Int. J. Open Problems Compt. Math., Vol. 18, No. 1, March 2025
Print ISSN: 1998-6262, Online ISSN: 2079-0376
Copyright ©ICSRS Publication, 2025

The poisson Fibonacci binomial q-Pascal matrix of triple difference operator of fractional

C.Priya[1], Nagarajan Subramanian[1], Ayhan Esi[2], and M. Kemal Ozdemir[3]

[1]Department of Mathematics, Bishop Heber College, Trichirappalli – 620102, India
e-mail: priya.ma@bhc.edu.in, nsmaths@gmail.com
[2]Department of Basic Engineering Sciences, Malatya Turgut Ozal University, 44040, Malatya, Turkey
e-mail: aesi23@hotmail.com
[3]Department of Mathematics, Inonu University, 44280 Malatya, Turkey
e-mail: kozdemir73@gmail.com

Abstract

In this paper we introduce Poisson Fibonacci binomial q-Pascal matrix of triple difference operator F_{Δ}^{α} of fractional order α by the composition of poisson Fibonacci binomial q-Pascal matrix F and difference operator $\Delta^{(\alpha)}$ of fractional order α defined by

$$(\Delta^{\alpha} x_{mnk}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{u+v+w}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{u!v!w!\Gamma(\alpha-u+1)\Gamma(\beta-v+1)\Gamma(\gamma-w+1)}$$

and introduce sequence spaces $l_{m,n,k,F}(\Delta^{\alpha})$ and $l_{m,n,k,F}^{\infty}(\Delta^{\alpha})$. We present some topological properties, obtain Schauder basis and determine α-, β-, and γ-duals of the spaces $l_{F}^{3}(\Delta^{\alpha})$ and $l_{\infty,F}^{3}(\Delta^{\alpha})$.

Keywords: *Poisson, Fibonacci binomial, q-Pascal matrix, triple sequence, difference operator Δ^{α}, schauder basis, α-, β-, and γ-duals.*
2010 Mathematics Subject Classification: 46A45, 46B45, 47B07.

1 Introduction

The double sequence was introduced and investigated initial by Priya et al.[7, 9, 13], Subramanian et al.[8, 10, 11, 12] and many others. A Triple sequence (real or complex) can be defined as a function $X : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, real numbers, and complex numbers, respectively. The different types of notions of triple sequences were introduced and investigated at the initial by Esi et al.[6], Subramanian et al.[3, 5], Hazarika et al.[4] and many others.

Throughout $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and ω^3 is the space of all real-valued sequences. Any linear subspace of ω^3 is called sequence space. The set of all $(m, n, k)^{\text{th}}$ summable sequences (x_{mnk}) bounded sequences (l^{∞}_{mnk}), are few examples of classical sequence spaces. The space of all convergent series and bounded series shall be denoted by $(cs^3)_{m,n,k}$ and $(bs^3)_{m,n,k}$ respectively. A Banach space X is said to be a BK- space if it has continuous coordinates. The triple sequence spaces (l_{mnk}) is a BK-space under the norm $\|x\|_{l_{mnk}} = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |x_{mnk}|^{m+n+k}\right)^{\frac{1}{m+n+k}}$. Here and in the entire paper $1 \leq m, n, k \leq \infty$ unless stated otherwise. Moreover the triple sequence spaces (l^{∞}_{mnk}) is a BK-space under the same norm $\|x\|_{l^{\infty}} = \sup_{m,n,k \in \mathbb{N}} |x_{mnk}|$.

Let X and Y be two triple sequence spaces and $A = \left(a^{mnk}_{uvw}\right)$ be a three dimensional infinite matrix of real entries. We write A_{mnk} to denote the sequence in $[m, n, k]^{\text{th}}$ section of the matrix A. We say that the matrix A defines a mapping from X to Y if Ax is in Y for every x in X, where

$$Ax = \{(Ax)_{mnk}\} = \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^{mnk}_{uvw} x_{mnk}\right\}$$

is called A-Transform of the sequence $x = (x_{mnk})$ provided that the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^{mnk}_{uvw} x_{mnk}$ exists for each $u, v, w \in \mathbb{N}$. The notation (X, Y) denote the family of all matrices that map from X to Y. The sequence space X_A defined by

$$X_A = \left\{x \in \omega^3 : Ax \in X\right\} \tag{1.1}$$

is called the domain of matrix A in the space X. The Poisson matrix is defined by $A = T \otimes I + I \otimes T$.

Example 1.1 *If* $T = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *then*

$$A = T \otimes I + I \otimes T = \begin{bmatrix} T + 2I & -I & 0 \\ -I & T + 2I & -I \\ 0 & -I & T + 2I \end{bmatrix}.$$

Properties of poisson matrix of eigenvalues and eigenvectors of $A = T \otimes I + I \otimes T$:

1. We have $Ax_{jk} = \lambda x_{jk}$ for $j, k = 1, 2, 3, \dots, m$.

2. The eigenvectors are orthogonal.

3. A is symmetric.

4. A is is positive definite.

Example 1.2 $A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$. *Hence*

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

and so on.

The q-analog of a mathematical expression means the generalization of that expression using the parameter q. The generalized expression returns the original expression when q approaches 1. The study of q-calculus dates back to the time of Euler. It is a wide and an interesting area of research in recent times. Several researchers are engaged in the field of q-calculus due to its vast applications in mathematics, physics, and engineering sciences. In the field of mathematics, it is widely used by researchers in approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, etc.

Let $0 < q < 1$. Then the q-number $r(q)$ is defined by

$$r(q) = \begin{cases} \sum_{a=0}^{r-1} \sum_{b=0}^{s-1} \sum_{c=0}^{t-1} q^{a+b+c} & r, s, t = 1, 2, 3, \ldots \\ 0 & r, s, t = 0. \end{cases}$$

one can notice that $r(q) = r$ whenever $q \to 1$. The q-analog $\binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q$ of the binomial coefficient $\binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q$ is defined by

$$\binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q = \begin{cases} \frac{u(q)!v(q)!w(q)!}{(u-m)(q)!(v-n)(q)!(w-k)(q)!} \frac{1}{m!n!k!} & u \le m, n \le v, k \le w \\ 0 & m > u, n > v, k > w \end{cases}$$

where q-factorial $u(q)!$ of u, $v(q)!$ of v, and $w(q)!$ of w is given by

$$u(q)! = u(q)(u-1)(q) \cdots 2(q)1(q)$$

$$v(q)! = v(q)(v-1)(q) \cdots 2(q)1(q)$$

$$w(q)! = w(q)(w-1)(q)\cdots 2(q)1(q)$$

Also

$$\binom{0}{0}_q = \binom{u}{0}_q = \binom{u}{u}_q = 1.$$

Similarly

$$\binom{0}{0}_q = \binom{v}{0}_q = \binom{v}{v}_q = 1$$

and

$$\binom{0}{0}_q = \binom{w}{0}_q = \binom{w}{w}_q = 1.$$

Further $\binom{u}{u-m}_q = \binom{u}{m}_q$; $\binom{v}{v-n}_q = \binom{v}{n}$; $\binom{w}{w-k}_q = \binom{w}{k}$; which is a natural q-analog of its ordinary version $\binom{u}{u-m}\binom{v}{v-n}\binom{w}{w-k} = \binom{u}{m}\binom{v}{n}\binom{w}{k}$. The triple Pascal matrix in- fact is an infinite matrix composed of the binomial coefficients in its entries. To accomplish this goal, there exist three various ways, namely using a lower-triangular, upper-triangular or a symmetric matrix. The 4×4 truncation of those are shown below. The triple upper triangular is

$$U_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 27 & 96 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Triple lower triangular is

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 27 & 1 & 0 \\ 1 & 96 & 500 & 1 \end{pmatrix},$$

symmetric:

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 27 & 500 & 8575 \\ 1 & 96 & 3375 & 87808 \\ 1 & 250 & 15435 & 592704 \end{pmatrix}.$$

one can see the pleasing relationship $A_n = L_n U_n$ between those matrices one can also easily observe that all of those three matrices have a value of 1 as their determinants. The elements of the symmetric triple Pascal matrix consist of binomial coefficients, that is,

$$A_{ijk} = \binom{r}{m}\binom{s}{n}\binom{t}{k} = \frac{r!}{m!(r-m)!}\frac{s!}{n!(n-s)!}\frac{t!}{k!(k-t)!}$$

where $r, s, t = i+j+k$ and $m = i, n = j, k = t$. In otherwords $A_{ijk} = i+j+k$, $C_{ijk} = \frac{(i+j+k)!}{i!\,j!\,k!}$. Thus the trace of A_n is given by

$$\operatorname{tr}(A_n) = \sum_{m=0}^{r-1}\sum_{n=0}^{s-1}\sum_{k=0}^{t-1} \frac{2m!}{(m!)^2}\frac{2n!}{(n!)^2}\frac{2k!}{(k!)^2}.$$

with the first few terms are given by the sequence $1, 27, 729, 24389, \ldots$.

Let $0 < q < 1$. Then the q-analog of $P(q) = [P^{rst}_{mnk}]^q$ of Pascal matrix is defined by

$$P(q) = \left[P^{rst}_{mnk}\right]^q = \begin{cases} \binom{r}{m}_q \binom{s}{n}_q \binom{t}{k}_q & \text{if } 0 \le m \le r,\ 0 \le n \le s,\ 0 \le k \le t \\ 0 & \text{if } m > r, n > s, k > t \end{cases}$$

and the inverse of q-analog Pascal matrix is given by

$$\begin{aligned} P(q) &= \left[P^{rst}_{mnk}\right]^q \\ &= \begin{cases} (-1)^{(r-m)+(s-n)+(t-k)} \binom{r}{m}_q \binom{s}{n}_q \binom{t}{k}_q & \text{if } 0 \le m \le r, 0 \le n \le s, 0 \le k \le t \\ 0 & \text{if } m > r, n > s, k > t \end{cases} \end{aligned}$$

Then q-Pascal matrix $P(q)$ can be expressed in the explicit form as

$$P(q) = \begin{bmatrix} \binom{0}{0}_q & \binom{0}{0}_q & \binom{0}{0}_q & 0 & 0 & \cdots & \binom{1}{0}_q \\ \\ \binom{1}{0}_q & \binom{1}{0}_q & \binom{1}{0}_q & \binom{1}{1}_q & \binom{1}{1}_q & \binom{1}{1}_q & 0\cdots \\ \vdots \end{bmatrix}$$

clearly the q-Pascal matrix $P[q]$ reduces to P when q approaches 1. Also we observe that the sum of the elements of $[rst]^{th}$ section is $\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} [P^{rst}_{mnk}]^q = G_{rst}(q)$, where $G_{rst}(q)$ is called $[r, s, t]^{th}$ section Galois number. The Galois number plays significant role in determining the number of subspaces of a finite field. Fibonacci numbers are one of the most beautiful creations of nature. They are often known as nature's number and can be found everywhere around us, from the leaf arrangements in plants, to the pattern of the florest of flowers, the bracts of pinecones or the scale of pine apples. The sequence of integers $1, 1, 2, 3, 5, 8, \ldots,$ is known as the Fibonacci sequence.

We define the poisson Fibonacci binomial q-Pascal matrix $F = (f^{mnk}_{ijk})^{\infty}_{m,n,k=1}$, which differs from the existing poisson Fibonacci binomial Pascal matrix by using Fibonacci numbers f_{ijk} and introducing some new triple sequence space of Γ^3 and Λ^3. Now we define the poisson Fibonacci binomial q-Pascal matrix

$Ab^{rs} = Ab^{rs}_{uvw,mnk}$ where

$$b^{rs} = b^{rs}_{uvw,mnk} = \begin{cases} \frac{fr_{mnk}}{f}\left(\frac{1}{(s+r)^{u+v+w}}\right)\binom{u}{m}_q\binom{v}{n}_q\binom{w}{k}_q s^{(u-m)+(v-n)+(w-k)} & \text{if } 0 \le m \le u, 0 \le n \le v, 0 \le k \le w \\ 0 & \text{if } m > u, n > v, k > w \end{cases}$$

The Gamma function $\Gamma(n)$ of a real number $n \notin \{0, -1, -2, \ldots\}$. $\Gamma(n)$ can be expressed as an improper integral given by $\Gamma(n)$ can be expressed as $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, we state some properties of the Gamma function which are used throughout the text

1. For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.
2. For any real number $n \notin \{0, -1, -2, \ldots\}$, $\Gamma(n+1) = n\Gamma(n)$.
3. For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!; \Gamma(4) = 3!$.

The triple difference sequence space is defined as

$$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$$

and $\Delta^\circ x_{mnk} = \langle x_{mnk}\rangle$ and also the generalized difference triple notion has the following binomial representation

$$\Delta^m x_{mnk} = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{q=0}^{k}(-1)^{i+j+k}\binom{m}{i}\binom{n}{j}\binom{k}{q}x_{m+i,n+j,k+q}.$$

Now, we define the poisson Fibonacci binomial q-Pascal matrix Ab^{rs} and the fractional difference operator $\Delta^{(\alpha)}$, the product matrix

$$A\left(b^{rs}(\Delta^{(\alpha)})\right)_q = A\left(b^{rs}_{uvw,mnk}\Delta^{(\alpha)}\right)_q,$$

where

$$\left(b^{rs}_{u,v,w,m,n,k}(\Delta^{(\alpha)})\right)_q = \begin{cases} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{f_{mnk}}{f}\frac{1}{(s+r)^{u+v+w}}(-1)^{(u-m)+(v-n)+(w-k)} \binom{u}{m}_q\binom{v}{n}_q\binom{w}{k}_q s^{(u-m)+(v-n)+(w-k)}r^{m+n+k} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(u-m)!+(v-n)!+(w-k)!}\frac{1}{\Gamma(\alpha-u+m+1)\Gamma(\beta-v+n+1)\Gamma(\gamma-w+k+1)} & \text{if } 0 \le m \le u, 0 \le n \le v, 0 \le k \le w \\ 0 & \text{if } m > u, n > v, k > w \end{cases}$$

where $r, s > 0$, we define the poisson Fibonacci binomial q-Pascal matrix of triple sequence spaces of the difference operator of fractional

$$\left(A\left(b^{rs}\Delta^{(\alpha)}\right)\right)_{q,\Gamma^3} = \left\{x = (x_{mnk}) \in \omega^3 : A\left(b^{rs}(\Delta^{(\alpha)})\right)_q x \in \Gamma^3\right\},$$
$$\left(A\left(b^{rs}\Delta^{(\alpha)}\right)\right)_{q,\Lambda^3} = \left\{x = (x_{mnk}) \in \omega^3 : A\left(b^{rs}(\Delta^{(\alpha)})\right)_q x \in \Lambda^3\right\},$$
$$\left(A\left(b^{rs}\Delta^{(\alpha)}\right)\right)_{q,l^3} = \left\{x = (x_{mnk}) \in \omega^3 : A\left(b^{rs}(\Delta^{(\alpha)})\right)_q x \in l^3\right\},$$
$$\left(A\left(b^{rs}\Delta^{(\alpha)}\right)\right)_{q,l^3_\infty} = \left\{x = (x_{mnk}) \in \omega^3 : A\left(b^{rs}(\Delta^{(\alpha)})\right)_q x \in l^3_\infty\right\}.$$

We redefine spaces

$$\Gamma^3\left(A(b^{rs}\Delta^{(\alpha)})_q\right) = \left(\Gamma^3\right)_{A(b^{rs}\Delta^{(\alpha)})\ q}$$
$$\Lambda^3\left(A(b^{rs}\Delta^{(\alpha)})_q\right) = \left(\Lambda^3\right)_{A(b^{rs}\Delta^{(\alpha)})\ q}$$
$$l^3\left(A(b^{rs}\Delta^{(\alpha)})_q\right) = \left(l^3\right)_{A(b^{rs}\Delta^{(\alpha)})\ q}$$
$$l^3_\infty\left(A(b^{rs}\Delta^{(\alpha)})_q\right) = \left(l^3_\infty\right)_{A(b^{rs}\Delta^{(\alpha)})\ q} \tag{1.2}$$

2 Main Result

Theorem 2.1 *If Γ^3 is a complete metric space, then $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ is also a complete metric space with the metric*

$$d(x,y) = \sup\left\{\left|Ab^{rs}(\Delta^{(\alpha)})_q x_{mnk} - Ab^{rs}(\Delta^{(\alpha)})_q y_{mnk}\right| : m,n,k = 1,2,3,\dots\right\}_{\Gamma^3}$$

Proof. It is obvious. ∎

Theorem 2.2 $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q) \simeq \Gamma^3$.

Proof. Define the mapping $\pi : \Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q) \to \Gamma^3$ by $\pi x = y = Ab^{rs}(\Delta^{(\alpha)})_q x$ for all $x \in \Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ clearly π is linear and one-one. Let $y = (y_{abc})$ be any triple sequence in Γ^3 and $x = (x_{mnk})$ defined by then $y \in \Gamma^3$, we have

$$\lim_{u,v,w\to\infty} Ab^{rs}(\Delta^{(\alpha)})_q x_{mnk}^{\frac{1}{m+n+k}} = \lim_{u,v,w\to\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q \frac{f_{mnk}}{f}$$
$$\left[\sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{k} \frac{1}{(s+r)^{u+v+w}} (-1)^{(m-a)+(n-b)+(k-c)}\right.$$
$$\left. q^{\binom{m-a}{2}+\binom{n-b}{2}+\binom{k-c}{2}} \binom{m}{a}_q \binom{n}{b}_q \binom{k}{c}_q y_{abc}^{\frac{1}{a+b+c}}\right] r^{m+n+k} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(u-m)!(v-n)!(w-k)!}$$
$$\frac{1}{\Gamma(\alpha-u+m+1)\Gamma(\beta-v+n+1)\Gamma(\gamma-w+k+1)} = \lim_{u,v,w\to\infty} y_{uvw} = 0$$

Thus, we realize that x is a sequence in $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ and the mapping π is onto and metric preserving. Hence $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q) \simeq \Gamma^3$. ■

Theorem 2.3 *For every fixed $\Delta \in \mathbb{N}_\circ$, define the sequence $b^s(q) = \left(b_r^{(\Delta)}(q)\right)$ of the elements of the space $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ by*

$$(b_r^{(\Delta)}(q)) = \begin{cases} (-1)^{(u-m)+(v-n)+(w-k)} q^{\binom{u-m}{2}+\binom{v-n}{2}+\binom{w-k}{2}} \binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q & \text{if } m \le u, n \le v, k \le w \\ 0 & \text{if } m > u, n > v, k > w \end{cases}$$

Then the following statements hold:

(a) *The set $\{b^{(0)}(q), b^{(1)}(q), \ldots\}$ forms the basis for the space $\Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ and every $x \in \Gamma^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ has a unique representation*

$$x = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} y_{mnk}^{\frac{1}{m+n+k}} b^{(s)}(q).$$

Theorem 2.4 *The sequence spaces $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ and $l^3_\infty\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ are BK- spaces with the norms defined by*

$$\|x\|_{l^3(Ab^{rs}(\Delta^{(\alpha)})_q)} = \left\|Ab^{rs}(\Delta^{(\alpha)})_q x\right\|_{l^3} = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|Ab^{rs}(\Delta^{(\alpha)})_q x\right|\right) \quad (2.1)$$

and

$$\|x\|_{l^3_\infty(Ab^{rs}(\Delta^{(\alpha)})_q)} = \left\|Ab^{rs}(\Delta^{(\alpha)})_q\right\| = \sup_{m,n,k\in\mathbb{N}} \left|Ab^{rs}(\Delta^{(\alpha)})_q x\right|, \quad (2.2)$$

respectively.

Proof. The spaces l^3 and l^3_∞ are BK spaces with their natural norms. Since equation (1.2) holds and by Theorem 4.3.12 of Wilansky [1] we get that $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ and $l^3_\infty\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ are BK- spaces. This completes the proof. ■

Lemma 2.5 *The inverse of the product matrix $Ab^{rs}(\Delta^{(\alpha)})_q$ is given by*

$$(Ab^{rs}(\Delta^{(\alpha)})_q)^{-1} = \begin{cases} \sum_{i=u}^{m}\sum_{j=v}^{n}\sum_{t=w}^{k}(u+m)^{-(i+j+t)}(-1)^{(m-i)+(n-j)+(k-t)} \binom{m}{u}_q\binom{n}{v}_q\binom{k}{w}_q s^{(m-i)+(n-j)+(u-t)} r^{-(i+j+t)} \frac{\Gamma(-\alpha+1)\cdot\Gamma(-\beta+1)\cdot\Gamma(-\gamma+1)}{(m-i)!(n-j)!(k-t)!} \frac{1}{\Gamma(-\alpha-m+i+1)\Gamma(-\beta-n+j+1)\Gamma(-\gamma-k+t+1)} \frac{f_{ijk}}{f} & \text{if } 0 \le u \le m,\ 0 \le v \le n,\ 0 \le w \le k \\ 0 & \text{if } u > m, v > n, w > k \end{cases}$$

Theorem 2.6 *The sequence spaces* $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ *and* $l^3_\infty\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ *are linearly isomorphic to* l^3 *and* l^3_∞.

Proof. Define the mapping $T : l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right) \to l_3$ by $x \mapsto y = Tx = Ab^{rs}(\Delta^{(\alpha)})_q x$ clearly T is linear and $x = 0$ whenever $Tx = 0$. Thus T is injective. Let $y = (y_{mnk}) \in l^3$ and by using Lemma 2.5, we define the sequence $x = (x_{mnk})$ by

$$x_{mnk} = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}(\Delta + r)^{-(u+v+w)}(-1)^{(m-u)+(n-v)+(k-w)}\binom{m}{u}_q\binom{n}{v}_q\binom{k}{w}_q$$
$$s^{(m-u)+(n-v)+(k-w)}r^{-(m+n+k)}\frac{f_{mnk}}{f}\frac{\Gamma(-\alpha+1)\Gamma(-\beta+1)\Gamma(-\gamma+1)}{(m-u)!(n-v)!(k-w)!}$$
$$\frac{1}{\Gamma(-\alpha-m-u+1)\Gamma(-\beta-n-v+1)\Gamma(-\gamma-k-w+1)}y_{uvw},\ u,v,w \in \mathbb{N}.$$

Then

$$\|x\|_{l^3_{mnk}\left(Ab^{rs}(\Delta^{(\alpha)})\right)_q}$$
$$= \left\|Ab^{rs}(\Delta^{(\alpha)})_q\right\|_{l^3}$$
$$= s^{(u-m)+(v-n)+(w-k)}r^{(m+n+k)}\frac{f_{mnk}}{f}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(u-m)!(v-n)!(w-k)!}$$
$$\left(\frac{1}{\Gamma(-\alpha-u+m+1)\Gamma(-\beta-v+n+1)\Gamma(-\gamma-w+k+1)}|x_{mnk}|^k\right)^{1/k}$$
$$= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|y_{mnk}|^k\right)^{\frac{1}{k}}$$
$$= \|y\|_{l^3} < \infty.$$

Thus $x \in l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ and T is norm preserving. Consequently, T is surjective. Thus $l^3(Ab^{rs}(\Delta^{(\alpha)})_q) \cong l^3$. Similar proof can be given for the case $l^3_\infty(Ab^{rs}(\Delta^{(\alpha)})_q)$. ■

3 Schauder Basis

In this section we construct the schauder basis for the sequence space $l^3(Ab^{rs}(\Delta^{(\alpha)})_q)$. A sequence $x = (x_{mnk})$ of a normed space $(X, \|\cdot\|)$ is called a schauder basis for the space X if every element $u \in X$ there exists a unique sequence of scalars (a_{mnk}) such that $\lim_{u,v,w\to\infty}\|u - \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} a_{mnk}x_{mnk}\| = 0$.

Define the sequence $b^{(s)} = (b_r^{(s)}q)$ for every fixed $r, s \in \mathbb{N}$ by

$$b_r^{(s)}(q) = \begin{cases} (-1)^{(u-m)+(v-n)+(w-k)} q^{\binom{u-m}{2}+\binom{v-n}{2}+\binom{w-k}{2}} & \text{if } 0 \le m \le u, 0 \le n \le v, \\ \binom{u}{m}_q \binom{v}{n}_q \binom{w}{k}_q & 0 \le k \le w \\ & \\ 0 & \text{if } m > u, n > v, k > w \end{cases}$$

for each $r, s \in \mathbb{N}$. Then the following statement holds.

Theorem 3.1 *The set* $\{b^{(0)}(q), b^{(1)}(q), \ldots\}$ *forms the basis for the space* $l^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ *and every* $x \in l^3(Ab^{rs}(\Delta^{(\alpha)})_q)$ *has a unique representation* $x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{mnk} b^{(s)}(q)$ *where* $\lambda_{mnk} = (Ab^{rs}(\Delta^{(\alpha)})_q x)$ *for each* $m, n, k \in \mathbb{N}$.

Proof. It is clear from the definitions $Ab^{rs}(\Delta^{(\alpha)})_q$ of that $Ab^{rs}(\Delta^{(\alpha)})_q = e^{[mnk]} \in (l^3_{m,n,k})$. The set $\{e^{[mnk]} : m, n, k \in \mathbb{N}\}$ forms the Schauder basis for the space (l^3). Since the mapping T defined by $x \mapsto y = Tx$ (see Theorem 2.6) from the space $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ to l^3_{mnk} is a linear bijection, therefore the inverse image of the set $\{e^{[mnk]}\}$ forms the basis of $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ that is

$$\lim_{u,v,w\to\infty} \left\| x - \sum_{m=0}^{u} \sum_{n=0}^{v} \sum_{k=0}^{w} \lambda_{mnk} b^{(s)}(q) \right\| = 0, \quad x \in l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right).$$

To verify the uniqueness of the representation, we assume that $x = \sum_m \sum_n \sum_k \mu_{mnk} b^{(s)}$ then, we have

$$\begin{aligned} \left(Ab^{rs}(\Delta^{(\alpha)})_q\right)_{mnk} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mu_{mnk} \left(Ab^{rs}(\Delta^{(\alpha)})_q\right)_{mnk} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mu_{mnk} e^{[mnk]} = \mu_{mnk}, \end{aligned}$$

which is a contradiction to our assumption that $\lambda_{mnk} = \left(Ab^{rs}(\Delta^{(\alpha)})_q\right)_{mnk}$ for each $m, n, k \in \mathbb{N}$. ■

Corollary 3.2 *The sequence space* $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ *is separable.*

Proof. The result is immediate from Theorem 2.4 and 3.1. ■

4 Köthe Duals

In this section, we determine Köthe duals (α-, β-, γ-duals) of the spaces $\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ and $l^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$, before proceeding we recall the definitions of Köthe duals.

Definition 4.1 *The Köthe Toeplitz duals or α-, β-, and γ-duals X^α,X^β,X^γ of a sequence space X are defined by:*

$$X^\alpha = \left\{u = (u_{mnk}) \in \omega^3 : ux = (u_{mnk}x_{mnk}) \in l^3;\ \textit{for all } x \in X\right\};$$

$$X^\beta = \left\{u = (u_{mnk}) \in \omega^3 : ux = (u_{mnk}x_{mnk}) \in cs^3;\ \textit{for all } x \in X\right\};$$

$$X^\gamma = \left\{u = (u_{mnk}) \in \omega^3 : ux = (u_{mnk}x_{mnk}) \in bs^3;\ \textit{for all } x \in X\right\},\ \textit{respectively.}$$

The following lemma is essential to determine the dual spaces. Throughout the paper, we denote the collection of all finite subsets of $\mathbb{N}_0$ by $\mathcal{N}$.

Lemma 4.2 *The following statements hold:*

i) $A = \left(a^{mnk}_{uvw}\right) \in (\Gamma^3 : l^3)$ *if and only if*

$$\sup_{K \in N} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{\infty} \left|\sum_{m\in K}\sum_{n\in K}\sum_{k\in K} a^{mnk}_{uvw}\right| < \infty \tag{4.1}$$

ii) $A = \left(a^{mnk}_{uvw}\right) \in (\pi^3 : c)$ *if and only if*

$$\sup_{u,v,w\in\mathbb{N}_0} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|a^{mnk}_{uvw}\right| < \infty \tag{4.2}$$

$$\exists\,(\alpha_{mnk}) \in \mathbb{C} \ni: \lim_{u,v,w\to\infty} \left(a^{mnk}_{uvw}\right) = \alpha_{mnk} \textit{ for each } m,n,k \in \mathbb{N}_0. \tag{4.3}$$

iii) $A = \left(a^{mnk}_{uvw}\right) \in (\Gamma^3 : \Lambda^3)$ *if and only if (4.2) holds.*

Theorem 4.3 *The α-dual of the space $\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ is the set $\delta_1(q)$ which is defined by*

$$\delta_1(q) = \Bigg\{z = (z_{uvw}) \in \omega^3 : \sup_{R\in N} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{u\in\mathbb{R}}\sum_{v\in\mathbb{R}}\sum_{w\in\mathbb{R}} \left(Ab^{rs}(\Delta^{(\alpha)})_q\right)(z_{uvw})^{\frac{1}{u+v+w}}\right| < \infty\Bigg\}.$$

Proof. Consider the following equality

$$|z_{uvw}x_{uvw}|^{\frac{1}{u+v+w}} = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} Ab^{rs}(\Delta^{(\alpha)})_q\,(A(q)y)_{uvw}\,(z_{uvw})^{\frac{1}{u+v+w}}\,(y_{mnk})^{\frac{1}{m+n+k}}$$
$$\text{for all } u,v,w\in\mathbb{N}_0. \tag{4.4}$$

where the matrix $A(q) = b_r^{(s)}(q)$ is defined by

$$b_r^{(s)}(q) = b^{rs}(q) = \begin{cases} (-1)^{(u-m)+(v-n)+(w-k)} & \text{if } 0\le m\le u,\\ q^{\binom{u-m}{2}+\binom{v-n}{2}+\binom{w-k}{2}}\binom{u}{m}_q\binom{v}{n}_q\binom{w}{k}_q z_{uvw} & 0\le n\le v, 0\le k\le w\\ \\ 0 & \text{if } m>u, n>v, k>w\end{cases}$$

for all $u,v,w,m,n,k\in\mathbb{N}$. We realize by using (4.4) that $zx = (z_{uvw}x_{uvw})\in l^3$ whenever $x\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ if and only if $A^{(q)}(\alpha)y\in l^3$ whenever $y\in\Gamma^3$. Thus we deduce that $z=(z_{uvw})$ is a sequence in the α-dual of $\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ if and only if the matrix $A^{(q)}(\alpha)_q$ belongs to the class $(\Gamma^3 : l^3)$. Thus we conclude from (4.1) of Lemma 4.2 that $\left[\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)\right]^{\alpha} = \delta_1(q)$. ■

Theorem 4.4 *Define the sets $\delta_2(q)$, $\delta_3(q)$, and $\delta_4(q)$ by*

$$\delta_2(q) = \Bigg\{ z=(z_{uvw})\in\omega^3 : \sum_{u=m}^{\infty}\sum_{v=n}^{\infty}\sum_{w=k}^{\infty}(-1)^{(u-m)+(v-n)+(w-k)}$$
$$q\binom{u-m}{2}+\binom{v-n}{2}+\binom{w-k}{2}\binom{u}{m}_q\binom{v}{n}_q\binom{w}{k}_q z_{uvw}$$
$$\textit{exist for each } m,n,k\in\mathbb{N}_0\Bigg\}$$

$$\delta_3(q) = \Bigg\{ z=(z_{uvw})\in\omega^3 : \sup_{u,v,w\in\mathbb{N}_0}\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\Bigg|\sum_{r=m}^{u}\sum_{s=n}^{v}\sum_{t=k}^{w}(-1)^{(t-k)+(s-n)+(r-m)}$$
$$q\binom{t-k}{2}+\binom{s-n}{2}+\binom{r-m}{2}\binom{t}{k}_q\binom{s}{n}_q\binom{r}{m}_q z_{rst}\Bigg| < \infty\Bigg\}$$

$$\delta_4(q) = \Bigg\{ z=(z_{uvw})\in\omega^3 : \lim_{u,v,w\to\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{(r-m)+(s-n)+(t-k)}$$
$$q\binom{r-m}{2}+\binom{s-n}{2}+\binom{t-k}{2}\binom{r}{m}_q\binom{s}{n}_q\binom{t}{k}_q z_{rst}\ \textit{exists}\Bigg\}.$$

Then $\left[\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)\right]^{\beta} = \delta_2(q)\cap\delta_3(q)$.

Proof. Consider the following equality:

$$\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\left(z_{mnk}x_{mnk}\right)^{\frac{1}{m+n+k}}$$
$$=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\left[\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{t=0}^{k}(-1)^{(r-m)+(s-n)+(t-k)}\right.$$
$$\left. q\binom{r-m}{2}+\binom{s-n}{2}+\binom{t-k}{2}\binom{r}{m}_q\binom{s}{n}_q\binom{t}{k}_q\left(y_{rst}\right)^{\frac{1}{r+s+t}}\right] z_{mnk}^{\frac{1}{m+n+k}}$$
$$=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\left[\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}(-1)^{(r-m)+(s-n)+(t-k)}\right.$$
$$\left. q\binom{r-m}{2}+\binom{s-n}{2}+\binom{t-k}{2}\binom{r}{m}_q\binom{s}{n}_q\binom{t}{k}_q\left(z_{rst}\right)^{\frac{1}{r+s+t}}\right] y_{mnk}^{\frac{1}{m+n+k}}$$
$$=\left(B\left(q\right)y\right)_{uvw} \tag{4.5}$$

for each $u, v, w \in \mathbb{N}_0$, where the matrix $B(q) = b^{rs}(q)$ is defined by

$$b^{rs}(q)=\begin{cases}\sum_{r=m}^{u}\sum_{s=n}^{v}\sum_{t=k}^{w}(-1)^{(r-m)+(s-n)+(t-k)} & \text{if } 0\le m\le u, 0\le n\le v,\\ q^{\binom{r-m}{2}+\binom{s-n}{2}+\binom{t-k}{2}}\binom{r}{m}_q\binom{s}{n}_q\binom{t}{k}_q\left(z_{rst}\right)^{\frac{1}{r+s+t}} & 0\le k\le w,\\ & \\ 0 & \text{if } m>u, n>v, k>w.\end{cases}$$

for all $u, v, w, m, n, k \in \mathbb{N}_0$.

Thus, using (4.5), we realize that $zx = (z_{uvw}x_{uvw}) \in cs^3$. Whenever $x = (x_{uvw}) \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$, if and only if $B(q)y \in c^3$ whenever $y = (y_{mnk}) \in \Gamma^3$. This yields that $z = (z_{uvw})$ is a sequence in β-dual of $\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$ if and only if the matrix $B(q)$ belongs to the class $(\Gamma^3 : c^3)$. This in turn implies by using (4.2) of Lemma 4.2 that $\sup_{u,v,w\in\mathbb{N}_0}\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}|b^{rs}(q)| < \infty$ and $\lim_{u,v,w\to\infty} b^{rs}(q)$ exists for each $m, n, k \in \mathbb{N}_0$. Thus $[\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)]^\beta = \delta_2(q)\cap\delta_3(q)$. ■

5 Compact Operators and Hausdorff Measure of non-Compactness (HMNC)

Let X and Y be two Banach spaces. By $B(X, Y)$ we denote the set of all analytic linear operators from the space X into the space Y which is again a Banach space equipped with the metric

$$d(L, 0) = \sup_{x\in B_X}|Lx - Ly|$$

where $L \in B(X, Y)$ and B_X denotes the open ball in X. Further

$$d(z, 0)_X = \sup_{x \in X} \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} z_{mnk} x_{mnk} \right| .$$

In this case, we observe that $x = (x_{mnk}) \in X^{\beta}$ provided that the supremum exists. Now, we recall the definitions of compact operator and Hmnc of an analytic set.

Definition 5.1 *An operator $L : X \to Y$ is said to be compact if the domain of L is all of X and for every analytic sequence (x_{uvw}) in X, the sequence $(L(x_{uvw}))$ has a convergent sub-sequence in Y.*

Definition 5.2 *The Hmnc of an analytic set H in a metric space X is defined by*

$$\chi(H) = \inf\{(m = 0, 1, 2, \ldots, u), (n = 0, 1, 2, \ldots, v), (k = 0, 1, 2, \ldots, w), u, v, w \in \mathbb{N}_0\}$$

where, $B\left(x_{mnk}^{\frac{1}{m+n+k}}, a_{mnk}^{\frac{1}{m+n+k}}\right)$ is the open ball center d at x_{mnk} and of radius a_{mnk} for each $m = 0, 1, 2, \ldots, u, n = 0, 1, 2, \ldots, v, k = 0, 1, 2, \ldots, w$. The compact operator and Hmnc are closely related. An operator $L : X \to Y$ is compact if and only if $d(L, 0)_X = 0$ where $d(L, 0)_X$ denotes Hmnc of the operator L and is defined by $d(L, 0)_X = \chi(L(B_X))$. Using Hmnc, several authors obtained the necessary and sufficient conditions for matrix operators to be compact between BK-spaces. If X and Y are any two BK-spaces, then every matrix $\Phi \in (X, Y)$ defines a linear operator $L_{\Phi} \in B(X, Y)$, where $L_{\Phi} x = \Phi x$ for all $x \in X$. Moreover, if $X \supset \sigma$ is a BK-space and $\Phi \in (X, Y)$, then $d(L_{\Phi}, 0) = d(\Phi, 0)_{(X,Y)} = \sup_{u,v,w \in \mathbb{N}_0} |\Phi_{uvw} \cdot x - \Phi_{uvw} y|_X < \infty$, where σ represents the set of all sequences that terminate in zeros.

Lemma 5.3 *Let X be a triple sequence space and $\Phi = (a_{uvw}^{mnk})$ be a three-dimensional infinite matrix. If $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : X\right)$, then $\theta \in (\Gamma^3 : X)$ and $\Phi x = \theta y$ for all $x \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$.*

Proof. Let $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : X\right)$ and $x \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)$. Then $\Phi_{uvw} = (a_{uvw}^{mnk})_{m,n,k \in \mathbb{N}_0} \in \left[\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)\right]^{\beta}$ for all $u, v, w \in \mathbb{N}_0$. Consider the following equality:

$$\begin{aligned}(\theta y)_{uvw} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\theta_{uvw,mnk}y_{mnk}^{\frac{1}{m+n+k}}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}(-1)^{(r-m)+(s-n)+(t-k)}q^{\binom{r-m}{2}+\binom{s-n}{2}+\binom{t-k}{2}}\\ &\quad \binom{r}{m}_q\binom{s}{n}_q\binom{t}{k}_q(a_{uvw}^{rst})\left[\sum_{a=0}^{m}\sum_{b=0}^{n}\sum_{c=0}^{k}\binom{m}{a}_q\binom{n}{b}_q\binom{k}{c}_q x_{abc}^{\frac{1}{a+b+c}}\right]\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(a_{uvw}^{mnk})x_{mnk}^{\frac{1}{m+n+k}}\\ &= (\Phi x)_{uvw} \text{ for all } u,v,w\in\mathbb{N}_0\end{aligned}$$

Thus we realize that θ_{uvw} is absolutely summable for each $u,v,w\in\mathbb{N}_0$ and $\theta y\in X$. This yields the desired consequence $\theta\in(\Gamma^3:X)$. ∎

Theorem 5.4 *The following statements hold:*

(a) *If* $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q),\Gamma^3\right)$ *then*

$$d(L_\Phi,0)=\lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}\right|^{\frac{1}{m+n+k}}.$$

(b) *If* $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:c^3\right)$ *then*

$$\begin{aligned}\frac{1}{2}\lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}-\theta\right|^{\frac{1}{m+n+k}} &\le d(L_\Phi,0)_X\\ &\le \lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}-\theta\right|^{\frac{1}{m+n+k}}\end{aligned}$$

where $\theta=(\theta_{mnk})$ *and* $\theta_{mnk}=\lim_{uvw\to\infty}\theta_{(uvw),(mnk)}$ *for each* $m,n,k\in\mathbb{N}_0$.

(c) *If* $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:\Lambda^3\right)$ *then*

$$0\le d(L_\Phi,0)\le\lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}\right|^{\frac{1}{m+n+k}}.$$

(d) *If* $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)$ *then*

$$\begin{aligned}\lim_{mnk\to\infty}[d(\Phi,0)]^{[m,n,k]}_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)} &\le d(L_\Phi,0)_X\\ &\le 4\lim_{mnk\to\infty}[d(\Phi,0)]^{[m,n,k]}_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)},\end{aligned}$$

where

$$[d(\Phi,0)]^{[m,n,k]}_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)} = \sup_{R\in\mathbb{R}_{mnk}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{u\in\mathbb{R}}\sum_{v\in\mathbb{R}}\sum_{w\in\mathbb{R}} \theta_{uvw,mnk}\right|^{\frac{1}{m+n+k}}.$$

(e) *If* $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : cs^3_0\right)$, *then*

$$d(L_\Phi,0)_X = \lim_{uvw\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}\right|^{\frac{1}{m+n+k}}\right).$$

(f) *If* $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : cs^3\right)$ *then*

$$\frac{1}{2}\lim_{uvw\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}-\tilde{\theta}\right|^{\frac{1}{m+n+k}}\right) \le d(L_\Phi,0)_X$$
$$\le \lim_{uvw\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}-\tilde{\theta}\right|^{\frac{1}{m+n+k}}\right)$$

where, $\tilde{\theta} = (\tilde{\theta}_{mnk})$ *with* $\tilde{\theta}_{mnk} = \lim_{uvw\to\infty}\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}$ *for each* $m,n,k \in \mathbb{N}_0$.

(g) *If* $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : bs^3\right)$ *then*

$$0 \le d(L_\Phi,0) \le \lim_{uvw\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}\right|^{\frac{1}{m+n+k}}\right).$$

Proof.

(a) Let $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : \Gamma^3\right)$. We observe that

$$d(\Phi_{uvw},0)_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)} = d(\theta_{uvw},0)_{\Gamma^3} = d(\theta_{uvw},0)_{l^3}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}\right|^{\frac{1}{m+n+k}} \text{ for } u,v,w \in \mathbb{N}_0.$$

Then $d(L_\Phi,0) = \lim_{uvw\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{(uvw),(mnk)}\right|^{\frac{1}{m+n+k}}\right)$

(b) Notice that

$$d(\theta_{uvw}-\theta,0)_{\Gamma^3}=d(\theta_{uvw}-\theta,0)_{l^3}$$
$$=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\left|\theta_{uvw,mnk}-\theta_{mnk}\right|^{\frac{1}{m+n+k}} \text{ for each } u,v,w\in|N|_0. \tag{5.1}$$

Now, let $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:c^3\right)$. Then Lemma 5.3 implies that $\theta\in(\Gamma^3:c^3)$, we deduce that

$$\frac{1}{2}\lim_{uvw\to\infty}\sup\left|\theta_{uvw}-\theta\right|^{\frac{1}{u+v+w}}\leq d(L_\Phi,0)\leq\lim_{uvw\to\infty}\sup\left|\theta_{uvw}-\theta\right|_{\Gamma^3}^{\frac{1}{u+v+w}}$$

which yields the right of equation (5.1) that

$$\frac{1}{2}\lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{uvw,mnk}-\theta_{mnk}\right|^{\frac{1}{m+n+k}}\leq d(L_\Phi,0)_X$$
$$\leq\lim_{uvw\to\infty}\sup\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\theta_{uvw,mnk}-\theta_{mnk}\right|^{\frac{1}{m+n+k}}.$$

(c) The proof is analogous to that of part (a). So, we omit details.

(d) We have

$$\left|\sum_{u\in R}\sum_{v\in R}\sum_{w\in R}\theta_{uvw}\right|_{\Gamma^3}^{\frac{1}{u+v+w}}=\left|\sum_{u\in R}\sum_{v\in R}\sum_{w\in R}\theta_{uvw}\right|_{l^3}=$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{u\in R}\sum_{v\in R}\sum_{w\in R}\theta_{uvw,mnk}\right|^{\frac{1}{m+n+k}}. \tag{5.2}$$

Let $\Phi\in\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)}):l^3\right)$. Then Lemma 5.3 implies that $\theta\in(\Gamma^3:l^3)$ we get

$$\lim_{r,s,t\to\infty}\left(\sup_{R\in\mathbb{R}_{r,s,t}}\left|\sum_{u\in R}\sum_{v\in R}\sum_{w\in R}\theta_{uvw}\right|_{\Gamma^3}^{\frac{1}{r+s+t}}\right)\leq d(L_\Phi,0)$$
$$\leq 4\cdot\lim_{r,s,t\to\infty}\left(\sup_{R\in\mathbb{R}_{r,s,t}}\left|\sum_{u\in R}\sum_{v\in R}\sum_{w\in R}\theta_{uvw}\right|_{\Gamma^3}^{\frac{1}{r+s+t}}\right)$$

which is reduced by using (5.2) to

$$\lim_{r,s,t\to\infty}d(\Phi,0)_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)}^{[rst]}\leq d(L_\Phi,0)_X\leq 4\cdot\lim_{u,v,w\to\infty}d(\Phi,0)_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q:l^3\right)}^{[rst]}$$

as desired.

(e) Notice that

$$\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\Phi_{rst}\right|^{\frac{1}{r+s+t}}_{\Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q\right)} = \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\Phi_{rst}\right|_{\Gamma^3}$$

$$= \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\Phi_{rst}\right|_{l^3}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}\right|^{\frac{1}{m+n+k}}$$

which yields

$$d(L\Phi,0)_X = \lim_{u,v,w\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}\right|^{\frac{1}{m+n+k}}\right).$$

(f) We have

$$\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst}-\theta\right|^{\frac{1}{u+v+w}}_{\Gamma^3} = \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst}-\tilde{\theta}\right|_{l^3}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}-\tilde{\theta}_{mnk}\right|^{\frac{1}{m+n+k}} \tag{5.3}$$

for each $u, v, w \in \mathbb{N}_0$. Let $\Phi \in \Gamma^3\left(Ab^{rs}(\Delta^{(\alpha)})_q : cs^3\right)$. Then by Lemma 5.3, implies that $\theta \in (\Gamma^3, cs^3)$, we deduce that

$$\frac{1}{2}\lim_{u,v,w\to\infty}\sup\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst}-\tilde{\theta}_{mnk}\right|_{\Gamma^3} \le d(L_\Phi,0)$$

$$\le \lim_{u,v,w\to\infty}\sup\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst}-\tilde{\theta}_{mnk}\right|^{\frac{1}{u+v+w}}_{\Gamma^3}$$

which yields that

$$\frac{1}{2}\lim_{u,v,w\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}-\tilde{\theta}_{mnk}\right|^{\frac{1}{m+n+k}}\right) \le d(L_\Phi,0)_X$$

$$\le \lim_{u,v,w\to\infty}\sup\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w}\theta_{rst,mnk}-\tilde{\theta}_{mnk}\right|^{\frac{1}{m+n+k}}\right)$$

as defined.

(g) Since the proof is analogous to that of part (e),we omit the details.

■

Corollary 5.5 *The following statements hold:*

(a) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : \Gamma^3)$. Then L_Φ is compact if and only if

$$\lim_{u,v,w\to\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\theta_{uvw}^{mnk}\right|^{\frac{1}{m+n+k}} = 0.$$

(b) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : c)$. Then L_Φ is compact if and only if

$$\lim_{u,v,w\to\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\tilde{\theta}_{uvw}^{mnk} - \tilde{\theta}_{mnk}\right|^{\frac{1}{m+n+k}} = 0.$$

(c) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : \Lambda^3)$. Then L_Φ is compact if

$$\lim_{u,v,w\to\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\theta_{uvw}^{mnk}\right|^{\frac{1}{m+n+k}} = 0.$$

(d) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : l^3)$. Then L_Φ is compact if and only if

$$\lim_{a,b,c\to\infty} \left[\sup_{R\in\mathbb{R}_{abc}} \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{u\in\mathbb{R}}\sum_{v\in\mathbb{R}}\sum_{w\in\mathbb{R}} \theta_{uvw}^{mnk}\right|\right)^{\frac{1}{m+n+k}}\right] = 0.$$

(e) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : cs_0^3)$. Then L_Φ is compact if and only if

$$\lim_{u,v,w\to\infty} \sup \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w} \theta_{rst}^{mnk}\right|^{\frac{1}{m+n+k}}\right) = 0.$$

(f) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : cs^3)$. Then L_Φ is compact if and only if

$$\lim_{u,v,w\to\infty} \sup \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w} \theta_{rst}^{mnk} - \tilde{\theta}\right|^{\frac{1}{m+n+k}}\right) = 0.$$

(g) Let $\Phi \in (\Gamma^3 \left(Ab^{rs}(\Delta^{(\alpha)})\right)_q : bs^3)$. Then L_Φ is compact if

$$\lim_{u,v,w\to\infty} \sup \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left|\sum_{r=0}^{u}\sum_{s=0}^{v}\sum_{t=0}^{w} \theta_{rst}^{mnk}\right|^{\frac{1}{m+n+k}}\right) = 0.$$

6 Open Problem

In this study, we introduced Poisson Fibonacci binomial q-Pascal matrix of triple difference operator $\mathbf{F}^{\alpha}_{\mathbf{\Delta}}$ of fractional order α by the composition q-Pascal matrix F and difference operator $\mathbf{\Delta}^{(\alpha)}$ of fractional order α.Readers who are interested in such triple sequence spaces can define the structures within the defined sequence spaces in different sequence spaces and observe whether the theorems given in the study are satisfied.

References

[1] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, Notas de Mathematica [Mathematical Notes] 91. North-Holland Publishing Co: Amsterdam, (1984).

[2] P. K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Lecture Notes Pure and Applied Mathematics 65, Marcel Dekker, Inc., New York, (1981).

[3] N. Subramanian and A. Esi, On triple sequence space of Bernstein operator of χ^3 of rough λ-statistical convergence in probability defined by Musielak-Orlicz function of p-metric. *Electronic J. Math. Anal. Appl* 6(1) (2018) 198–203.

[4] B. Hazarika, N.Subramanian, and A. Esi, On rough weighted ideal convergence of triple sequence of Bernstein polynomials. *Proceedings of the Jangjeon Mathematical Society Memories of the Jangjeon Mathematical Society* 21(3) (2018) 497–506.

[5] N. Subramanian and A. Esi, Rough variables of convergence. *Sci. Stud. Res. Ser. Math. Inform.* 27(2) (2017) 65–72.

[6] A. Esi, N. Subramanian and Ayten Esi, On triple sequence space of Bernstein operator of rough I-convergence pre-Cauchy sequences. *Proyecciones (Journal of Mathematics)* 36(4) (2017) 567–587.

[7] C. Priya, N. Saivaraju and N. Subramanian, The ideal convergent sequence spaces over np-metric spaces defined by sequence of modulus. *Far East Journal of Mathematical Sciences (FJMS)* 92(2) (2014) 173–203.

[8] N. Subramanian, C. Priya and N. Saivaraju, Randomness of lacunary statistical convergence of χ^2 over p-metric spaces defined by sequence of modulus. *Far East Journal of Mathematical Sciences (FJMS)* 94(1) (2014) 89–111.

[9] C. Priya, N. Saivaraju and N. Subramanian, The Fibonacci numbers of χ^2over p-metric spaces defined by sequence of modulus. *Far East Journal of Mathematical Sciences (FJMS)* 93(1) (2014) 1–21.

[10] N. Subramanian, C. Priya and N. Saivaraju, The χ^2 sequence space over p-metric spaces defined by Musielak modulus. *Songklanakarin Journal of science and Technology* 36(5) (2014) 591–598.

[11] N. Subramanian, C. Priya and N. Saivaraju, The $\int \chi^{2I}$ of real numbers over Musielak p-metric space. *Southeast Asian Bulletin of Mathematics* 39 (2015) 133–148.

[12] N. Subramanian, N. Saivaraju and C. Priya, The ideal of χ^2 of fuzzy real numbers over fuzzy p-metric spaces defined by Musielak. *Journal of Mathematical Analysis* 6(1) (2015) 1–12.

[13] C. Priya, N. Saivaraj and N. Subramanian, The Cesàro lacunary ideal double sequence χ^2- of ϕ-statistical defined by a Musielak-Orlicz function. *Applied Mathematics and Information Science* 10(4) (2016) 1–7.