on C and λ. If Equation 22 is partially differentiated with w.r.t λ along $\lambda = \lambda_0$, we find

$$\xi(x,y)\phi_1 + \eta(x,y)\phi = \left(\frac{\partial C'}{\partial \lambda}\right)_{\lambda=\lambda_0} \tag{3}$$

The RHS of Equation 3 is a function of C only, hence we can denote it by $F(C)$. Using Equation 2, Equation Equation 22 can be written as

$$\xi(x,y)\phi_x + \eta(x,y)\phi_y = f(\phi). \tag{4}$$

The representation of Equation 4 can equivalently be represented as

$$\psi(x,y) = G \tag{5}$$

for which ψ is a function of ϕ, that is, for which

$$\psi = G(\phi). \tag{6}$$

Similarly, an equivalence to representation Equation 5 from equation Equation 4 gives

$$\xi(Q_x + \eta Q_y) = (\xi\phi_x + \eta\phi_y)\frac{dG}{d\phi} = \frac{dG}{d\phi}F(\phi). \tag{7}$$

We therefore choose the function G and we thus have

$$G(\phi) = \int \frac{d\phi}{f\phi}. \tag{8}$$

The RHS of Equation 8 becomes 1, that is,

$$\xi\psi_x + \eta\psi_y = 1. \tag{9}$$

3.2 Invariance of Differential Equation

Known as invariant solutions of differential equations (DEs), invariant curves from the LGTs that these equations allow provide a helpful way to find their solutions. If there isn't a constant solution, we can make the difficulty of a regular differential equation easier by using the constants. In general, for any its invariance can be explained as follows:
Considering the ODE below,

$$F_n(x, y, y_1, ..., y_n) \tag{10}$$

admitting Lie groups with infinitesimal generators G_1, G_2 and G_3 :

(i). Determine $x_1(u,v)$, $v_1(u,v,v_1)$ and hence $v_{(1)1}$, invariants of G_1^2.

(ii). Apply G_2^2 to x_1, y_1, $y_{(1)1}$ to determine $\alpha_1(x_1)$, $\alpha_2(x_1, y_1)$, $\alpha_3(x_1, y_1, y_{(1)1})$, so that $G_2^2 = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial y_1} + \alpha_3 \frac{\partial}{\partial y_{(1)1}}$.

(iii). Find the invariants $x_2(x_1, y_1)$, $y_2(x_1, y_1, y_{(1)1})$ of G_2^2 and hence $y_{(2)1}G_2^3$.

(iv). Apply G_3^3 to x_2, y_2, $y_{(2)1}$ to find $\beta_1(x_2)$, $\beta_2(x_2, y_2)$, $\beta_3(x_2, y_2, y_{(2)1})$, so that $G_3^3 = \beta_1 \frac{\partial}{\partial x_2} + \beta_2 \frac{\partial}{\partial y_2} + \beta_3 \frac{\partial}{\partial y_{(2)1}}$.

3.3 Integrating Factors

The provided function takes the systems and multiplies them with the ordinary differential equation (ODE) to transform it into a precise form known as a total derivative form. Integrating factors provide a methodical way to reduce the complexity of an ODE by locating a primary integral [5]. Compared to reducing using point symmetries, the simpler ODE includes both the independent variables we get from it. Also, any starting integral can be found using an integrating part. This factor is decided by a math formula that looks at the variables and their rates of change up to a certain level. The factor is then multiplied using an ODE to change it into a specific form called a total derivative.

3.4 Adjoint symmetry

The ODE's solutions from linearization that hold true are known as adjoint symmetry (AS). For instance, for an AS to be an IF, there must be sufficient and necessary additional determining equations. Consequently, the study of the first integral of ODEs heavily relies on AS [10].
Given the following ODE

$$y^{(n)} = f(x, y, y'......y^{(n-1)}) \tag{11}$$

represented by the surface

$$F(x, y, y_1, ..., y_n) = y_n - f(x, y, y_1, ..., y_{n-1}) = 0. \tag{12}$$

The linearization operator (LO) of Equation 11 is given by

$$T_g = P^m - \sum_{j=0}^{m-1} f_{yi} P^j, \tag{13}$$

where

$$P = \frac{\partial}{\partial x} + \sum_{j=1}^{\infty} y_i \frac{\partial}{\partial y_{j-1}}.$$

The adjoint LO is given through integration by parts as

$$T_g^* = (-1)^m P^m - \sum_{j=0}^{m-1} \sum_{i=0}^{j} (-1)^j \frac{j!}{(j-i)!i!} (P^{j-f} f_{yj}) p^i. \tag{14}$$

Now, the operator in Equation 14 can be equated to

$$MT_g U - UT_g^* M = PQ[M, U, g], \tag{15}$$

where Q is the trilinear function defined by

$$Q[M, U; V] = \sum_{j-0}^{m-1} \sum_{i=0}^{j} (-1)^i (P^{j-i} UMV_{y_{j+1}}), \tag{16}$$

for arbitrary functions

$$V(x, y, y_1, ..., y_m), U(x, y, y_1, ..., y_m), M(x, y, y_1, ..., y_m).$$

In particular, self adjointness is equivalent to the $m + 1$ conditions

$$(-1)^m = 1, \tag{17}$$

that is to say m is even

$$f_{yj} = \sum_{i=0}^{m-j-1} (-1)^{j+i} \frac{(j+i)!}{j!i!} P^i f_{yj+i}, j = 0, 1...m-1 \tag{18}$$

Let

$$T_g = T_g|_{g=0} = P^m - \sum_{j=0}^{m-1} f y_j P^j \tag{19}$$

and

$$T_g^* = T_g^*|_{g=0} = (-1)P^m - \sum_{j=0}^{m-1} \sum_{i=0}^{j} (-1)^j \frac{j!}{(j-i)!i!} (P^{j-i} f_{yj}) P^i \tag{20}$$

where

$$P = P|_{g=0} = \frac{\partial}{\partial x} + \sum_{j=1}^{m-1} Yj \frac{\partial}{y_{j-1}} = f(x, y, y_1, ..., y_{m-1}) \frac{\partial}{\partial y_{m-1}}.$$

Operators Equation 19 and Equation 20 are restricted to the surface by infinitesimal operators. As discussed in [2], we can observe that the solutions $\hat{\eta}(x, y, y_1.....y_\varrho)$ of

$$T_g \hat{\eta} = P^m \hat{\eta} - \sum_{j=0}^{m-1} f y_j D^j \hat{\eta} = 0 \tag{21}$$

correspond to the symmetries in characteristics form of order $0 \leq \varrho \leq n$ of ODE in Equation 22.

3.5 Infinitesimal transformations

Suppose we have a single parameter LGT ε defined by

$$U^* = U(u; \varepsilon), \tag{22}$$

with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding (3.0.57) about $\varepsilon = 0$, in the neighborhood of $\varepsilon = 0$, we get

$$u* = u + \varepsilon \left[\frac{\delta U(u : \varepsilon)}{\delta \varepsilon} |_{\varepsilon=0} \right] + \frac{1}{2\varepsilon^2} \left[\frac{\delta^2 u(u; \varepsilon)}{\delta \varepsilon^2} |_\varepsilon = 0 \right] + ...$$

$$= u + \varepsilon \left[\frac{\delta u(u : \varepsilon}{\delta \varepsilon} |_{\varepsilon=0} \right] + 0(\varepsilon^2) \tag{23}$$

Let

$$Z_2(u) = \left[\frac{\delta U(U : \varepsilon)}{\delta \varepsilon} |_{\varepsilon=0} \right]. \tag{24}$$

The function $u + \varepsilon z_2(u)$ is called the infinitesimal transformation (IT) of the Equation 41 .
A one-parameter Lie Group Transformation (LGT) is defined by its small changes, which can also be described by its small generator.
Theorem
The one parameter LGT is equivalent to

$$U^* = \varrho^{\epsilon u} U = U + \epsilon U u + \frac{\epsilon^2}{2} U^2 u + ...$$

$$[1 + \varepsilon U + \frac{\varepsilon^2 U^2}{2} + ...] U$$

$$= \sum_{h=0}^{\infty} \frac{\varepsilon^h}{h!} U^h u, \tag{25}$$

where the operator ($U = U(u)$) is well defined by and the operator $U^h = UU^{h-1}, H = 1, 2...$, in particular, $U^h f(u), H = 1, 2...$ with $U^0 f(u) \equiv f(u)$.

3.6 Infinitesimal generators (IG)

These are properties of transformational Lie groups. Suppose that a one parameter ε LGTs is parameterized in such a way that $\phi(a, b) = a + b$ gives its law of compositions, and $\varepsilon^{-1} = -\varepsilon$ and $\lceil(\varepsilon) \equiv 1$. This means that the one parameter LGT (3.10) becomes

$$\frac{du^*}{d\varepsilon} = \xi^*. \tag{26}$$

with

$$U^* = Uat\varepsilon = 0, \tag{27}$$

in terms of its infinitesimals ξ. Therefore, he IG of the one-parameter LGT is the operator

$$U = U(u) = (\xi) \cdot \nabla = \sum_{j=l}^{m} \varepsilon_j(u) \frac{\delta}{\delta uj}, \tag{28}$$

where ∇ is the gradient operator

$$\nabla = [\frac{\delta}{\delta u_1}, \frac{\delta}{\delta u_2} .... \frac{\delta}{\delta u_n}]. \tag{29}$$

For any differentiable function $F(U) = f(U_1, U_2...U_n)$, one has $UF(u) = \xi.\nabla F(U) = \sum_{j=1}^{n} \xi_j(u) \frac{\delta F(u)}{\delta uj}$.

3.7 Lie point symmetries of ODE

A symmetry where the infinitesimals solely rely on coordinates is known as a point symmetry. Our description pertains to a Lie point symmetry that is contingent upon a minimum of one parameter, meaning that the parameter can fluctuate continuously within a certain range.Lie point symmetries of ODEs are of the form

$$D = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}, \tag{30}$$

in which α and β, the coefficients, are functions only of x and y. To be able to apply a point transformation to an n^{th} order ODE

$$f(x, y, y', y''..., y^{(n)}) = 0. \tag{31}$$

As explained in [3], understanding how the derivatives change as a result of the following infinitesimal transformations (IT) is necessary. So, $\overline{x} = x + \epsilon\alpha(x, y)$

$$\overline{y} = y + \epsilon\beta(x, y) \tag{32}$$

having a generator

$$D = \alpha(x, y) \frac{\partial}{\partial x} + \beta(x, y) \frac{\partial}{\partial y}. \tag{33}$$

Therefore for the first derivative,

$$= \frac{d\overline{y}}{d\overline{x}} = \frac{d(y + \epsilon\beta)}{d(x + \epsilon\beta)} = \frac{d(y + \epsilon\beta)}{d(x + \epsilon\alpha)} \tag{34}$$

$$= \frac{\frac{dy}{dx} + \frac{\epsilon d\beta}{dx}}{(1 + \frac{\epsilon d\alpha}{dx})} \tag{35}$$

$$= (y' + \epsilon\beta')(1 - \epsilon\alpha' + \epsilon^2\alpha^2 - ...)$$

$$= y' + \epsilon(\beta' - y'\alpha') \quad (36)$$

which the term $O(\epsilon^2)$ indicates where the termination is done. Note that the total derivatives with respect to x is mentioned by the primes. We have the second derivative for which

$$\frac{d^2\overline{y}}{d\overline{x}^2} = \frac{d}{d\overline{x}}(\frac{d\overline{y}}{d\overline{x}})$$

$$= \frac{d[(y' + \epsilon(\beta' - y'\alpha'))]}{d(x + \epsilon\alpha')}$$

$$= \frac{\frac{dy'}{dx} + \epsilon\frac{d}{dx}(\beta' - y'\alpha')}{(1 + \epsilon\alpha')}$$

$$= y'' + \epsilon(\beta'' - 2y''\alpha' - y'\alpha'') \quad (37)$$

For the third and the fourth derivatives, we have

$$\frac{d^3\overline{y}}{d\overline{x}^3} = y''' + \epsilon(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''') \quad (38)$$

$$\frac{d^4\overline{y}}{d\overline{x}^4} = y^{iv} + \epsilon(\beta^{iv} - 4y^{iv}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{iv}) \quad (39)$$

and so on. Thus in general

$$\frac{d^n\overline{y}}{d\overline{x}^n} = y^{(n)} + \epsilon(\beta^{(n)} - \sum_{i=l}^{n} C_i^n y^{(i+l)}\alpha^{(n-i)}) \quad (40)$$

4 Main results

We analyzed a fourth order non-linear ordinary differential equation using symmetry method. Let us consider the following equation

$$F(x, y, y', y'', y''', y^{(4)}) = 0 \quad (41)$$

which arises due to collapse of shafts in power transmission systems. In particular, we have in this study attempted to solve a particular form of Equation 41 given as

$$y^{(4)} - \frac{4}{3}(y'')^{-1}(y''')^2 = 0 \quad (42)$$

using Lie symmetry analysis approach. This case arises in the study of equations due to collapse of shafts in power transmission system. we can observe that Equation 42 can alternatively be written in the form

$$y^{(4)} = f(x, y, y', y'', y''') = 0 \quad (43)$$

and thus we obtain

$$y^{(4)} = \frac{4}{3}(y'')^{-1}(y''')^2 = 0 \tag{44}$$

We first consider prolongations in the next steps. Since our equation is of 4^{th} order, we apply the fourth prolongation whose n^{th} extension is of the form

$$G^{[n]} = G + \sum_{i=l}^{n}\{\beta^{(i)} - \sum_{i=l}^{n}\{\tbinom{i}{j}y^{(i+1-j)}\alpha^{(i)}\}\frac{\partial}{\partial y^{(i)}} \tag{45}$$

We can now use Equation 45 the n^{th} extension of G to find the fourth extension. That is $G^{(4)}$, thus

$$G^{[4]} = G^{[3]} + (\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}$$

$$= G^{[2]} + (\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''')\frac{\partial}{\partial y'''}$$

$$+(\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}$$

$$= G^{[1]} + (\beta'' - 2y''\alpha' - y'\alpha'')\frac{\partial}{\partial y''}) + (\beta''' - 3y'''\alpha' - 3y''\alpha''$$

$$-y'\alpha''')\frac{\partial}{\partial y'''} + (\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}$$

$$G^{[4]} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + (\beta' - \alpha' y')\frac{\partial}{\partial y'} + (\beta'' - 2y''\alpha' - y'\alpha'')\frac{\partial}{\partial y''})$$

$$+(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''')\frac{\partial}{\partial y'''} + (\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}} \tag{46}$$

Applying the fourth prolongation of the generator $G^{(4)}$ on the differential Equation 44, that is,

$$G^{[4]}[y^{(4)} - \frac{4}{3}(y'')^{-1}(y''')^2] = 0,$$

leads to

$$\Rightarrow [\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + (\beta' - \alpha' y')\frac{\partial}{\partial y'} + (\beta'' - 2y''\alpha' - y'\alpha'')\frac{\partial}{\partial y''}$$

$$+(\beta''' - 3y'''\alpha' - 3y''\alpha'' - y'\alpha''')\frac{\partial}{\partial y'''} + (\beta^{(4)} - 4y^{(4)}\alpha'$$

$$-6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)})[\frac{\partial}{\partial y^4}]y^{(4)} - \frac{4}{3}(y'')^{-1}(y''')^2] = 0 \tag{47}$$

On expansion Equation 47 we obtain

$$\Rightarrow \alpha\frac{\partial}{\partial x}[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]+\beta\frac{\partial}{\partial y}[(y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]$$

$$+(\beta'-\alpha'y')\frac{\partial}{\partial y'}[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]+$$

$$(\beta''-2y''\alpha'-y'\alpha'')\frac{\partial}{\partial y''}[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]$$

$$+(\beta'''-3y'''\alpha'-3y''\alpha''-y'\alpha''')\frac{\partial}{\partial y'''}[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]+(\beta^4$$

$$-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)})\frac{\partial}{\partial y^{(4)}}[y^{(4)}-\frac{4}{3}(y'')^{-1}(y''')^2]=0 \quad (48)$$

On differentiating partially Equation 47) we obtain

$$\alpha[y^{(5)}+\frac{4}{3}(y'')^{-2}(y''')^3-\frac{8}{3}(y''')(y'')^{-1}y^{(4)}]+\beta[0]$$

$$+(\beta'-\alpha'y')[0]+(\beta''-2y''\alpha'-y'\alpha''')[\frac{4}{3}(y'')^{-2}(y''')^2-0]$$

$$+[\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}](1)=0 \quad (49)$$

But

$$y^{(5)}=(y^{(4)})'$$

$$y^{(5)}=[\frac{4}{3}(y'')^{-1}(y''')^2]'$$

$$y^{(5)}=\frac{4}{3}[-1(y'')^{-2}(y''')(y''')^2+2(y'')^{-1}(y''')^1y^{(4)}]$$

Therefore

$$y^{(5)}=-\frac{4}{3}(y'')^{-2}(y''')^3+\frac{8}{3}(y'')^{-1}(y''')y^{(4)} \quad (50)$$

Substituting Equation 48 into Equation 49 we obtain the following equations

$$\alpha[-\frac{4}{3}(y'')^{-2}(y''')^3+\frac{8}{3}(y'')^{-1}(y''')y^{(4)}+\frac{4}{3}(y'')^{-2}(y''')^3-\frac{8}{3}(y''')(y'')^{-1}y^{(4)}]$$

$$+\beta[0]+(\beta'-\alpha'y')[0]+(\beta''-2y''\alpha'-y'\alpha''')[\frac{4}{3}(y'')^{-2}(y''')^2-0]$$

$$+[\beta^{(4)}-4y^{(4)}\alpha'-6y'''\alpha''-4y''\alpha'''-y'\alpha^{(4)}](1)=0 \quad (51)$$

On simplifying Equation 51 we obtain

$$\frac{4}{3}(y'')^{-2}(y''')^2\beta''-\frac{8}{3}(y'')^{-2}(y''')^2(y'')\alpha'-\frac{4}{3}(y')(y'')^{-2}(y''')^2\alpha'''$$

$$+\beta^{(4)} - 4y^{(4)}\alpha' - 6y'''\alpha'' - 4y''\alpha''' - y'\alpha^{(4)} = 0 \tag{52}$$

The first, second, third, and fourth total derivatives of α and β can therefore be stated in terms of partial derivatives as follows. The primes in Equation 52 correspond to the total derivatives.

$$\alpha' = \frac{\partial \alpha}{\partial x} + y\frac{\partial \alpha}{\partial y} from\, d(\alpha) = (\frac{\partial \alpha}{\partial x})dx + (\frac{\partial \alpha}{\partial y})dy \tag{53}$$

$$\alpha'' = \frac{d}{dx}(\frac{\partial \alpha}{\partial x} + y'\frac{\partial x}{\partial y}) + \frac{d}{dy}(\frac{\partial \alpha}{\partial x} + y'\frac{\partial \alpha}{\partial y})y'$$

$$= \frac{\partial^2 \alpha}{\partial x^2} + y'\frac{\partial^2 \alpha}{\partial y \partial x} + y''\frac{\partial x}{\partial y} + y'\frac{\partial^2 \alpha}{\partial x \partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2} + 0$$

$$= \frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2} + y''\frac{\partial \alpha}{\partial y} \tag{54}$$

$$\alpha''' = \frac{\partial}{\partial x}(\frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial x}{\partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2})$$

$$+y'\frac{\partial}{\partial y}(\frac{\partial^2 \alpha}{\partial x^2} + 2y'\frac{\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial x}{\partial y} + y'^2\frac{\partial^2 \alpha}{\partial y^2})$$

$$= \frac{\partial^3 \alpha}{\partial x^3} + \frac{2y'\partial^3 \alpha}{2x\partial x \partial y} + \frac{2y''\partial^2 \alpha}{\partial x \partial y} + y''\frac{\partial^2 \alpha}{\partial y \partial x} + y'''\frac{\partial x}{\partial y} + \frac{y'^2\partial^3 \alpha}{\partial x \partial y^2} + 2y'\frac{y''\partial^2 \alpha}{\partial y^2}$$

$$+\frac{y'\partial^3 \alpha}{\partial y \partial x^2} + \frac{2y'\partial^3 \alpha}{\partial y \partial x \partial y} + 0 + y'y''\frac{\partial^2 \alpha}{\partial y^2} + 0 + y'^3\frac{\partial^3 \alpha}{\partial y^3} + 0 = \frac{\partial^3 \alpha}{\partial x^3}$$

$$+\frac{3y'\partial^3 \alpha}{\partial x^2 \partial y} + \frac{3y''\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial \alpha}{\partial y} + 3y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^3 \beta}{\partial y^3} \tag{55}$$

and

$$\alpha^4 = \frac{\partial}{\partial x}(\frac{\partial^3 \alpha}{\partial x^3} + \frac{3y'\partial^3 \alpha}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial x}{\partial y} + 3y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^3 \alpha}{\partial y^3})$$

$$+y'\frac{\partial}{\partial y}(\frac{\partial^3 \alpha}{\partial x^3} + \frac{3y'\partial^3 \alpha}{\partial x^2 \partial y} + \frac{3y''\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial x}{\partial y} + 3y'^2\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^3 \alpha}{\partial y^3})$$

$$= \frac{\partial^4 \alpha}{\partial x^4} + 3y'\frac{\partial^4 \alpha}{\partial x^3 \partial y} + 3y''\frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y''\frac{\partial^3 \alpha}{\partial x^2 \partial y} + 3y'''\frac{\partial^2 \alpha}{\partial x \partial y} + y'''\frac{\partial^2 x}{\partial x \partial y} + y^{(4)}\frac{\partial x}{\partial y}$$

$$+3y'^2\frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 6y'y''\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y''\frac{\partial^3 \alpha}{\partial x \partial y^2} + 3y'y'''\frac{\partial^2 \alpha}{\partial y^2} + 3y''^2\frac{\partial^2 \alpha}{\partial y^2} + y'^3\frac{\partial^4 \alpha}{\partial x \partial y^3}$$

$$+3y'^2y''\frac{\partial^3 \alpha}{\partial y^3} + y'\frac{\partial^4 \alpha}{\partial y \partial x^3} + 3y'^2\frac{\partial^4 \alpha}{\partial x^2 \partial y^2} + 3y'y''\frac{\partial^3 \alpha}{\partial x \partial y^2} + y'y'''\frac{\partial^2 \alpha}{\partial y^2} + 3y'^3\frac{\partial^4 \alpha}{\partial x \partial y^3}$$

$$+3y'^2y''\frac{\partial^3\alpha}{\partial y^3}+y'^4\frac{\partial^4\alpha}{\partial y^4}$$

$$=\frac{\partial^4\alpha}{\partial x^4}+\frac{4y'\partial^4\alpha}{\partial x^3\partial y}+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^4\frac{\partial\alpha}{\partial y}+\frac{3y'3\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$+4y'y'''\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2}+4y'\frac{\partial^4\alpha}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}+3y'^2\frac{\partial^4\alpha}{8x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4} \tag{56}$$

and

$$\beta'=\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y}\{from\ d(\beta)=(\frac{\partial\beta}{\partial x})dx+(\frac{\partial\beta}{\partial y})dy\}$$

$$\beta''=\frac{d}{dx}(\frac{\partial\beta}{\partial x}+y'\frac{\partial\beta}{\partial y})+\frac{d}{dy}(\frac{\partial\beta}{\partial x}+y'(\frac{\partial\beta}{\partial y})y'$$

$$=\frac{\partial^2\beta}{\partial x^2}+y'\frac{\partial^2\beta}{\partial y\partial x})+y''\frac{\partial\beta}{\partial y}+y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+0$$

$$=\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y})+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y} \tag{57}$$

$$\beta'''=\frac{\partial}{\partial x}(\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y''\frac{\partial\beta}{\partial y}+y'\frac{\partial^2\beta}{\partial y^2})+y'\frac{\partial}{\partial y}(\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y''\frac{2\beta}{\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2})$$

$$=\frac{\partial^3\beta}{\partial x^3}+2y'\frac{\partial^3\beta}{\partial x\partial x\partial y}+\frac{2y''\partial^2\beta}{\partial x\partial y}+\frac{y''\partial^2\beta}{\partial x\partial y}+\frac{y'''\partial\beta}{\partial y}4+y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+2y'y''\frac{\partial^2\beta}{\partial y^2}$$

$$+y'\frac{\partial^3\beta}{\partial y\partial x^2}+2y'^2\frac{\partial^3\beta}{\partial y\partial x\partial y}+0+y'y''\frac{\partial^2\beta}{\partial y^2}+0+y'^3\frac{\partial^3\beta}{\partial y^3}+0$$

$$=\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}+3y^{-2}\frac{\partial^2\beta}{\partial x\partial y^2}+3y'y''^2\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3} \tag{58}$$

$$\beta^4=\frac{\partial}{\partial x}(\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y''^2\frac{\partial^2\beta}{\partial y}$$

$$+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2})+3y'y''\frac{\partial^3\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3})+y'\frac{\partial}{\partial y}(\frac{\partial^3\beta}{\partial x^3}+3y'\frac{\partial^3\beta}{\partial x^2\partial y}+3y''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial\beta}{\partial y}$$

$$+3y'^2\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y''\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^3\beta}{\partial y^3})$$

$$=\frac{\partial^4\beta}{\partial x^4}+3y'\frac{\partial^4\beta}{\partial x^3\partial y}+3y''\frac{\partial^3\beta}{\partial x\partial y}+3y''\frac{\partial^3\beta}{\partial x^2\partial y}+3y'''\frac{\partial^2\beta}{\partial x\partial y}+y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}$$

$$+\frac{3y'^2\partial^4\beta}{\partial x^2\partial y^2}+\frac{\partial x^2\partial y\partial^3\beta}{\partial x\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+3y'y'''\frac{\partial^2\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+3y'^2y''\frac{\partial^3\beta}{\partial y^3}+y'\frac{\partial^4\beta}{\partial y\partial x^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}$$

$$+y'y'''\frac{\partial^2\beta}{\partial y^2}+3y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+3y'^2y''\frac{\partial^3\beta}{\partial y^3}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$=\frac{\partial^4\beta}{\partial x^4}+$$

$$4y'\frac{\partial^4\beta}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^3\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4} \quad (59)$$

Substituting the above results in Equation 54 we obtain the following

$$\frac{4}{3}(y'')^{-2}(y''')^2(\frac{\partial^2\beta}{\partial x^2}+2y'\frac{\partial^2\beta}{\partial x\partial y}+y'^2\frac{\partial^2\beta}{\partial y^2}+y''\frac{\partial\beta}{\partial y})-\frac{8}{3}(y'')^{-1}(y''')^2+[\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y}]$$

$$-\frac{4}{3}y'(y'')^{-2}(y''')^2[\frac{\partial^2\alpha}{\partial x^2}+2y'\frac{\partial^2\alpha}{\partial x\partial y}+y'^2\frac{\partial^2\alpha}{\partial y^2}+y''\frac{\partial\alpha}{\partial y}]+\frac{\partial^4\beta}{\partial x^4}+4y'\frac{\partial^4\beta}{\partial x^3\partial y}$$

$$+6y''\frac{\partial^3\beta}{\partial x^2\partial y}+\frac{4y'''\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+6y''\frac{\partial^3\beta}{\partial x\partial y}+4y'y'''\frac{\partial^2\beta}{\partial y^2}$$

$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+y'^4\frac{\partial^4\beta}{\partial y^4}$$

$$-4y^{(4)}(\frac{\partial\alpha}{\partial x}+y'\frac{\partial\alpha}{\partial y})-6y''(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y})+3y''\frac{\partial^2\alpha}{\partial x\partial y^2}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$+3y'y''\frac{\partial^3\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}-4y''(\frac{\partial^3\alpha}{\partial x^3}+3y'\frac{\partial^3\alpha}{\partial x^2\partial y})+3y''\frac{\partial^2\alpha}{\partial x\partial y}+y'''\frac{\partial\alpha}{\partial y}+3y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}$$

$$+3y'y''\frac{\partial^2\alpha}{\partial y^2}+y'^3\frac{\partial^3\alpha}{\partial y^3}-y'(\frac{\partial^4\alpha}{\partial x^4}+4y'\frac{\partial^4\alpha}{\partial x^3\partial y})+6y''\frac{\partial^3\alpha}{\partial x^2\partial y}+4y'''\frac{\partial^2\alpha}{\partial x\partial y}+y^4\frac{\partial x}{\partial y}$$

$$+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+4y'y''(\frac{\partial^2\alpha}{\partial y^2}+3y''^2\frac{\partial^2\alpha}{\partial y^2})+4y'^3\frac{\partial^4\alpha}{\partial x\partial y^3}$$

$$+6y'^2y''\frac{\partial^3\alpha}{\partial y^3}+3y'^2\frac{\partial^4\alpha}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\alpha}{\partial x\partial y^2}+y'^4\frac{\partial^4\alpha}{\partial y^4}) \quad (60)$$

On expansion of Equation 60 and simplifying it, we obtain

$$\frac{4}{3}(y'')^{-2}(y''')^2\frac{\partial^2\beta}{\partial x^2}+\frac{4}{3}(y'')^{-2}(y''')^2 2y'\frac{\partial^2\beta}{\partial x\partial y}$$

$$+\frac{4}{3}(y'')^{-2}(y''')^2(y'^2)\frac{\partial^2\beta}{\partial y^2}+\frac{4}{3}(y'')^{-2}(y''')^2(y'')\frac{\partial\beta}{\partial y}$$

$$-\frac{8}{3}(y'')^{-1}(y''')^2\frac{\partial\alpha}{\partial x}-\frac{8}{3}(y'')^{-1}(y''')^2y'\frac{\partial\alpha}{\partial y}-\frac{4}{3}(y)'(y'')^{-2}(y''')^2$$

$$\frac{\partial^2\alpha}{\partial x^2}-\frac{4}{3}(y')(y'')^{-2}(y''')^2 2y'\frac{\partial^2\alpha}{\partial x\partial y}-\frac{4}{3}(y')(y'')^{-2}(y''')^2 y'^2\frac{\partial^2\alpha}{\partial y^2}$$
$$-\frac{4}{3}(y')(y'')^{-2}(y''')^2(y'')\frac{\partial\alpha}{\partial y}+\frac{\partial^4\beta}{\partial x^4}+\frac{4y'\partial^4\partial}{\partial x^3\partial y}+6y''\frac{\partial^3\beta}{\partial x^2\partial y}$$
$$+4y'''\frac{\partial^2\beta}{\partial x\partial y}+y^{(4)}\frac{\partial\beta}{\partial y}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+9y'y''\frac{\partial^3\beta}{\partial x\partial y^2}+4y'y'''\frac{\partial^2\beta}{\partial y^2}$$
$$+3y''^2\frac{\partial^2\beta}{\partial y^2}+4y'^3\frac{\partial^4\beta}{\partial x\partial y^3}+6y'^2y''\frac{\partial^3\beta}{\partial y^3}+3y'^2\frac{\partial^4\beta}{\partial x^2\partial y^2}+3y'y''\frac{\partial^3\beta}{\partial x\partial y^2}$$
$$+y'^{(4)}\frac{\partial^4\beta}{\partial y^4}-4y^4\frac{\partial\alpha}{\partial x}-4y^{(4)}y'\frac{\partial\alpha}{\partial y}-6y''\frac{\partial^3\beta}{\partial x^3}-18y''y'^2\frac{\partial^3\alpha}{\partial x^2\partial y}$$
$$-18(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}-6(y'')(y''')\frac{\partial\alpha}{\partial y}-18y''y'^2\frac{\partial^3\alpha}{\partial x\partial y^2}-18(y'')^2y'\frac{\partial^2\alpha}{\partial y^2}$$
$$-6y''(y')^3\frac{\partial^3\alpha}{\partial y^3}-4(y'')\frac{\partial^3\alpha}{\partial x^3}-12(y'')y'\frac{\partial^3\alpha}{\partial x^3\partial y}-12(y'')^2\frac{\partial^2\alpha}{\partial x\partial y}$$
$$-4(y'')(y''')\frac{\partial\alpha}{\partial y}-12y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-12y'(y'')^2\frac{\partial x^2\alpha}{\partial y^2}-4y'^3y''\frac{\partial^3\alpha}{\partial y^3}-y'\frac{\partial^4\alpha}{\partial x^4}$$
$$-4y'^2\frac{\partial^4\alpha}{\partial x^3\partial y}-6y'y''\frac{\partial^3\alpha}{\partial x^3\partial y}-4y'y'''\frac{\partial^2\alpha}{\partial x\partial y}-y'y^{(4)}\frac{\partial\alpha}{\partial y}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}$$
$$-9y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-4y'^2y''\frac{\partial^2\alpha}{\partial y^2}-3y'y''^2\frac{\partial^2\alpha}{\partial y^2}-4y'^4\frac{\partial^4\alpha}{\partial x\partial y^3}$$
$$-6y'^3y''\frac{\partial^3\alpha}{\partial y^3}-3y'^3\frac{\partial^4\alpha}{\partial x^2\partial y^2}-3y'^2y''\frac{\partial^3\alpha}{\partial x\partial y^2}-y'^5\frac{\partial^4\alpha}{\partial y^4}=0 \tag{61}$$

In x, y, y'', and y''', Equation 61 is an identity, meaning that it holds true for any arbitrary chance of x, y, y', y'', and y'''. Being functions of only x and y, α and β must equal zero for the co-efficient of the powers of y', y'', y''', and their combinations. The following partial differential equation systems, sometimes referred to as determining equations, are obtained.

$$(y')^3(y'')^{-2}(y''')^3:\frac{4}{3}\frac{\partial^2\alpha}{\partial y^2}=0 \tag{62}$$

$$(y')^2(y'')^{-2}(y''')^3:\frac{4}{3}\frac{\partial^2\beta}{\partial y^2}-\frac{8}{3}\frac{\partial^2\alpha}{\partial x\partial y}=0 \tag{63}$$

$$(y')^1(y'')^{-2}(y''')^3:-\frac{4}{3}\frac{\partial^2\alpha}{\partial x^2}+\frac{8}{3}\frac{\partial^2\beta}{\partial x\partial y}=0 \tag{64}$$

$$(y')^0(y'')^{-2}(y''')^3:\frac{\partial^2\beta}{\partial x^2} \tag{65}$$

Solving determining equations.
Intergrating Equation 62 we find

$$\frac{4}{3}\frac{\partial^2\alpha}{\partial y^2} = 0 \Longrightarrow \frac{\partial^2\alpha}{\partial y^2} = 0$$

$$\frac{\partial\alpha}{\partial y} = C_1 \Longrightarrow \alpha = C_1 y + C_2..., \tag{66}$$

where C_1 and C_2 are arbitrary functions of x. We substitute Equation 66 in Equation 63 and solve to find

$$\frac{4}{3}\frac{\partial^2\beta}{\partial y^2} - \frac{8}{3}\frac{\partial^2\alpha}{\partial x\partial y} = 0 \Longrightarrow \frac{\partial^2\beta}{\partial y^2} - 2\frac{\partial^2\alpha}{\partial x\partial y} = 0$$

$$\frac{\partial^2\beta}{\partial y^2} - 2\frac{\partial}{\partial x}C_1 \Longrightarrow \frac{\partial^2\beta}{\partial y^2} = 2C_1^1 \Longrightarrow \frac{\partial\beta}{\partial y} = 2C_1^1 y + C_3$$

$$\Longrightarrow \beta = C_1^1 y^2 + C_3 y + C_4 \tag{67}$$

Where C_3 and C_4 are arbitrary functions of x, substituting Equation 66 and Equation 67 into Equation 64 gives

$$-\frac{\partial^2\alpha}{\partial x^2} + \frac{2\partial^2\beta}{\partial x\partial y} = 0$$

$$\frac{2\partial^2\beta}{\partial x\partial y} - \frac{\partial^2\alpha}{\partial x^2} = 0$$

$$\frac{2\partial}{\partial x}\frac{\partial\beta}{\partial y} - \frac{\partial^2\alpha}{\partial x^2} = 0$$

$$2\frac{\partial}{\partial x}(2C_1' y + C_3) - (C_1'' y + C_2'') = 0$$

$$2(2C_1'' y + C_3') - (C_1'' y + C_2'') = 0$$

$$4C_1'' y + 2C_3' - C_1'' y - C_2) = 0$$

$$3C_1'' y + 2C_3' - C_2'' = 0 \tag{68}$$

Since C_1, C_2 and C_3 depends on X only, we can now equate the coefficients of powers of y to zero. This yields

$$y^1 : 3C_1'' = 0. \tag{69}$$

$$y^0 : 2C_3' - C_2'' = 0. \tag{70}$$

Now, we substitute Equation 68 into Equation 69 and this yields

$$\frac{\partial^2 \beta}{\partial x^2} = 0$$

Therefore,

$$C_1'''y^2 + C_3''y + C_4'' = 0 \tag{71}$$

we equate the coefficient of y to zero and obtain

$$y^2 : C_1''' = 0 \tag{72}$$

$$y' : C_3'' = 0 \tag{73}$$

$$y^0 : C_4'' = 0 \tag{74}$$

We now solve the differential equations as follows:
From Equation 72 we have

$$C_1''' = 0 \Rightarrow C_1'' = H_1 \Rightarrow C_1' = H_1 x = H_2 \Rightarrow C_1 = \frac{1}{2}H_1 x^2 + H_2 x + H_3, \tag{75}$$

where H_1, H_2 and H_3 are arbitrary constant. Now from Equation 73

$$C_3'' = 0 \Rightarrow C_3' = H_4 \Rightarrow C_3 = H_4 x + H_5 \tag{76}$$

then considering Equation 72, we have

$$C_4'' = 0 \Rightarrow C_4' = H_6 \Rightarrow C_4 = H_6 x + H_7 \tag{77}$$

Where $H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 are arbitrary constants. Thus we have

$$\alpha = C_1 y + C_2$$

$$\beta = C_1' y^2 + C_3 y + C_4$$

$$\alpha(x, y) = (\frac{1}{2}H_1 x^2 + H_2 x + H_3)y + 0$$

Where $C_2 = 0$

$$\alpha(x, y) = \frac{1}{2}H_1 x^2 y + H_2 xy + H_3)y \tag{78}$$

We also apply to produce

$$\beta(x, y) = (H_1 x + H_2)y^2 + (H_4 x + H_5)y + (H_6 x + H_7)$$

Therefore

$$\beta(x, y) = H_1 xy^2 + H_2 y^2 + H_4 xy + H_5 x + H_6 x + H_7. \tag{79}$$

Now we consider infinitesimals for the equation. As a result, the generator G of infinitesimal transformation is

$$G = (\frac{1}{2}H_1x^2y + H_2xy + (H_3y)\frac{\partial}{\partial x} + (H_1xy^2 + H_2y^2 + H_4xy + H_5y + H_6x + H_7)\frac{\partial}{\partial y}$$

which is written as

$$G = H_1(\frac{1}{2}x^2y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}) + H_2(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}) + H_3(y\frac{\partial}{\partial}) + H_4(xy\frac{\partial}{\partial y}H_5(y\frac{\partial}{\partial y}) + H_6(x\frac{\partial}{\partial y})$$
$$+H_7\frac{\partial}{\partial y}, \tag{80}$$

which is a seven parameter symmetry of the equation of concern.

5 Open Problems

We have analyzed the problematic phenomena of an apparently unintentional beating and the potential collapse of shafts in power transmission systems was discovered by motor ship constructors. We examined a fourth order ODE using Lie symmetry which demonstrates how the collapse of shafts in power transmission networks occurs dynamically. This study exposes certain two open problems.

Problem 1: Can one develop an efficient algorithm to solve this forth order ODE numerically?

Problem 2: Using Lie symmetry analysis, determine the solution of higher order non-linear ODES.

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