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New integral formulas inspired by an old integral result

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Abstract

In this article, we establish new integral formulas inspired by an old integral result listed in two key books. Some of these formulas depend on the Euler-Mascheroni constant and extend existing results, broadening their scope and applicability. The Laplace transforms of certain original functions are emphasized. The proofs are presented in full detail, using rigorous analytical techniques. These include various geometric and power series expansions, and asymptotic analysis. An open problem is given at the end.

Keywords: *Integral formulas, Euler-Mascheroni constant, power series expansion, infinite product, Fubini-Tonelli integral theorem, Laplace transform.*
2010 Mathematics Subject Classification: 33B15, 33B20.

1 Introduction

Over the past 150 years, numerous integral formulas have been established. Many of these are documented in the essential book of tables [5], which itself extracts formulas from older books of tables, such as [4]. These formulas are fundamental in various branches of mathematics, including analysis, differential equations, optimization, signal processing, probability and statistics. Additionally, they play a key role in multiple fields of physics, such as classical mechanics, electromagnetism, quantum mechanics, statistical mechanics and general relativity. The study of integrals continues to be an active area of research, as demonstrated by recent advancements in [6, 7, 8, 9, 10, 11, 2, 1].

The integral formulas in [5] and [4] are given clearly but roughly, without proof. They may therefore appear somewhat obscure or unintuitive to the reader. In some cases, their derivations rely on techniques that are not immediately obvious and require a deeper study of mathematical methods. Revisiting these formulas with modern analytical tools can provide greater clarity and ideas about their possible extensions. With this in mind, we focus on an interesting formula in [5, entry 3.4311, page 361], which in turn refers to [4, entry 144(6)]. It is presented formally below.

Proposition 1.1 *([4, entry 144(6)] and [5, entry 3.4311, page 361]). For any $p > 0$, we have*

$$\int_0^{+\infty} \left[x^{-1} - \frac{1}{2}x^{-2}(x+2)(1-e^{-x})\right] e^{-px}dx = \left(p + \frac{1}{2}\right)\log\left(1 + \frac{1}{p}\right) - 1.$$

There are three main reasons why this formula has attracted our attention:

1. It involves power, exponential and logarithmic functions within an original integral relation.
2. It offers flexibility through the parameter p, which can be chosen as any positive value.
3. It has a natural connection with the Laplace transform.

However, to the best of our knowledge, there is no explicit proof of Proposition 1.1 in the literature. In the process of filling this gap, we have discovered several new integral formulas. They are all related in different ways and can depend on an adjustable parameter. Some of them involve the Euler-Mascheroni constant and serve as generalizations of existing results. The study is completed by a new result on an exponential inequality, which can be seen as of independent interest, and an open problem. The proofs are given in full detail, using classical mathematical techniques, such as geometric and power series expansions, and asymptotic analysis. By revisiting and refining the "old" formula in Proposition 1.1, we thus contribute to the development of new integral formulas.

The remainder of this article is structured as follows: Section 2 presents two fundamental lemmas. Section 3 contains the main results. The mentioned exponential inequality is introduced in Section 4. Section 5 formulates the open problem. Finally, concluding remarks are provided in Section 6.

2 Two fundamental lemmas

The lemma below concerns the study of a particular function that will play a central role in most of our results.

Lemma 2.1 *For any $x \in \mathbb{R}$, let us set*

$$f(x) = x - \frac{1}{2}(x+2)(1-e^{-x}). \tag{1}$$

1. *Then f is non-decreasing, with $f(x) \geq 0$ for any $x \geq 0$, and $f(x) \leq 0$ for any $x \leq 0$.*
2. *The following equivalence of $f(x)$ at $x = 0$ holds:*

$$f(x) \sim \frac{x^3}{12}.$$

3. *The following equivalence of $f(x)$ at $x \to +\infty$ holds:*

$$f(x) \sim \frac{x}{2}.$$

Proof.

1. Using standard differentiation rules, a factorization and the basic exponential inequality $e^x \geq 1 + x$ for any $x \in \mathbb{R}$, we have

$$f'(x) = \frac{1}{2}e^{-x}(e^x - 1 - x) \geq 0.$$

Therefore, $f(x)$ is non-decreasing for any $x \in \mathbb{R}$. Since

$$f(0) = 0 - \frac{1}{2}(0+2)(1-e^{-0}) = 0,$$

we have $f(x) \geq f(0) = 0$ for any $x \geq 0$, and $f(x) \leq f(0) = 0$ for any $x \leq 0$,

2. At $x = 0$, the series expansion of the exponential function and a simple product development give

$$(x+2)(1-e^{-x}) \quad \sim (x+2)\left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}\right) = 2x - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{24}$$
$$\sim 2x - \frac{x^3}{6}.$$

We therefore have

$$f(x) = x - \frac{1}{2}(x+2)(1-e^{-x}) \sim x - \frac{1}{2}\left(2x - \frac{x^3}{6}\right) = \frac{x^3}{12}.$$

3. At $x \to +\infty$, we have

$$f(x) = x - \frac{1}{2}(x+2)(1-e^{-x}) \sim x - \frac{1}{2}(x+2)(1-0) = \frac{x}{2} - 1 \sim \frac{x}{2}.$$

This ends the proof. □

This result will be needed mainly to justify the existence of the future integrals under consideration (more details are given at the beginning of Section 3) and the use of the Fubini-Tonelli integral theorem.

The lemma below concerns a basic integral formula obtained using standard mathematical techniques.

Lemma 2.2 *For any $p > 0$, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] e^{-px}dx = \frac{1}{2p^2} - \frac{1}{p} + \frac{1}{2(1+p)^2} + \frac{1}{1+p}.$$

Proof. Using a development of the integrand, the linearity of the integral, the changes of variables $y = px$ and $z = (1+p)x$, $\int_0^{+\infty} xe^{-x}dx = 1$ and $\int_0^{+\infty} e^{-x}dx = 1$, we have

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] e^{-px}dx \\
&= \int_0^{+\infty} xe^{-px}dx - \frac{1}{2}\int_0^{+\infty} xe^{-px}dx - \int_0^{+\infty} e^{-px}dx + \frac{1}{2}\int_0^{+\infty} xe^{-(1+p)x}dx \\
&+ \int_0^{+\infty} e^{-(1+p)x}dx \\
&= \frac{1}{2}\int_0^{+\infty} xe^{-px}dx - \int_0^{+\infty} e^{-px}dx + \frac{1}{2}\int_0^{+\infty} xe^{-(1+p)x}dx + \int_0^{+\infty} e^{-(1+p)x}dx \\
&= \frac{1}{2p^2}\int_0^{+\infty} ye^{-y}dy - \frac{1}{p}\int_0^{+\infty} e^{-y}dy + \frac{1}{2(1+p)^2}\int_0^{+\infty} ze^{-z}dz \\
&+ \frac{1}{1+p}\int_0^{+\infty} e^{-z}dz \\
&= \frac{1}{2p^2} - \frac{1}{p} + \frac{1}{2(1+p)^2} + \frac{1}{1+p}.
\end{aligned}$$

This concludes the proof. □

Further developments, including the proof of Proposition 1.1 and new propositions, will be based more or less directly on this integral result.

3 New results

This section introduces the new integral formulas. Most of them depend on the function f defined by Equation (1). Thanks to the second and third results in Lemma 2.1, we are able to show that these integrals exist (or converge), which obviously makes sense since analytical expressions are given for them. For the sake of redundancy, we will omit this aspect of existence in the proofs, but provide the details that lead to the desired formulas.

The result below presents a new one-parameter integral formula in the same spirit as Proposition 1.1. Lemma 2.2 is central to the proof.

Proposition 3.1 *For any $q > 0$, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-qx} dx = \frac{1}{2q} + \frac{1}{2(1+q)} - \log\left(1 + \frac{1}{q}\right).$$

Proof. Using the exponential primitive, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Lemma 2.2, and usual primitives and limits, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-qx} dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \left[\int_q^{+\infty} e^{-px} dp\right] dx \\
&= \int_q^{+\infty} \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] e^{-px} dx\right\} dp \\
&= \int_q^{+\infty} \left[\frac{1}{2p^2} - \frac{1}{p} + \frac{1}{2(1+p)^2} + \frac{1}{1+p}\right] dp \\
&= \left[-\frac{1}{2p} - \log(p) - \frac{1}{2(1+p)} + \log(1+p)\right]_{p=q}^{p\to+\infty} \\
&= \left[-\frac{1}{2p} - \frac{1}{2(1+p)} + \log\left(1 + \frac{1}{p}\right)\right]_{p=q}^{p\to+\infty} \\
&= \frac{1}{2q} + \frac{1}{2(1+q)} - \log\left(1 + \frac{1}{q}\right).
\end{aligned}$$

This ends the proof. □

Some special cases are given below.

- If we take $q = 1$, then we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-x} dx = \frac{1}{2} + \frac{1}{4} - \log(2) \approx 0.05685.$$

- If we set, for any $q > 0$ and $x > 0$,

$$g(x) = \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-qx},$$

then the Laplace transform of g at $\lambda > 0$ is given by

$$\begin{aligned}\mathcal{L}(g)(\lambda) &= \int_0^{+\infty} g(x)e^{-\lambda x}dx \\ &= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-(q+\lambda)x}dx \\ &= \frac{1}{2(q+\lambda)} + \frac{1}{2(1+q+\lambda)} - \log\left(1 + \frac{1}{q+\lambda}\right).\end{aligned}$$

To the best of our knowledge, this is a new Laplace transform formula in the literature.

The result below is a new one-integer-parameter integral result derived from Proposition 3.1. It has the feature to involve the Euler-Mascheroni constant.

Proposition 3.2 *For any* $m \in \mathbb{N}\backslash\{0\}$*, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{e^{-mx}}{x(1-e^{-x})}dx = \gamma - \sum_{q=1}^{m-1} \frac{1}{q} + \log(m) - \frac{1}{2m},$$

with the convention $\sum_{q=1}^{0} = 0$*, where* γ *is the Euler-Mascheroni constant, i.e.,* $\gamma \approx 0.57721$*.*

Proof. Using the geometric series expansion, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.1, a famous series expansion of γ and telescopic developments, we get

$$\begin{aligned}
&\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{e^{-mx}}{x(1-e^{-x})}dx\\
&=\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x}\left(\sum_{q=m}^{+\infty}e^{-qx}\right)dx\\
&=\sum_{q=m}^{+\infty}\left\{\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x}e^{-qx}dx\right\}\\
&=\sum_{q=m}^{+\infty}\left[\frac{1}{2q}+\frac{1}{2(1+q)}-\log\left(1+\frac{1}{q}\right)\right]\\
&=\sum_{q=m}^{+\infty}\left[\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right]-\frac{1}{2}\sum_{q=m}^{+\infty}\left(\frac{1}{q}-\frac{1}{1+q}\right)\\
&=\sum_{q=1}^{+\infty}\left[\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right]-\sum_{q=1}^{m-1}\left[\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right]-\frac{1}{2}\sum_{q=m}^{+\infty}\left(\frac{1}{q}-\frac{1}{1+q}\right)\\
&=\sum_{q=1}^{+\infty}\left[\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right]-\sum_{q=1}^{m-1}\frac{1}{q}+\sum_{q=1}^{m-1}[\log(q+1)-\log(q)]\\
&-\frac{1}{2}\sum_{q=m}^{+\infty}\left(\frac{1}{q}-\frac{1}{1+q}\right)\\
&=\gamma-\sum_{q=1}^{m-1}\frac{1}{q}+\log(m)-\frac{1}{2m}.
\end{aligned}$$

This concludes the proof. □

Some special cases are given below.

- If we take $m=1$, then we have

$$\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{e^{-x}}{x(1-e^{-x})}dx=\gamma-\frac{1}{2}\approx 0.07721.$$

- If we take $m=2$, then we have

$$\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{e^{-2x}}{x(1-e^{-x})}dx=\gamma-1+\log(2)-\frac{1}{4}\approx 0.02036.$$

The result below is a new integral result derived from Proposition 3.2. It generalizes a classical integral formula from the literature.

Proposition 3.3 *For any* $m \in \mathbb{N}\backslash\{0\}$*, we have*

$$\int_0^{+\infty} \left[\frac{e^{-x}}{m(1-e^{-x/m})} - \frac{1}{x}e^{-x} \right] dx = \gamma - \sum_{q=1}^{m-1} \frac{1}{q} + \log(m),$$

with the convention $\sum_{q=1}^{0} = 0$*, where* γ *is the Euler-Mascheroni constant, i.e.,* $\gamma \approx 0.57721$*.*

Proof. Using a development of the integrand, the linearity of the integral, the change of variables $y = mx$ and $\int_0^{+\infty} e^{-x}dx = 1$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{e^{-mx}}{x(1-e^{-x})} dx \\
&= \int_0^{+\infty} \left[\frac{e^{-mx}}{1-e^{-x}} - \frac{1}{2}(x+2)\frac{e^{-mx}}{x} \right] dx \\
&= \int_0^{+\infty} \left(\frac{e^{-mx}}{1-e^{-x}} - \frac{1}{x}e^{-mx} \right) dx - \frac{1}{2}\int_0^{+\infty} e^{-mx} dx \\
&= \int_0^{+\infty} \left(\frac{e^{-y}}{1-e^{-y/m}} - \frac{m}{y}e^{-y} \right) \frac{1}{m} dy - \frac{1}{2}\int_0^{+\infty} e^{-y}\frac{1}{m} dy \\
&= \int_0^{+\infty} \left[\frac{e^{-y}}{m(1-e^{-y/m})} - \frac{1}{y}e^{-y} \right] dy - \frac{1}{2m}.
\end{aligned}$$

By this and Proposition 3.2, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[\frac{e^{-x}}{m(1-e^{-x/m})} - \frac{1}{x}e^{-x} \right] dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{e^{-mx}}{x(1-e^{-x})} dx + \frac{1}{2m} \\
&= \gamma - \sum_{q=1}^{m-1} \frac{1}{q} + \log(m) - \frac{1}{2m} + \frac{1}{2m} \\
&= \gamma - \sum_{q=1}^{m-1} \frac{1}{q} + \log(m).
\end{aligned}$$

This ends the proof. □

Some special cases are given below.

- If we take $m = 1$, then we have

$$\int_0^{+\infty} \left(\frac{e^{-x}}{1-e^{-x}} - \frac{1}{x}e^{-x} \right) dx = \gamma \approx 0.57721.$$

  This is a well-known integral formula for γ. See, for example, [5, Entry 3.4272].

- If we take $m = 2$, then we have

$$\int_0^{+\infty} \left[\frac{e^{-x}}{2(1-e^{-x/2})} - \frac{1}{x}e^{-x}\right] dx = \gamma - 1 + \log(2) \approx 0.27036.$$

- For a more original example, if we take $m = 10$, then we have

$$\int_0^{+\infty} \left[\frac{e^{-x}}{10(1-e^{-x/10})} - \frac{1}{x}e^{-x}\right] dx = \gamma - \frac{7129}{2520} + \log(10) \approx 0.05083.$$

The result below gives a new one-integer-parameter integral formula involving power, exponential and logarithmic functions, and a new constant not mentioned in the literature.

Proposition 3.4 *We have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} \log\left(\frac{1}{1-e^{-x}}\right) dx = \frac{\pi^2}{12} + \frac{1}{2} - \log(\theta),$$

where

$$\theta = \prod_{q=1}^{+\infty} \left(1 + \frac{1}{q}\right)^{1/q} \approx 3.51749.$$

Proof. Using the logarithmic series expansion, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.1, the classical formula $\sum_{q=1}^{+\infty}(1/q^2) = \pi^2/6$ and telescopic developments, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} \log\left(\frac{1}{1-e^{-x}}\right) dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} \left(\sum_{q=1}^{+\infty} \frac{1}{q} e^{-qx}\right) dx \\
&= \sum_{q=1}^{+\infty} \frac{1}{q} \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-qx} dx\right\} \\
&= \sum_{q=1}^{+\infty} \frac{1}{q} \left[\frac{1}{2q} + \frac{1}{2(1+q)} - \log\left(1 + \frac{1}{q}\right)\right] \\
&= \frac{1}{2}\sum_{q=1}^{+\infty} \frac{1}{q^2} + \frac{1}{2}\sum_{q=1}^{+\infty} \frac{1}{q(q+1)} - \sum_{q=1}^{+\infty} \frac{1}{q} \log\left(1 + \frac{1}{q}\right) \\
&= \frac{\pi^2}{12} + \frac{1}{2}\sum_{q=1}^{+\infty} \left(\frac{1}{q} - \frac{1}{q+1}\right) - \log\left[\prod_{q=1}^{+\infty} \left(1 + \frac{1}{q}\right)^{1/q}\right] \\
&= \frac{\pi^2}{12} + \frac{1}{2} - \log(\theta).
\end{aligned}$$

This concludes the proof. □

The one-parameter integral result in [4, entry 144(6)] and [5, entry 3.4311, page 361], as recalled in Proposition 1.1, is demonstrated below. It is mainly based on Proposition 3.1.

Proposition 3.5 *For any $r > 0$, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} e^{-rx} dx = \left(r + \frac{1}{2}\right) \log\left(1 + \frac{1}{r}\right) - 1.$$

Proof. Using the exponential primitive, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.1, and usual primitives and limits, we have

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} e^{-rx} dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} \left[\int_r^{+\infty} e^{-qx} dq\right] dx \\
&= \int_r^{+\infty} \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x} e^{-qx} dx\right\} dq \\
&= \int_r^{+\infty} \left[\frac{1}{2q} + \log(q) + \frac{1}{2(1+q)} - \log(1+q)\right] dq \\
&= \left[\frac{1}{2}\log(q) + q\log(q) - q + \frac{1}{2}\log(1+q) - (1+q)\log(1+q) + q\right]_{q=r}^{q\to+\infty} \\
&= \left[-\left(q + \frac{1}{2}\right)\log\left(1 + \frac{1}{q}\right)\right]_{q=r}^{q\to+\infty} \\
&= \left(r + \frac{1}{2}\right)\log\left(1 + \frac{1}{r}\right) - 1.
\end{aligned}$$

This ends the proof. □

This fills a certain gap in [5, entry 3.4311, page 361] by providing a rigorous proof. However, we do not claim that it is unique; some other mathematical ways are certainly possible.

The result below goes one step further. It gives an integral formula for the same integrand as in the result above, but multiplied by $1/x$.

Proposition 3.6 *For any $s > 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} e^{-sx} dx \\
&= \frac{1}{4}\left[1 + 2s - 2s(s+1)\log\left(1 + \frac{1}{s}\right)\right].
\end{aligned}$$

Proof. Using the exponential primitive, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.5, and usual primitives and limits, we have

$$\begin{aligned}
&\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^3}e^{-sx}dx\\
&=\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^2}\left[\int_s^{+\infty}e^{-rx}dr\right]dx\\
&=\int_s^{+\infty}\left\{\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^2}e^{-rx}dx\right\}dr\\
&=\int_s^{+\infty}\left[\left(r+\frac{1}{2}\right)\log\left(1+\frac{1}{r}\right)-1\right]dr\\
&=\left\{\frac{1}{2}r\left[(r+1)\log\left(1+\frac{1}{r}\right)-1\right]\right\}_{r=s}^{r\to+\infty}\\
&=\frac{1}{4}\left[1+2s-2s(s+1)\log\left(1+\frac{1}{s}\right)\right].
\end{aligned}$$

This concludes the proof. □

Some special cases are given below.

- If we take $s=1$, then we have

$$\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^3}e^{-x}dx=\frac{1}{4}\left[3-4\log(2)\right]\approx 0.05685.$$

- If we set, for any $s>0$ and $x>0$,

$$h(x)=\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^3}e^{-sx},$$

  then the Laplace transform of h at $\lambda>0$ is given by

$$\begin{aligned}
\mathcal{L}(h)(\lambda)&=\int_0^{+\infty}h(x)e^{-\lambda x}dx\\
&=\int_0^{+\infty}\left[x-\frac{1}{2}(x+2)(1-e^{-x})\right]\frac{1}{x^3}e^{-(s+\lambda)x}dx\\
&=\frac{1}{4}\left[1+2(s+\lambda)-2(s+\lambda)(s+\lambda+1)\log\left(1+\frac{1}{s+\lambda}\right)\right].
\end{aligned}$$

  To the best of our knowledge, this is a new Laplace transform formula in the literature.

The result below complements the above proposition by considering $1-e^{-tx}$ instead of e^{-sx} in the integrand, which leads to a new integral formula.

Proposition 3.7 *For any $t > 0$, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3}(1-e^{-tx})dx$$
$$= \frac{1}{2}t\left[(t+1)\log\left(1+\frac{1}{t}\right)-1\right].$$

Proof. Using the exponential primitive, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.5, and usual primitives and limits, we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3}(1-e^{-tx})dx$$
$$= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2}\left[\int_0^t e^{-rx}dr\right]dx$$
$$= \int_0^{t} \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2}e^{-rx}dx\right\}dr$$
$$= \int_0^{t} \left[\left(r+\frac{1}{2}\right)\log\left(1+\frac{1}{r}\right)-1\right]dr$$
$$= \left\{\frac{1}{2}r\left[(r+1)\log\left(1+\frac{1}{r}\right)-1\right]\right\}_{r\to 0}^{r=t}$$
$$= \frac{1}{2}t\left[(t+1)\log\left(1+\frac{1}{t}\right)-1\right].$$

This ends the proof. □

Some special cases are given below.

- If we take $t = 1$, then we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3}(1-e^{-x})dx = \frac{1}{2}\left[2\log(2)-1\right]$$
$$\approx 0.19314.$$

- If we take $t = 2$, then we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3}(1-e^{-2x})dx = 3\log\left(\frac{3}{2}\right)-1$$
$$\approx 0.21639.$$

The result below proposed to combine Propositions 3.6 and 3.7 for a concise formula.

Proposition 3.8 *We have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} dx = \frac{1}{4}.$$

Proof. It follows from Propositions 3.6 and 3.7 with $s = 1$ and $t = 1$, respectively, that

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} [(1-e^{-x}) + e^{-x}] dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} (1-e^{-x}) dx \\
&+ \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} e^{-x} dx \\
&= \frac{1}{2}\left[2\log(2) - 1\right] + \frac{1}{4}\left[3 - 4\log(2)\right] = \frac{1}{4}.
\end{aligned}$$

The choice of $s = 1$ and $t = 1$ can in fact be replaced by $s = t$ without affecting the development.

An more direct proof independent of Propositions 3.6 and 3.7 is given below.

Alternative proof. Identifying the (non-trivial) primitive of the integrand and using limits, in particular $e^{-x} \sim 1 - x + x^2/2$ at $x = 0$, we get

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} dx = \left[-\frac{e^{-x} - 1 + x}{2x^2}\right]_{x\to 0}^{x\to+\infty} = \frac{1}{4}.$$

This ends the proof. □

The result below gives a new one-integer-parameter integral formula involving power, exponential and logarithmic functions. We also mention the important role of an original infinite product sequence, which can be determined explicitly for some values of the parameter involved.

Proposition 3.9 *For any $n \in \mathbb{N}\backslash\{0\}$, we have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{e^{-nx}}{x^2(1-e^{-x})} dx = \log(\upsilon_n),$$

where

$$\upsilon_n = \prod_{r=n}^{+\infty} \left[e^{-1} \left(1 + \frac{1}{r} \right)^{r+1/2} \right].$$

In particular, for $n = 1, \ldots, 8$, we have

$$\upsilon_1 = \frac{e}{\sqrt{2\pi}} \approx 1.08444, \quad \upsilon_2 = \frac{e^2}{4\sqrt{\pi}} \approx 1.04221, \quad \upsilon_3 = \frac{e^3}{9}\sqrt{\frac{2}{3\pi}} \approx 1.02806,$$

$$\upsilon_4 = \frac{3e^4}{64\sqrt{2\pi}} \approx 1.02101, \quad \upsilon_5 = \frac{12}{625} e^5 \sqrt{\frac{2}{5\pi}} \approx 1.01678,$$

$$\upsilon_6 = \frac{5e^6}{648\sqrt{3\pi}} \approx 1.01397, \quad \upsilon_7 = \frac{360e^7}{117649}\sqrt{\frac{2}{7\pi}} \approx 1.01196$$

and

$$\upsilon_8 = \frac{315e^8}{524288\sqrt{\pi}} \approx 1.01046.$$

Proof. Using the geometric series expansion, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1 and Proposition 3.5, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{e^{-nx}}{x^2(1-e^{-x})} dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{1}{x^2} \left(\sum_{r=n}^{+\infty} e^{-rx} \right) dx \\
&= \sum_{r=n}^{+\infty} \left\{ \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{1}{x^2} e^{-rx} dx \right\} \\
&= \sum_{r=n}^{+\infty} \left[\left(r + \frac{1}{2} \right) \log \left(1 + \frac{1}{r} \right) - 1 \right] \\
&= \log \left\{ \prod_{r=n}^{+\infty} \left[e^{-1} \left(1 + \frac{1}{r} \right)^{r+1/2} \right] \right\} = \log(\upsilon_n).
\end{aligned}$$

This concludes the proof. □

We mention that υ_n tends to 1 as n tends to $+\infty$, which is consistent with the fact that e^{-nx} tends to 0 in the integrand.

Some special cases are given below.

- If we take $n = 1$, then we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x}) \right] \frac{e^{-x}}{x^2(1-e^{-x})} dx = \log(\upsilon_1) \approx 0.08106.$$

- If we take $n = 2$, then we have

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{e^{-2x}}{x^2(1-e^{-x})} dx = \log(\upsilon_2) \approx 0.04134.$$

The result below exploits Proposition 3.5 to derive a new integral formula. An unreferenced constant, defined as an infinite product, emerged naturally.

Proposition 3.10 *We have*

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} \log\left(\frac{1}{1-e^{-x}}\right) dx = \log(\xi),$$

where

$$\xi = \prod_{r=1}^{+\infty} \left[e^{-1/r}\left(1+\frac{1}{r}\right)^{(r+1/2)/r}\right] \approx 1.05302.$$

Proof. Using the logarithmic series expansion, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1 and Proposition 3.5, we get

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} \log\left(\frac{1}{1-e^{-x}}\right) dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} \left(\sum_{r=1}^{+\infty} \frac{1}{r} e^{-rx}\right) dx \\
&= \sum_{r=1}^{+\infty} \frac{1}{r} \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^2} e^{-rx} dx\right\} \\
&= \sum_{r=1}^{+\infty} \frac{1}{r} \left[\left(r+\frac{1}{2}\right) \log\left(1+\frac{1}{r}\right) - 1\right] \\
&= \log\left\{\prod_{r=1}^{+\infty} \left[e^{-1/r}\left(1+\frac{1}{r}\right)^{(r+1/2)/r}\right]\right\} = \log(\xi).
\end{aligned}$$

This concludes the proof. □

The result below is our final proposition. It gives a new one-parameter integral formula, still based on power, exponential and logarithmic functions.

Proposition 3.11 *For any $u > 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^4}(1-e^{-ux}) dx \\
&= \frac{1}{12}\left[2u(u+1) - u^2(2u+3)\log\left(1+\frac{1}{u}\right) + \log(1+u)\right].
\end{aligned}$$

Proof. Using the exponential primitive, the Fubini-Tonelli integral theorem, which is validated because the integrand is non-negative by the first result of Lemma 2.1, Proposition 3.6, and usual primitives and limits, we have

$$\begin{aligned}
&\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^4}(1-e^{-ux})dx \\
&= \int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} \left[\int_0^u e^{-sx} ds\right] dx \\
&= \int_0^u \left\{\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^3} e^{-sx} dx\right\} ds \\
&= \frac{1}{4}\int_0^u \left[1 + 2s - 2s(s+1)\log\left(1+\frac{1}{s}\right)\right] ds \\
&= \frac{1}{4}\left\{\frac{1}{3}\left[2s(s+1) - s^2(2s+3)\log\left(1+\frac{1}{s}\right) + \log(1+s)\right]\right\}_{s\to 0}^{s=u} \\
&= \frac{1}{12}\left[2u(u+1) - u^2(2u+3)\log\left(1+\frac{1}{u}\right) + \log(1+u)\right].
\end{aligned}$$

This ends the proof. □

Some special cases are given below.

- If we take $u = 1$, then we have
$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^4}(1-e^{-x})dx = \frac{1}{3}\left[1 - \log(2)\right] \approx 0.10228.$$

- If we take $u = 2$, then we have
$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right] \frac{1}{x^4}(1-e^{-2x})dx = \frac{1}{12}\left[12 - 28\log\left(\frac{3}{2}\right) + \log(3)\right] \approx 0.14546.$$

4 An independent result

An inequality of independent interest is given in this section. It is related to Lemma 2.1, but focuses on an original bound for the exponential function e^{-x}. Two different proofs are given.

Lemma 4.1 *For any $x \geq 0$, we have*

$$e^{-x} \geq 1 - \frac{2x}{2+x}.$$

For any $x \leq 0$, this inequality is reversed.

Proof. Two proofs are given, with different mathematical bases.
Proof 1. For any $x \in \mathbb{R}$, let us set

$$f(x) = x - \frac{1}{2}(x+2)(1-e^{-x}).$$

Using standard differentiation rules, a factorization and the basic exponential inequality $e^x \geq 1 + x$ for any $x \in \mathbb{R}$, we have

$$f'(x) = \frac{1}{2}e^{-x}(e^x - 1 - x) \geq 0.$$

Therefore, $f(x)$ is non-decreasing for any $x \in \mathbb{R}$. Since $f(0) = 0$, we have $f(x) \geq f(0) = 0$ for any $x \geq 0$, which is equivalent to

$$e^{-x} \geq 1 - \frac{2x}{2+x}.$$

We also have $f(x) \leq f(0) = 0$ for any $x \leq 0$, which is equivalent to

$$e^{-x} \leq 1 - \frac{2x}{2+x}.$$

The desired inequalities are established.
Proof 2. For any $y > -1$, the following logarithmic inequality is well-known:

$$\frac{\log(1+y)}{y} \geq \frac{2}{2+y}.$$

See, for instance, [3], in which nine different proofs are proposed. Applying it with $y = e^x - 1$ for any $x \in \mathbb{R}$ satisfying $y > -1$, we get

$$\frac{x}{e^x - 1} = \frac{\log(1+y)}{y} \geq \frac{2}{2+y} = \frac{2}{1+e^x}.$$

For any $x \geq 0$, the following equivalences hold:

$$\frac{x}{e^x - 1} \geq \frac{2}{1+e^x} \Leftrightarrow \frac{x}{2} \geq \frac{e^x - 1}{1+e^x} \Leftrightarrow \frac{x}{2} \geq \frac{1-e^{-x}}{1+e^{-x}} \Leftrightarrow \frac{x}{2} \geq 1 - \frac{2e^{-x}}{1+e^{-x}}$$
$$\Leftrightarrow \frac{2e^{-x}}{1+e^{-x}} \geq 1 - \frac{x}{2} \Leftrightarrow 2e^{-x} \geq \left(1 - \frac{x}{2}\right)(1+e^{-x})$$
$$\Leftrightarrow \left(1 + \frac{x}{2}\right) e^{-x} \geq 1 - \frac{x}{2} \Leftrightarrow e^{-x} \geq \frac{1 - x/2}{1 + x/2} = 1 - \frac{x}{1+x/2} = 1 - \frac{2x}{2+x}.$$

On the same mathematical basis, for any $x \leq 0$, the following equivalences hold:

$$\frac{x}{e^x - 1} \geq \frac{2}{1+e^x} \Leftrightarrow \frac{x}{2} \leq \frac{e^x - 1}{1+e^x} \Leftrightarrow \frac{x}{2} \leq \frac{1-e^{-x}}{1+e^{-x}} \Leftrightarrow \frac{x}{2} \leq 1 - \frac{2e^{-x}}{1+e^{-x}}$$
$$\Leftrightarrow \frac{2e^{-x}}{1+e^{-x}} \leq 1 - \frac{x}{2} \Leftrightarrow 2e^{-x} \leq \left(1 - \frac{x}{2}\right)(1+e^{-x})$$
$$\Leftrightarrow \left(1 + \frac{x}{2}\right) e^{-x} \leq 1 - \frac{x}{2} \Leftrightarrow e^{-x} \leq \frac{1 - x/2}{1 + x/2} = 1 - \frac{x}{1+x/2} = 1 - \frac{2x}{2+x}.$$

This ends the proof. □

For any $x \geq 0$, since $x+2 \geq 2$, we obviously have

$$e^{-x} \geq 1 - \frac{2x}{2+x} \geq 1 - x.$$

This improves a well-known exponential inequality, which is mainly of interest for $x \in (0,1)$.

As an interpretation of the first proof, this exponential inequality is equivalent to the first result in Lemma 2.1. It can be used for various mathematical purposes, where a sharp lower or upper bound of the exponential function e^{-x} is needed, depending on $x \geq 0$ or $x \leq 0$.

5 Open problem

Based on the statements of the previous propositions, one can think of searching formulas for integrals of the following form:

$$\int_0^{+\infty} \left[x - \frac{1}{2}(x+2)(1-e^{-x})\right]^{\varepsilon} \frac{1}{x^{\zeta}} e^{-vx} dx,$$

where $\varepsilon > 0$, $\zeta > 0$ and $v > 0$. This article has focused on the special case $\epsilon = 1$. However, the combined presence of ε and ζ can ensure the convergence of this integral under certain assumptions. To the best of our knowledge, there is no general formula for this.

6 Conclusion

In this article, we have completed the collection of formulas involving power, exponential and logarithmic functions within an original integral relation. Many new formulas are given, including one from an old result that inspired this research. Some of them involve the Euler-Mascheroni constant, extending existing results. The expression of the Laplace transform of certain original power-exponential functions are also obtained. The techniques used in the proofs are quite understandable and can be reused beyond the scope of the article. An open problem concludes the article, broadening its horizon with a solid mathematical challenge.

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