Int. J. Open Problems Compt. Math., Vol. 18, No. 2, June 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376

${W_0}^*$- Curvature Tensor Characterizations on ϵ- Kenmotsu Metric Space.

Yashu Prada N and H G Nagaraja

Department of Mathematics, Jnana Bharathi Campus, Bangalore University,
Bengaluru-560056 Karnataka, India.
e-mail:yashupradabu@gmail.com, hgnraj@yahoo.com

Received 10 January 2025; Accepted 13 March 2025

Abstract

The objective of the present paper is to study ${W_0}^*$ curvature tensor on ϵ-Kenmotsu manifold. We have analyzed $\xi - {W_0}^*$-flat, ϕ-${W_0}^*$- projectively semisymmetric and ${W_0}^*.Q = 0$ on ϵ-Kenmotsu manifold. Also, we have considered the conditions $R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R = 0$ and $R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R = L_M Q(g, {W_0}^*)$ in ϵ-Kenmotsu manifold. Finally, an example of 5-dimensional ϵ-Kenmotsu manifold has been constructed which verifies certain results.

Keywords: *ϵ-Kenmotsu manifold ,${W_0}^*$ curvature tensor, η-Einstein manifold, concurrent vector field.*
2010 Mathematics Subject Classification: 53B30, 53C25.

1 Introduction

The geometry of indefinite metric has significant use in physics and relativity in the credential of Einstein's theory of general relativity, as the nature of metric depend on the geometric properties of the manifold.
The notion of indefinite metrics of almost contact manifolds was firstly initiated by Takahashi[16] in 1969. One of the importance of indefinite metric is, that it allows tangent vectors to be classified into timelike, null, and spacelike vectors. These circumstances lead many authors to investigate and explore the importance and applications of the manifolds with indefinite metrics.

Kenmotsu[6] introduced a special class of contact Riemannian manifolds, satisfying certain conditions, which was later named as Kenmotsu manifold. Further De and Sarkar[3] studied on ϵ-Kenmotsu manifolds and proved the existence of new indefinite structure on the manifold which influences the curvature of the manifold. Later on several other authors[18, 9, 4, 5] studied ϵ-Kenmotsu manifolds and have found interesting results on indefinite structures.
Pokarial and Mishra[13] introduced new tensor fields in 1970 such as W_2 and E tensor fields on Riemannian manifolds. Later on Matsumoto, Ianus and Mihai[7] extended the study of these tensor fields on para Sasakian manifolds. The concept of $W_0{}^*$ curvature tensor on $2n+1$ dimensional manifold was explained by Pokhariyal[13] which is defined as follows

$$W_0{}^*(X,Y)Z = R(X,Y)Z + \frac{1}{2n}[S(Y,Z)X - g(X,Z)QY], \tag{1}$$

where R and S are the curvature tensor and Ricci tensor respectively and $X, Y, Z \in \chi(M)$. Further the studies have been made by Uygun, Dirik and Atceken on $W_0{}^*$ curvature tensor.
Motivated by the above study, in the present paper we study about $W_0{}^*$ curvature tensor on ϵ-Kenmotsu manifold. After introduction section 2 gives the basic formulas and notations of ϵ-Kenmotsu manifold. In section 3, we study $\xi - W_0{}^*$- flat ϵ-Kenmotsu manifold. Section 4 deals with ϕ-$W_0{}^*$- projectively semisymmetric condition which makes ϵ-Kenmotsu manifold an Einstein manifold. In Section 5 we study ϵ- Kenmotsu manifold satisfying $W_0{}^*.Q = 0$. Section 6 deals with the study of conditions satisfying $R(\xi,X).W_0{}^* - W_0{}^*(\xi,X).R = 0$ and $R(\xi,X).W_0{}^* - W_0{}^*(\xi,X).R = L_M Q(g, W_0{}^*)$. Finally a 5-dimensional example of ϵ-Kenmotsu manifold has been constructed which verifies certain results.

2 Preliminaries

We give some definitions and basic formulas which are necessary for proving the results. An almost contact smooth manifold (M,g) of dimension $2n+1$ is said to be almost contact metric manifold if it is a triple (ϕ,ξ,η),where ϕ is a $(1,1)$ tensor field, ξ is a characteristic vector field (Reeb vector field), η is a global 1-form, g is an associated metric of η satisfying the following ,

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0 \tag{2}$$

where I denotes the identity endomorphism.
We obtain a Riemannian metric g and a (1,1)-tensor field ϕ of contact metric on doing polarization of $d\eta$ on the contact subbundle $\eta = 0$ such that

$$d\eta(X,Y) = g(X,\phi Y), \tag{3}$$

where X, Y denote arbitrary vector fields on M. A semi-Riemannian metric[3] g on M is said to be compatible with almost contact structure if it satisfies,

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \quad g(X, \xi) = \epsilon\eta(X), \tag{4}$$

$\forall\ X, Y \in \chi(M)$, where $\chi(M)$ is a lie algebra of smooth vector fields and $\epsilon = \pm 1$.

An ϵ-contact metric manifold M satisfying

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{5}$$

where ∇ denotes the Levi-civita connection with respect to g is said to be an ϵ-Kenmotsu manifold [7].
An ϵ-almost contact metric manifold is said to be an ϵ-Kenmotsu manifold if the following relations hold [3],

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi), \tag{6}$$

$$(\nabla_X \eta)Y = [g(X, Y) - \epsilon\eta(X)\eta(Y)], \tag{7}$$

where ∇ is the Levi-Civita connection with respect to g.
In an ϵ-Kenmotsu manifold the following relations hold[3]

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$

$$R(\xi, X)Y = -\eta(Y)X - \epsilon g(X, Y)\xi, \tag{9}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{10}$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{11}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{12}$$

$$Q\xi = -\epsilon(n-1)\xi, \tag{13}$$

$$S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y), \tag{14}$$

where X,Y acts as vector fields on M.

Definition 2.1 *[6] An ϵ-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S of type (0, 2) is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{15}$$

where a and b are smooth functions on M. If $b = 0$, then an η-Einstein manifold becomes an Einstein manifold.

Definition 2.2 *[3] If the tensors $R.{W_0}^*$ and $Q(g, {W_0}^*)$ are linearly dependent then the manifold is called Weyl-pseudosymmetric which is given by*

$$R.{W_0}^* = L_M Q(g, {W_0}^*), \tag{16}$$

where, L_M is some function and

$$\begin{aligned} Q((g, {W_0}^*)(X_1, X_2, ..., X_K; X, Y) = -{W_0}^*((X \wedge Y)X_1, X_2, ..., X_K) - ... \\ -{W_0}^*(X_1, ..., X_K - 1, (X \wedge Y)X_K) \end{aligned} \tag{17}$$

$$(X \wedge Y)Z = g(Y, Z) - g(X, Z)Y \tag{18}$$

for $(0, K)$-tensor field, $K \geq 1$ on M.

3 ξ-${W_0}^*$- flat ϵ- Kenmotsu manifold

Definition 3.1 *An ϵ-Kenmotsu manifold is said to be ξ-${W_0}^*$-flat if*

$${W_0}^*(X, Y)\xi = 0, \tag{19}$$

for any vector fields X, Y on M.

Theorem 3.2 *Let M be an ϵ- Kenmotsu manifold of dimension $2n + 1$ satisfying ξ-${W_0}^*$- flat condition. Then the manifold reduces to η-Einstein manifold.*

Proof 3.1 *Consider Equation(1) and replacing Z by ξ we get,*

$$R(X, Y)\xi + \frac{1}{2n}[S(Y, \xi)X - g(X, \xi)QY] = 0 \tag{20}$$

By making use of equations (8) and (12), we obtain

$$\eta(X)Y - \eta(Y)X - \frac{1}{2n}[(n-1)\eta(Y)X + \epsilon\eta(X)QY] = 0. \tag{21}$$

Taking inner product with respect to Z the above equation reduces to

$$\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \frac{n-1}{2n}\eta(Y)g(X, Z) - \frac{1}{2n}\epsilon\eta(X)S(Y, Z) = 0. \tag{22}$$

Replacing X by ξ and solving for $S(Y, Z)$, the above equation becomes

$$S(Y, Z) = 2n\epsilon g(Y, Z) - (3n-1)\eta(Y)\eta(Z). \tag{23}$$

Thus the result follows.

4 ϕ-${W_0}^*$- projectively semisymmetric condition on ϵ- Kenmotsu manifold

Definition 4.1 *An ϵ-Kenmotsu manifold is said to be ϕ-${W_0}^*$- projectively semisymmetric if*

$$ {W_0}^*(X,Y).\phi = 0, \tag{24} $$

for any vector fields X, Y on M.

Theorem 4.2 *Let M be an ϵ- Kenmotsu manifold of dimension $(2n+1)$ satisfying ϕ-${W_0}^*$-projectively semisymmetric condition, then the manifold becomes an Einstein manifold.*

Proof 4.1 *Now Equation(24) can be written as*

$$ {W_0}^*(X,Y)\phi Z - \phi {W_0}^*(X,Y)Z = 0. \tag{25} $$

By making use of (1) the above equation reduces to

$$ \begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{2n}S(Y,\phi Z)X - \frac{1}{2n}g(X,\phi Z)QY \\ -\frac{1}{2n}S(Y,Z)\phi X + \frac{1}{2n}g(X,Z)\phi QY = 0. \end{aligned} \tag{26} $$

Replacing $X = \xi$ and using (2) the preeceding equation becomes

$$ \frac{1}{2n}S(Y,Z)\xi - \epsilon g(Y,Z)\xi + \eta(Z)\phi Y + \frac{1}{2n}\epsilon\eta(Z)\phi QY = 0 \tag{27} $$

Now, by taking inner product of the above with X, we get

$$ \frac{1}{2n}\epsilon S(Y,Z)\eta(X) - g(Y,Z)\eta(X) + \eta(Z)g(\phi Y,X) + \frac{1}{2n}\epsilon\eta(Z)S(\phi Y,X) = 0 \tag{28} $$

Replacing X by ξ the above equation reduces to

$$ S(Y,Z) = 2n\epsilon g(Y,Z.) \tag{29} $$

Thus the result follows.

Theorem 4.3 *An ϵ-Kenmotsu manifold is $\phi - {W_0}^*$-projectively semisymmetric if and only if it is ${W_0}^*$-projectively flat.*

Proof 4.2 *Considering (26) by plugging $Y = \xi$, we get*

$$ R(X,\xi)\phi Z + \frac{1}{2n}[S(\xi,\phi Z)X - g(X,\phi Z)Q\xi] - \phi[R(X,\xi)Z + \frac{1}{2n}[S(\xi,Z)X - g(X,Z)Q\xi] = 0. \tag{30} $$

By making use of (2) and (9), we obtain

$$[\epsilon + \frac{\epsilon(n-1)}{2n}]g(X, \phi Z)\xi - [\epsilon + \frac{\epsilon(n-1)}{2n}]g(Z, \xi)\phi X = 0. \tag{31}$$

Now, by taking inner product with W, the above equation reduces to

$$[\epsilon + \frac{\epsilon(n-1)}{2n}][\epsilon g(X, \phi Z)\eta(W) - \epsilon\eta(Z)g(\phi X, W)] = 0. \tag{32}$$

Replacing $W = \xi$ the above equation becomes

$$\frac{3n-1}{2n}g(X, \phi Z) = 0. \tag{33}$$

Since $g(X, \phi Z) \neq 0$, we have $\frac{3n-1}{2n} = 0$. Hence the manifold is ${W_0}^*$-projectively flat.
Conversely, suppose that M is ${W_0}^*$-projectively flat, then we have ${W_0}^* = 0$ and hence ${W_0}^*(X, Y).\phi = 0$. Thus the result follows.

5 ϵ- Kenmotsu manifold satisfying ${W_0}^*.Q = 0$.

In this section we study ϵ-Kenmotsu manifold satisfying,

$${W_0}^*.Q = 0. \tag{34}$$

Theorem 5.1 *An ϵ-Kenmotsu manifold satisfying ${W_0}^*.Q = 0$ is an η-Einstein manifold.*

Proof 5.1 *Now, Equation(34) can be written as,*

$${W_0}^*(X, Y)QZ - Q({W_0}^*(X, Y)Z) = 0. \tag{35}$$

Replacing Y by ξ, the above equation becomes

$${W_0}^*(X, \xi)QZ - Q({W_0}^*(X, \xi)Z) = 0. \tag{36}$$

Now, by making use of equation (1), we get

$$R(X, \xi)QZ + \frac{1}{2n}[S(\xi, QZ)X - g(X, QZ)Q\xi] - Q[R(X, \xi)Z + \frac{1}{2n}[s(\xi, Z)X - g(X, Z)Q\xi]] = 0. \tag{37}$$

Using (2), (9) and (13), we obtain

$$(1 - \frac{n-1}{2n})\eta(QZ)X + \epsilon(1 + \frac{n-1}{2n})g(X, QZ)\xi + (1 + \frac{n-1}{2n})\eta(Z)QX - \epsilon(1 + (n-1))g(X, Z)Q\xi = 0. \tag{38}$$

Taking inner product of the above with Y we get

$$(1-\frac{n-1}{2n})\eta(QZ)g(X,Y)+(1+\frac{n-1}{2n})S(X,Z)\eta(Y)+(1+\frac{n-1}{2n})\eta(Z)S(X,Y)$$
$$+\epsilon n(n-1)g(X,Z)\eta(Y)=0. \tag{39}$$

Replacing Z by ξ and using (13) the above equation becomes

$$\frac{3n-1}{2n}S(X,Y)-\frac{n+1}{2n}\epsilon(n-1)g(X,Y)-(n-1)\frac{3n-1}{2n}\eta(X)\eta(Y)+n(n-1)\eta(X)\eta(Y)=0. \tag{40}$$

Rearranging the above equation we get

$$S(X,Y)=\epsilon(n-1)\frac{n+1}{3n-1}g(X,Y)+(n-1)\frac{1-2n^2}{3n-1}\eta(X)\eta(Y). \tag{41}$$

Thus the result follows.

6 ϵ- Kenmotsu manifold satisfying $R(\xi,X).{W_0}^*-{W_0}^*(\xi,X).R=0$ and $R(\xi,X).{W_0}^*-{W_0}^*(\xi,X).R=L_MQ(g,{W_0}^*)$ conditions

Theorem 6.1 *An ϵ-Kenmotsu manifold satisfying $R(\xi,X).{W_0}^*-{W_0}^*(\xi,X).R=0$ condition represents an η-Einstein manifold.*

Proof 6.1 *Writing the terms $R(\xi,X).{W_0}^*$ and ${W_0}^*(\xi,X).R=0$ explicitly, we have*

$$\begin{aligned}(R(\xi,X).{W_0}^*)(Y,Z)W &= R(\xi,X){W_0}^*(Y,Z)W-{W_0}^*(R(\xi,X)Y,Z)W\\ &\quad -{W_0}^*(Y,R(\xi,X)Z)W-{W_0}^*(Y,Z)R(\xi,X)W,\end{aligned} \tag{42}$$

and

$$\begin{aligned}({W_0}^*(\xi,X).R)(Y,Z)W &= {W_0}^*(\xi,X)R(Y,Z)W-R({W_0}^*(\xi,X)Y,Z)W\\ &\quad -R(Y,{W_0}^*(\xi,X)Z)W-R(Y,Z){W_0}^*(\xi,X)W.\end{aligned} \tag{43}$$

By making use of equations (9) and (1), (42) takes the form

$$(R(\xi,X).W_0{}^*)(Y,Z)W = -R(Y,Z)X\eta(W) - \frac{1}{2n}S(Z,W)\eta(Y)X + \frac{1}{2n}g(Y,W)\eta(QZ)X$$
$$-\epsilon g(X,W)R(Y,Z)\xi - \frac{1}{2n}\epsilon g(X,W)S(Z,\xi)Y + \frac{1}{2n}\epsilon g(X,W)g(Y,\xi)QZ + \eta(Y)R(X,Z)W$$
$$+\frac{1}{2n}\eta(Y)S(Z,W)X - \frac{1}{2n}g(X,W)\eta(Y)QZ + \epsilon g(X,Y)R(\xi,Z)W + \frac{1}{2n}\epsilon g(X,Y)S(Z,W)\xi$$
$$-\frac{1}{2n}\epsilon g(X,Y)g(\xi,W)QZ + \eta(Z)R(Y,X)W + \frac{1}{2n}S(X,W)\eta(Z)Y - \frac{1}{2n}g(Y,W)\eta(Z)QX$$
$$+\epsilon g(X,Z)R(Y,\xi)W + \frac{1}{2n}\epsilon g(X,Z)S(\xi,W)Y - \frac{1}{2n}\epsilon g(X,Z)g(Y,W)Q\xi + \eta(W)R(Y,Z)X$$
$$+\frac{1}{2n}\eta(W)S(Z,X)Y - \frac{1}{2n}\eta(W)g(Y,X)QZ + \epsilon g(X,W)R(Y,Z)\xi + \frac{1}{2n}\epsilon g(X,W)S(Z,\xi)Y$$
$$-\frac{1}{2n}\epsilon g(X,W)g(Y,\xi)QZ. \tag{44}$$

Similarly (43) takes the form

$$(W_0{}^*(\xi,X).R)(Y,Z)W = -R(Y,Z)X\eta(W) - \epsilon\eta(Y)Zg(X,W) + \epsilon\eta(Z)Yg(X,W)$$
$$+\frac{1}{2n}\eta(Y)ZS(X,\xi) - \frac{1}{2n}\eta(Z)YS(X,\xi) - \frac{1}{2n}R(Y,Z)QXg(\xi,W) + \eta(Y)R(X,Z)W$$
$$-\epsilon g(X,Y)\eta(W)Z - g(X,Y)g(Z,W)\xi + \frac{1}{2n}S(X,Y)\eta(W)Z + \frac{1}{2n}\epsilon S(X,Y)g(Z,W)\xi$$
$$+\frac{1}{2n}g(\xi,Y)R(QX,Z)W + \eta(Z)R(Y,X)W + \epsilon g(X,Z)\eta(W)Y + g(X,Z)g(Y,W)\xi$$
$$-\frac{1}{2n}S(X,Z)\eta(W)Y + \frac{1}{2n}S(X,Z)\epsilon g(Y,W)\xi + \frac{1}{2n}g(\xi,Z)R(Y,QX)W + \eta(W)R(Y,Z)X$$
$$+\epsilon\eta(Y)g(X,W)Z - \epsilon g(X,W)\eta(Z)Y - \frac{1}{2n}S(X,W)\eta(Y)Z + \frac{1}{2n}S(X,W)\eta(Z)Y$$
$$+\frac{1}{2n}g(\xi,W)R(Y,Z)QX \tag{45}$$

By taking inner product with ξ for the equations (44) and(45), using the condition $R(\xi,X).W_0{}^* - W_0{}^*(\xi,X).R = 0$ and replacing $Y = W = \xi$, we obtain

$$-g(Z,X) + \frac{(n-1)}{2n}\epsilon\eta(X)\eta(Z) + \epsilon\eta(X)\eta(Z) - g(X,Z) + \frac{(n-1)}{2n}\epsilon\eta(X)\eta(Z)$$
$$+\epsilon\eta(X)\eta(Z) + \frac{\epsilon}{2n}S(X,Z) = 0, \tag{46}$$

which on simplification gives,

$$S(X,Z) = 4n\epsilon g(X,Z) - 2(n+1)\eta(X)\eta(Y). \tag{47}$$

Thus the result follows.

Theorem 6.2 *An ϵ-Kenmotsu manifold M with the condition $R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R = L_M Q(g, {W_0}^*)$ is an η-Einstein manifold if $L_M = 0$ or, $R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R = -Q(g, {W_0}^*)$.*

Proof 6.2 *We now consider,*

$$R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R = L_M Q(g, {W_0}^*) \tag{48}$$

From previous theorem, it is noticed that the value for $R(\xi, X).{W_0}^* - {W_0}^*(\xi, X).R$ is given by equation (47), now we find the value for $L_M Q(g, {W_0}^*)$.

$$\begin{aligned} L_M Q(g, {W_0}^*) = L_M[&(\xi \wedge X){W_0}^*(Y, Z)W - {W_0}^*((\xi \wedge X)Y, Z)W \\ &-{W_0}^*(Y, (\xi \wedge X)Z)W - {W_0}^*(Y, Z)(\xi \wedge X)W] \end{aligned} \tag{49}$$

By making use of equations (18) and (1) in previous equation, we obtain

$$\begin{aligned} L_M Q(g, {W_0}^*) &= L_M[-R(\xi, Z)Wg(X, Y) - \frac{1}{2n}S(Z, W)g(X, Y)\xi \\ &+\frac{1}{2n}g(\xi, W)g(X, Y)QZ + g(\xi, Y)R(X, Z)W + \frac{1}{2n}S(Z, W)g(\xi, Y)X \\ &-\frac{1}{2n}g(X, W)g(\xi, Y)QZ - g(X, Z)R(Y, \xi)W - \frac{1}{2n}g(X, Z)S(\xi, W)Y \\ &+\frac{1}{2n}g(Y, W)g(X, Z)Q\xi + R(Y, X)Wg(\xi, Z) + \frac{1}{2n}S(X, W)g(\xi, Z)Y \\ &-\frac{1}{2n}g(Y, W)g(\xi, Z)QX]. \end{aligned} \tag{50}$$

Using equation (2) and (9), the above equation becomes

$$\begin{aligned} L_M Q(g, {W_0}^*) &= L_M[\eta(W)Zg(X, Y) + \epsilon g(Z, W)\xi g(X, Y) - \frac{1}{2n}S(Z, W)g(X, Y)\xi \\ &+\frac{1}{2n}\epsilon\eta(W)g(X, Y)QZ + \epsilon\eta(Y)R(X, Z)W + \frac{1}{2n}\epsilon\eta(Y)S(Z, W)X - \frac{1}{2n}\epsilon\eta(Y)g(X, W)QZ \\ &-\eta(W)g(X, Z)Y - \epsilon g(Y, W)\xi g(X, Z) + \frac{(n-1)}{2n}\eta(W)g(X, Z)Y - \frac{(n-1)}{2n}\epsilon g(Y, W)g(X, Z)\xi \\ &+\epsilon\eta(Z)R(Y, X)W + \frac{1}{2n}\epsilon\eta(Z)S(X, W)Y - \frac{1}{2n}\epsilon\eta(Z)g(Y, W)QX] \end{aligned} \tag{51}$$

Taking inner product with ξ, and replacing $Y = W = \xi$, we get

$$L_M Q(g, {W_0}^*) = [-2\epsilon g(X, Z) + (2 - \epsilon)\eta(Z)\eta(X)] \tag{52}$$

From (47) and (52), (48) becomes

$$S(X, Z) - 4n\epsilon g(X, Z) + 2(n+1)\eta(X)\eta(Z) = L_M[-2\epsilon g(X, Z) + (2 - \epsilon)\eta(Z)\eta(X)]. \tag{53}$$

On simplification the above equation reduces to

$$S(X, Z) = (1 - L_M)(2(2n+1)\epsilon g(X, Z) - (2n + \epsilon)\eta(X)\eta(Z)). \tag{54}$$

7 Example

We consider 5-dimensional manifold $\{M = (x_1, x_2, y_1, y_2, z) \in R^5\}$ where (x_1, x_2, y_1, y_2, z) are the standard co-ordinates in R^5. Let e_1, e_2, e_3, e_4, e_5 be the vector fields on M given by
$e_1 = \epsilon z\frac{\partial}{\partial x_1}, \quad e_2 = \epsilon z\frac{\partial}{\partial x_2}, \quad e_3 = -\epsilon z\frac{\partial}{\partial y_1}, \quad e_4 = -\epsilon z\frac{\partial}{\partial y_2}, \quad e_5 = -\epsilon z\frac{\partial}{\partial z},$
which are linearly independent forming a basis of T_pM.
Let g be a semi- Riemannian metric on M defined as

$$g(e_i, e_j) = \begin{cases} 0, & \text{if } i \neq j, \\ \epsilon, & \text{if } i = j = 5. \end{cases} \tag{55}$$

Set $e_5 = \xi$. Let η be a 1-form on M defined by $\eta(X) = \epsilon g(X, e_5) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$.
Also, we define (1,1)-tensor ϕ as

$$\phi(e_1) = \epsilon e_2, \quad \phi(e_2) = -\epsilon e_1, \quad \phi(e_3) = \epsilon e_4, \quad \phi(e_4) = -\epsilon e_3, \quad \phi(e_5) = 0.$$

In consequence of the above equations, the linearity property of ϕ and g yields

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \forall X, Y \in \chi(M).$$

The above relations imply that the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact structure on M.
Now by direct computations, we obtain

$$\begin{array}{ccccc} [e_1, e_1] = 0, & [e_1, e_2] = 0, & [e_1, e_3] = 0, & [e_1, e_4] = 0, & [e_1, e_5] = \epsilon e_1, \\ [e_2, e_1] = 0, & [e_2, e_2] = 0, & [e_2, e_3] = 0, & [e_2, e_4] = 0, & [e_2, e_5] = \epsilon e_2, \\ [e_3, e_1] = 0, & [e_3, e_2] = 0, & [e_3, e_3] = 0, & [e_3, e_4] = 0, & [e_3, e_5] = \epsilon e_3, \\ [e_4, e_1] = 0, & [e_4, e_2] = 0, & [e_4, e_3] = 0, & [e_4, e_4] = 0, & [e_4, e_5] = \epsilon e_4, \end{array}$$
$$[e_5, e_1] = -\epsilon e_1, \quad [e_5, e_2] = -\epsilon e_2, \quad [e_5, e_3] = -\epsilon e_3, \quad [e_5, e_4] = -\epsilon e_4,$$
$$[e_5, e_5] = 0.$$

By making use of Koszul's formula, we can easily calculate

$$\begin{array}{ccccc} \nabla_{e_1} e_1 = -\epsilon e_5, & \nabla_{e_2} e_1 = 0, & \nabla_{e_3} e_1 = 0, & \nabla_{e_4} e_1 = 0, & \nabla_{e_5} e_1 = 0, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = -\epsilon e_5, & \nabla_{e_3} e_2 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_5} e_2 = 0, \\ \nabla_{e_1} e_3 = 0, & \nabla_{e_2} e_3 = 0, & \nabla_{e_3} e_3 = -\epsilon e_5, & \nabla_{e_4} e_3 = 0, & \nabla_{e_5} e_3 = 0, \\ \nabla_{e_1} e_4 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_3} e_4 = 0, & \nabla_{e_4} e_4 = -\epsilon e_5, & \nabla_{e_5} e_4 = 0, \\ \nabla_{e_1} e_5 = \epsilon e_1, & \nabla_{e_2} e_5 = \epsilon e_2, & \nabla_{e_3} e_5 = \epsilon e_3, & \nabla_{e_4} e_5 = \epsilon e_4, & \nabla_{e_5} e_5 = 0. \end{array}$$

Also, from the above relations we can verify that

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi) \text{ and } (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X.$$

Therefore the manifold $M(\phi, \xi, \eta, \mathrm{g})$ represents an ϵ-kenmotsu manifold. Now we calculate Riemann curvature tensor from the well known formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ which can be obtained as follows.

$$\begin{array}{l} R(e_1,e_2)e_2 = -e_1, R(e_1,e_3)e_3 = -e_1, R(e_1,e_4)e_4 = -e_1, R(e_1,e_5)e_5 = -e_1, \\ R(e_2,e_1)e_1 = -e_2, R(e_2,e_3)e_3 = -e_2, R(e_2,e_4)e_4 = -e_2, R(e_2,e_5)e_5 = -e_2, \\ R(e_3,e_1)e_1 = -e_3, R(e_3,e_2)e_2 = -e_3, R(e_3,e_4)e_4 = -e_3, R(e_3,e_5)e_5 = -e_3, \\ R(e_4,e_1)e_1 = -e_4, R(e_4,e_2)e_2 = -e_4, R(e_4,e_3)e_3 = -e_4, R(e_4,e_5)e_5 = -e_4, \\ R(e_5,e_1)e_1 = -e_5, R(e_5,e_2)e_2 = -e_5, R(e_5,e_3)e_3 = -e_5, R(e_5,e_4)e_4 = -e_5, \end{array}$$

From the above values the Ricci tensor can be calculated as follows,

$$S(e_1,e_1) = -4, \quad S(e_2,e_2) = -4, \quad S(e_3,e_3) = -4,$$
$$S(e_4,e_4) = -4, \quad S(e_5,e_5) = -4, \text{ which verifies Theorem[3.2].}$$

8 Open Problem

In this paper we have found ${W_0}^*$- curvature tensor on ϵ-Kenmotsu manifold. Further the charecterizations can be done by using other forms of curvature tensors where the results will be analysed differently.

References

[1] D. E. Blair., Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics 509, Springer- Verlag Berlin-New York, (1976).

[2] U. C. De., A., Sarkar, On ϵ-Kenmotsu manifolds, Hardonic J 32(2),(2009) 231-242.

[3] R. Deszcz, On pseudosymmetric spaces, Bull. Soc. Math. Belg., 49,134-145,1990.

[4] A. Haseeb., Some results on projective curvature tensor in an ϵ-Kenmotsu manifold, Palestine J. Math. 6(Special Issue:II),(2017), 196–203.

[5] A. Haseeb.,M. A. Khan., M. D. Siddiqi., Some more results on an ϵ-Kenmotsu manifold with a semi-symmetric metric connection. Acta Math. Univ. Comenianae 85(2016), 9–20.

[6] K. Kenmotsu ., A class of almost contact Riemannian manifold. Tohoku Math. J. 24(1972), 93–103.

[7] K. Matsumoto ., S. Ianus ., Ion Mihai. On P-Saskian manifolds which admit certain tensor fields. Publ. Math. Debrecen. 1986;33:61-65.

[8] S. K. Moindi, B. M. Nzimbi, Study of W_4-Curvature tensor in Para-Kenmotsu manifolds, Int. journal of Mathematics Trends and Technology. Vol (68)(11)(2022), 73-78.

[9] R. N. Singh ., S. K. Pandey., G. Pandey., K. Tiwari ., On a semi-symmetric metric connection in an ϵ-Kenmotsu manifold. Commun. Korean Math. Soc. 29(2014), 331– 343.

[10] K. L. Sai Prasad, S. Sunitha Devi and G. V. S. R. Deekshitulu, Study on a class of P- Kenmotsu manifolds admitting Weyl-projective Curvature tensor of type $(1, 3)$, Recent advances in Mathematical Research and Computer Science, Vol.(1). DOI:10.9734/bpi/ramrcs/v1/4441F.

[11] D. G. Prakash, M. R. Amruthalakshmi, F. Mofarreh and A. Haseeb, Generalized Lorentzian Sasakian-Space-Forms with M-Projective curvature tensor, Mathematics, 10(2022),2869. https://doi.org/10.3390/math10162869.

[12] G. P. Pokhariyal and R. S. Mishra, "Curvature Tensors and Their Relativistic Significance II," Yokohama Mathematical Journal, vol.19, no. 2, pp. 97-103, 1971.

[13] G. P. Pokhariyal, "Curvature Tensors in a Lorentzian Para-Sasakian Manifold," Quaestiones Math, vol. 19, no. 12 , pp. 129-136, 1996.

[14] G.P. Singh, Rajan, A. K. Mishra and P. Prajapati, W_8- Curvature tensor in generalized Sasakian- space- forms, Ratio Mathematica, Vol. (48)(2023).

[15] K. L. Sai Prasad, S. Sunitha Devi and G. V. S. R. Deekshitulu, Study on a class of P- Kenmotsu manifolds admitting Weyl-projective Curvature tensor of type $(1, 3)$, Recent advances in Mathematical Research and Computer Science, Vol.(1). DOI:10.9734/bpi/ramrcs/v1/4441F.

[16] Takashi, T., Sasakian manifold with pseudo-Riemannian metric, Tohuku Math.J.,Second Series, 21(1969), 271-290.

[17] P. Uygun, S. Dirik and M.Atceken, Some curvature Characterizations on Kenmotsu Metric Spaces, Gulf Journal of mathematics, Vol. 13(2)(2022), 78-86.

[18] Venkatesha, Vishnuvardhana, S. V., ϵ-Kenmotsu manifolds admitting a semisymmetric metric connection. Italian J. Pure Appl Math. 38(2017), 615–623 .

[19] X. Xu, X. Chao, Two theorems on ϵ-sasakian manifolds, internet.J.Math.Sci. 21(2)(1998), 249-254.