*Int. J. Open Problems Compt. Math., Vol. 18, No. 2, June 2025*
*Print ISSN: 1998-6262, Online ISSN: 2079-0376*


# Several New Algebraic Inequalities for Arithmetic sequence

**Mingcong Li and Li Yin**

Department of Mathematics, Shandong University of Aeronautics,
Binzhou City, Shandong Province P.O.Box 256603 China.
e-mail:yinli_79@163.com

Received 19 January 2025; Accepted 5 March 2025

**Abstract**

***By applying $c_p$ inequality, this paper mainly proves several new algebraic inequalities for arithmetic sequence.***

**Keywords:** *Arithmetic sequence; Geometric sequence; Nesbitt-type inequality*

**2010 Mathematics Subject Classification:** *33B15.*

## 1 Introduction

The inequalities of arithmetic sequence are an interesting research topic. Recently, in reference [1], the authors discussed the Nesbitt-type inequalities of the general term and the sum of the first $n$ terms of arithmetic sequence. Before the discussion, it is necessary to extend the definition of the arithmetic sequence. Suppose $a_n$ is an arithmetic sequence with the first term $a_1$ and the common difference $d$. On this basis, it is stipulated that $n$ can take any real number, and the corresponding forms of its general term formula and the formula for the sum of the first $n$ terms can be written similarly. Such a sequence can be called a generalized arithmetic sequence. The extension of the geometric sequence is similar.

As is well known, the Nesbitt inequality is an extremely important inequality in algebraic inequalities, and its form can be described as: when $a, b, c$ are positive numbers, the following holds

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}. \tag{1}$$

In September 2022, M. Afifurrahman[[2]] proposed a problem 4764 in the 9th issue of the Crux Math.: Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$. Then, for any $p, q, r \in \mathbb{N}^+$, the following holds

$$\frac{a_{2p}}{a_{q+r}}+\frac{a_{2q}}{a_{r+p}}+\frac{a_{2r}}{a_{p+q}}\geq 3. \tag{2}$$

Next, we examine some similar Nesbitt-type algebraic inequalities.

# 2 Main results

**Theorem 2.1** *Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$. Then, for any $p, q, r \in \mathbb{N}^+, \alpha \geq 1$, the following holds*

$$\left(\frac{a_{2p}}{a_{q+r}}\right)^{\alpha}+\left(\frac{a_{2q}}{a_{r+p}}\right)^{\alpha}+\left(\frac{a_{2r}}{a_{p+q}}\right)^{\alpha}\geq 3. \tag{3}$$

*Proof.* By using $c_p$ inequality and the inequality (2), we have

$$\begin{aligned}&\left(\tfrac{a_{2p}}{a_{q+r}}\right)^{\alpha}+\left(\tfrac{a_{2q}}{a_{r+p}}\right)^{\alpha}+\left(\tfrac{a_{2r}}{a_{p+q}}\right)^{\alpha}\\ &\geq \tfrac{1}{3^{\alpha-1}}\left(\tfrac{a_{2p}}{a_{q+r}}+\tfrac{a_{2q}}{a_{r+p}}+\tfrac{a_{2r}}{a_{p+q}}\right)^{\alpha}\\ &\geq \tfrac{1}{3^{\alpha-1}}3^{\alpha}=3.\end{aligned}$$

**Remark 2.2** *In fact, here $p, q, r > 1$ is sufficient. The general term formula is still defined by $a_n = a_1 + (n-1)d$. The above inequality still holds.*

**Theorem 2.3** *Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$, and let the sum of the first $n$ terms be denoted as $S_n$. Then, for any $p, q, r \in \mathbb{N}^+, \alpha \geq 1$, the following holds*

$$\left(\frac{S_{2p}}{S_{q+r}}\right)^{\alpha}+\left(\frac{S_{2q}}{S_{r+p}}\right)^{\alpha}+\left(\frac{S_{2r}}{S_{p+q}}\right)^{\alpha}\geq 3. \tag{4}$$

*Proof.* By using $c_p$ inequality and the inequality(See the proposition 2 in reference [1])

$$\frac{S_{2p}}{S_{q+r}}+\frac{S_{2q}}{S_{r+p}}+\frac{S_{2r}}{S_{p+q}}\geq 3,$$

we have

$$\begin{aligned}&\left(\tfrac{S_{2p}}{S_{q+r}}\right)^{\alpha}+\left(\tfrac{S_{2q}}{S_{r+p}}\right)^{\alpha}+\left(\tfrac{S_{2r}}{S_{p+q}}\right)^{\alpha}\\ &\geq \tfrac{1}{3^{\alpha-1}}\left(\tfrac{S_{2p}}{S_{q+r}}+\tfrac{S_{2q}}{S_{r+p}}+\tfrac{S_{2r}}{S_{p+q}}\right)^{\alpha}\\ &\geq 3.\end{aligned}$$

**Theorem 2.4** *Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$. Then, for any $m, n, p, q \in \mathbb{N}^+, \alpha \geq 1$, the following holds*

$$\left(\frac{a_{3p}}{a_{q+m+n}}\right)^\alpha + \left(\frac{a_{3q}}{a_{m+n+p}}\right)^\alpha + \left(\frac{a_{3m}}{a_{n+p+q}}\right)^\alpha + \left(\frac{a_{3n}}{a_{p+q+m}}\right)^\alpha \geq 4. \tag{5}$$

*Proof.* Using $c_p$ inequality, we have

$$\begin{aligned}&\left(\frac{a_{3p}}{a_{q+m+n}}\right)^\alpha + \left(\frac{a_{3q}}{a_{m+n+p}}\right)^\alpha + \left(\frac{a_{3m}}{a_{n+p+q}}\right)^\alpha + \left(\frac{a_{3n}}{a_{p+q+m}}\right)^\alpha \\ &\geq \frac{1}{4^{\alpha-1}}\left(\frac{a_{3p}}{a_{q+m+n}} + \frac{a_{3q}}{a_{m+n+p}} + \frac{a_{3m}}{a_{n+p+q}} + \frac{a_{3n}}{a_{p+q+m}}\right)^\alpha.\end{aligned}$$

Therefore, to prove that inequality (5) holds, it is only necessary to prove

$$\frac{a_{3p}}{a_{q+m+n}} + \frac{a_{3q}}{a_{m+n+p}} + \frac{a_{3m}}{a_{n+p+q}} + \frac{a_{3n}}{a_{p+q+m}} \geq 4. \tag{6}$$

In fact, from the general term formula of the arithmetic sequence, we know that inequality (6) is equivalent to

$$\begin{aligned}&\frac{a_1+(3p-1)d}{a_1+(q+m+n-1)d} + \frac{a_1+(3q-1)d}{a_1+(m+n+p-1)d} \\ &+\frac{a_1+(3m-1)d}{a_1+(n+p+q-1)d} + \frac{a_1+(3n-1)d}{a_1+(p+q+m-1)d} \geq 4 \\ &\Leftrightarrow \frac{(3p-q-m-n)d}{a_1+(q+m+n-1)d} + \frac{(3q-m-n-p)d}{a_1+(m+n+p-1)d} \\ &+\frac{(3m-n-p-q)d}{a_1+(n+p+q-1)d} + \frac{(3n-p-q-m)d}{a_1+(p+q+m-1)d} \geq 0.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&(p-q)^2 d\frac{1}{[a_1+(q+m+n-1)d][a_1+(m+n+p-1)d]} \\ &+(q-m)^2 d\frac{1}{[a_1+(m+n+p-1)d][a_1+(n+p+q-1)d]} \\ &+(p-m)^2 d\frac{1}{[a_1+(q+m+n-1)d][a_1+(n+p+q-1)d]} \\ &+(p-n)^2 d\frac{1}{[a_1+(q+m+n-1)d][a_1+(p+q+m-1)d]} \\ &+(q-n)^2 d\frac{1}{[a_1+(m+n+p-1)d][a_1+(p+q+m-1)d]} \\ &+(m-n)^2 d\frac{1}{[a_1+(n+p+q-1)d][a_1+(p+q+m-1)d]} > 0.\end{aligned}$$

It is obvious that this expression is greater than 0, and the theorem is proved.

**Theorem 2.5** *Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$, and let the sum of the first $n$ terms be denoted as $S_n$. Then, for any $p, q, m, n \in \mathbb{N}^+$, the following holds*

$$\frac{S_{3p}}{S_{q+m+n}} + \frac{S_{3q}}{S_{m+n+p}} + \frac{S_{3m}}{S_{n+p+q}} + \frac{S_{3n}}{S_{p+q+m}} \geq 4. \tag{7}$$

*Proof.* Simple computation yields

$$\frac{S_{3p}}{S_{q+m+n}}+\frac{S_{3q}}{S_{m+n+p}}+\frac{S_{3m}}{S_{n+p+q}}+\frac{S_{3n}}{S_{p+q+m}}=\frac{3p\left[a_1+\left(\frac{3}{2}p-\frac{1}{2}\right)d\right]}{(q+m+n)\left[a_1+\left(\frac{q+m+n}{2}-\frac{1}{2}\right)d\right]}$$
$$+\frac{3q\left[a_1+\left(\frac{3}{2}q-\frac{1}{2}\right)d\right]}{(m+n+p)\left[a_1+\left(\frac{m+n+p}{2}-\frac{1}{2}\right)d\right]}+\frac{3m\left[a_1+\left(\frac{3}{2}m-\frac{1}{2}\right)d\right]}{(n+p+q)\left[a_1+\left(\frac{q+m+n}{2}-\frac{1}{2}\right)d\right]}$$
$$+\frac{3n\left[a_1+\left(\frac{3}{2}n-\frac{1}{2}\right)d\right]}{(p+q+m)\left[a_1+\left(\frac{p+q+m}{2}-\frac{1}{2}\right)d\right]}.$$

Without loss of generality, we may assume that $p>q>m>n>0$, and it is known that the sequence $a_n$ is an increasing arithmetic sequence. we easily obtain

$$\frac{p}{q+m+n}\geq\frac{q}{m+n+p}\geq\frac{m}{n+p+q}\geq\frac{n}{p+q+m}$$

and

$$\frac{a_{\frac{3}{2}p+\frac{1}{2}}}{a_{\frac{q+m+n+1}{2}}}\geq\frac{a_{\frac{3}{2}q+\frac{1}{2}}}{a_{\frac{m+n+p+1}{2}}}\geq\frac{a_{\frac{3}{2}m+\frac{1}{2}}}{a_{\frac{n+p+q+1}{2}}}\geq\frac{a_{\frac{3}{2}n+\frac{1}{2}}}{a_{\frac{p+q+m+1}{2}}}.$$

By applying Chebyshev's inequality, we can get

$$\frac{S_{3p}}{S_{q+m+n}}+\frac{S_{3q}}{S_{m+n+p}}+\frac{S_{3m}}{S_{n+p+q}}+\frac{S_{3n}}{S_{p+q+m}}$$
$$\geq\frac{3}{4}\left(\frac{p}{q+m+n}+\frac{q}{m+n+p}+\frac{m}{n+p+q}+\frac{n}{p+q+m}\right)$$
$$\left(\frac{a_{\frac{3}{2}p+\frac{1}{2}}}{a_{\frac{q+m+n+1}{2}}}+\frac{a_{\frac{3}{2}q+\frac{1}{2}}}{a_{\frac{m+n+p+1}{2}}}+\frac{a_{\frac{3}{2}m+\frac{1}{2}}}{a_{\frac{n+p+q+1}{2}}}+\frac{a_{\frac{3}{2}n+\frac{1}{2}}}{a_{\frac{p+q+m+1}{2}}}\right).$$

Considering to the identity

$$\frac{a_{\frac{3}{2}p+\frac{1}{2}}}{a_{\frac{q+m+n+1}{2}}}+\frac{a_{\frac{3}{2}q+\frac{1}{2}}}{a_{\frac{m+n+p+1}{2}}}+\frac{a_{\frac{3}{2}m+\frac{1}{2}}}{a_{\frac{n+p+q+1}{2}}}+\frac{a_{\frac{3}{2}n+\frac{1}{2}}}{a_{\frac{p+q+m+1}{2}}}$$
$$=\frac{a_{3\left(\frac{1}{2}p+\frac{1}{6}\right)}}{a_{\frac{1}{2}q+\frac{1}{6}+\frac{1}{2}m+\frac{1}{6}+\frac{1}{2}n+\frac{1}{6}}}+\frac{a_{3\left(\frac{1}{2}q+\frac{1}{6}\right)}}{a_{\frac{1}{2}m+\frac{1}{6}+\frac{1}{2}n+\frac{1}{6}+\frac{1}{2}p+\frac{1}{6}}}+\frac{a_{3\left(\frac{1}{2}m+\frac{1}{6}\right)}}{a_{\frac{1}{2}n+\frac{1}{6}+\frac{1}{2}p+\frac{1}{6}+\frac{1}{2}q+\frac{1}{6}}}+\frac{a_{3\left(\frac{1}{2}n+\frac{1}{6}\right)}}{a_{\frac{1}{2}p+\frac{1}{6}+\frac{1}{2}q+\frac{1}{6}+\frac{1}{2}m+\frac{1}{6}}}.$$

By applying the quaternary Nesbitt inequality and the formula (6), we can complete the proof.

Similar to the proof of the Theorem 2.2. we can get the following theorem.

**Theorem 2.6** *Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$, and let the sum of the first $n$ terms be denoted as $S_n$. Then, for any $p,q,m,n\in\mathbb{N}^+,\alpha\geq 1$, the following holds*

$$\left(\frac{S_{3p}}{S_{q+m+n}}\right)^\alpha+\left(\frac{S_{3q}}{S_{m+n+p}}\right)^\alpha+\left(\frac{S_{3m}}{S_{n+p+q}}\right)^\alpha+\left(\frac{S_{3n}}{S_{p+q+m}}\right)^\alpha\geq 4. \quad (8)$$

The above similar conclusions can be extended to the geometric sequence. Finally, we pose an open problem.

**Open Problem 2.1** Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$. Then, for any $l_1, l_2, \cdots l_n \in \mathbb{N}^+, \alpha \geq 1$, the following holds

$$\sum \left( \frac{a_{(n-1)l_i}}{a_{l_1+l_2+\cdots l_{i-1}+l_{i+1}+\cdots+l_n}} \right)^{\alpha} \geq n. \tag{9}$$

**Open Problem 2.2** Let $a_n$ be an arithmetic sequence with the first term $a_1$ and the common difference $d$ and let the sum of the first $n$ terms be denoted as $S_n$. Then, for any $l_1, l_2, \cdots l_n \in \mathbb{N}^+, \alpha \geq 1$, the following holds

$$\sum \left( \frac{S_{(n-1)l_i}}{S_{l_1+l_2+\cdots l_{i-1}+l_{i+1}+\cdots+l_n}} \right)^{\alpha} \geq n. \tag{10}$$

# References

[1] J. Li, Z. -M. Song, L. Yin, *Nesbitt-Type Inequalities for Arithmetic Progressions,* Math. Comp.(Shùxuétōngxùn), **2**(2023), 38-39.

[2] M. Afifurrahman, *Problem 4764,* Crux. Math., **9**(2022).