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Defining Novel Integral Operator on the class of Multivalent Functions

A. A. Yusuf & A. O AbdulKareem

Department of Mathematics, College of Physical Sciences,
Federal University of Agriculture, Abeokuta. Ogun , Nigeria.
e-mail:yusufaa@funaab.edu.ng
Department of Mathematics, Lagos State University Ojo, Lagos, Nigeria
e-mail:afeezokareem@gmail.com

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Abstract

In this study, we focus on the class of multivalent functions of a negative coefficient by introducing a new subclass of multivalent functions defined by a novel integral operator denoted as $\mathbf{I}_{\zeta,\mu}^{n,t}f(\xi)$. We investigated the following properties such as coefficient estimate, radii of starlikeness, convexity and close-to-convexity, closure theorems, extreme points, integral means, neighborhood and convolution for the class of multivalent functions denoted as $\mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$. The interesting results obtained remarked the existing ones.

Keywords: *Analytic functions, multivalent functions, Opoola integral operator, coefficient inequality, radii and neighborhoods properties.*
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1 Introduction

Geometric function theory is one of the branches of complex analysis, that investigates the features of the geometric properties of the image domain of analytic functions. As such, a geometric function is an analytic and normalized function of the form $q(0) = 0$, $q'(0) = 1$ defined as

$$q(\xi) = \xi + \sum_{k=2}^{\infty} d_k \xi^k, \tag{1}$$

having certain geometric properties. Several subclasses of the normalized analytic functions have been defined and their properties studied in various areas of research.
The study of these features of the geometric properties of the image domain have been extended to multivalent function defined as follows:
Let $q(\xi) \in \mathcal{A}_\alpha$ be the class of the functions of the form

$$q(\xi) = \xi^\alpha + \sum_{k=\alpha+1}^{\infty} d_k \xi^k, (\xi \in \mathbb{U}, k \geq \alpha + 1, \alpha \in \mathbb{N}). \tag{2}$$

analytic in the unit disk $\{\xi : |\xi| < 1\}$.
Let $\mathcal{M}_\alpha$ be the subclass of $\mathcal{A}_\alpha$ functions of the form

$$f(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} e_k \xi^k, (\xi \in \mathbb{U}, k \geq \alpha + 1, \alpha \in \mathbb{N}). \tag{3}$$

analytic in the unit disk $\{\xi : |\xi| < 1\}$.
The class and subclass of the function of form (2) and (3), referred to as multivalent functions with change of notations, p-valent functions with both positive and negative coefficients has it origin in 1944 by Shigeo Ozaki who studied some properties of the classes of the functions such as coefficient inequalities, sufficient conditions, (see [5]). These classes have captured the attention of many researcher in the area of geometric function theory with various geometric properties investigated such as coefficient inequalities, growth and distortions, convolution, integral means and many more (see [15, 16, 17, 18, 19, 20]).
Given the functions $f(\xi)$ and $k(\xi)$ be of the form (3), the Hadamard convolution of $f(\xi)$ and $k(\xi)$ is given as

$$(f * k)(\xi) = \sum_{k=\alpha+1}^{\infty} \Xi_k e_k \xi^k. \tag{4}$$

and the principle of subordination of two functions $f(z)$ and $g(z)$ expressed as follows: $f \prec k$ if

$$f = k \circ w = k(w(z)), z \in \mathbb{U}$$

where $w(z)$ is a unit bound functions, $w(0) = 0$ and $|w(z)| < 1$, alternatively for g to be univalent in $\mathbb{U}$, then $f \prec k \Leftrightarrow f(0) = k(0)$ and $f(\mathbb{U}) \subset k(\mathbb{U})$.

Several classes of the functions of the form (3) have been introduced to have satisfy the geometric conditions of starlikeness, convexity, close-to-convexity, starlikeness of order ρ, convexity of order ρ, close-to-convexity of order ρ with different geometric properties studied via families of both differentials and integrals operators (see [5, 6, 9, 9]).

This research study is motivated by the work of [10, 11] by introducing a new class of multivalent function via a novel integral operator denoted as $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$ and investigate the properties namely: coefficient estimate, radii of starlikeness, convexity and close-to-convexity, closure theorems, extreme points, integral means, neighborhood and convolution for the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$.
The novel Opoola integral operator introduced in [14] on the class of the functions (1) defined as follows:

$$\mathbf{I}^{0,t}_{\zeta,\mu}q(\xi) = q(\xi).$$

$$\mathbf{I}^{1,t}_{\zeta,\mu}q(\xi) = \frac{1}{t\eta^{(\frac{1}{t}+\zeta-\mu-1)}}\int_0^\xi \xi^{(\frac{1}{t}+\zeta-\mu-2)}[(t\zeta - t\mu)\xi + q(\xi)]d\xi = (\mathbf{I}_t q(\xi)).$$

So that by mathematical inductions,

$$\mathbf{I}^{n,t}_{\zeta,\mu}q(\xi) = \mathbf{I}_t(\mathbf{I}^{n-1,t}_{\zeta,\mu}q(\xi)).$$

Then

$$\mathbf{I}^{n,t}_{\zeta,\mu}q(\xi) = \xi + \sum_{k=2}^{\infty}\frac{1}{[1+(k+\zeta-\mu-1)t]^n}d_k\xi^k. \tag{5}$$

Clearly, varying the parameter n, t, ζ, μ, the integral operator reduces to Salagean integral operator, Al-Oboudi-Al-Qahtani integral operator, (see [1, 2, 3].
Defining the integral operator on the class $f \in \mathcal{M}_\alpha$, we have

$$\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) = z^\alpha - \sum_{k=\alpha+1}^{\infty}\frac{1}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n d_k\xi^k}.$$

A new class of multivalent functions is introduced using the operator $\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi)$ on the class of the functions $\mathcal{M}_\alpha$, defined as follows:

Definition 1.1 *The class of functions $f \in \mathcal{M}_\alpha$ is contained in the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$ if the condition is satisfied*

$$\begin{aligned}&\Re e\left\{\frac{\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))' + \beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''}{(1-\beta)\xi^\alpha + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}\right\}\\ &> \theta\left|\frac{\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))' + \beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''}{(1-\beta)\xi^\alpha + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'} - \alpha\right|\\ &+ \gamma, \xi \in \mathbf{U}.\end{aligned} \tag{6}$$

$n \in \mathbb{N}_0, \zeta, t \geq 0, \mu \in [0,\zeta]$, $0 \leq \beta \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \gamma < \alpha$, $\theta \geq 0$ *and* $\alpha \in \mathbb{N}$.

Remark 1.2 *Varying the parameters $n, \zeta, t, \mu, \beta, \eta, \gamma, \theta, \alpha$, the condition (6), reduces to the classes of multivalent functions introduced and studied by the following authors in [5, 6, 11, 12, 13, 17, 20].*

2 Preliminaries Lemmas

Definition 2.1 *[8]*
The $(k-\rho)$ neighborhoods of functions $f(\xi) \in \mathcal{M}_\alpha$ is defined as

$$\mathcal{G}_{k,\rho}(f) = \left\{ k(\xi) \in \mathcal{M}_\alpha : k(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} \Xi_k \xi^\alpha and \sum_{k=\alpha+1}^{\infty} k|e_k - \Xi_k| \leq \rho, 0 \leq \rho < 1 \right\} \tag{7}$$

For identity $l(\xi) = \xi^\alpha$, $\alpha \in \mathbb{N}$

$$\mathcal{G}_{k,\rho}(l) = \left\{ k(\xi) \in \mathcal{M}_\alpha : k(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} \Xi_k \xi^\alpha and \sum_{k=\alpha+1}^{\infty} k|\Xi_k| \leq \rho, 0 \leq \rho < 1 \right\} \tag{8}$$

Definition 2.2 *The function $f(\xi) \in \mathcal{M}_\alpha$ is contained in the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \lambda, \alpha)$, if there exists $k(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \lambda, \alpha)$ such that*

$$\left| \frac{f(\xi)}{k(\xi)} - 1 \right| < \alpha - \Phi, (0 < \Phi < 1).$$

Lemma 2.3 *[8] Let $\phi = u + iv$ be a complex number and $\theta, \Theta \in \mathbb{R}$. Then $\Re e(\phi) \geq \Theta$ if and only if $|\phi - (\alpha + \Theta)| \leq |\phi + (\alpha - \Theta)|$, where $\Theta \geq 0$.*

Lemma 2.4 *[8] Let $\phi = u + iv$ be a complex number and $\theta, \Theta \in \mathbb{R}$. Then $\Re e(\phi) \geq \theta|\phi - \xi| + \Theta$ if and only if $\Re e(\phi(1 + \theta e^{i\delta}) - \alpha\theta e^{i\delta}) \geq \Theta$.*

Lemma 2.5 *[15] If f and g are analytic in $\mathbb{U}$, with $f \prec g$, then*

$$\int_0^{2\pi} |f(re^{i\delta})|' d\delta \leq \int_0^{2\pi} |g(re^{i\delta})|' d\delta,$$

where $\prime > 0$, $\xi = re^{i\delta}$, $(0 < r < 1)$.

The next section contains the properties investigated namely: coefficient estimate, radii of starlikeness, convexity and close-to-convexity, closure theorems, extreme points, integral means, neighborhood and convolution for the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \theta, \alpha)$.

3 Main results

3.1 Coefficient Estimate

Theorem 3.1 *The function $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \theta, \alpha)$, if and only if*

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta) - \beta(\gamma+\theta\alpha)}{\left[1 + \left(\frac{k}{\alpha} + \zeta - \mu - 1\right) t\right]^n} d_k \leq (1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta). \tag{9}$$

Where $n \in \mathbb{N}_0, \zeta, t \geq 0, \mu \in [0, \zeta]$, $0 \leq \beta \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \gamma < \alpha$, $\theta \in \mathbb{R}$ *and* $\alpha \in \mathbb{N}$.
The inequality is sharp for the extremal function

$$f(\xi) = \xi^{\alpha} - \frac{(1+\theta)(\alpha + \eta\xi^2\eta\alpha) - (\gamma + \theta\alpha)(1 - \eta\beta + \alpha\eta\beta)\left[1 + \left(\frac{k}{\alpha} + \zeta - \mu - 1\right)t\right]^n}{(1 - \eta + \eta k)(k(1+\theta) - \beta(\gamma + \theta\alpha)}\xi^k. \tag{10}$$

Proof Since $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \alpha)$, using the lemma 2.4, the inequality (6) is equivalent to

$$\Re e\left\{\frac{\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))' + \beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''}{(1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'} - \xi\theta e^{i\delta}\right\} > \gamma$$

$n \in \mathbb{N}_0, \zeta, t \geq 0, \mu \in [0, \zeta]$, $0 \leq \beta \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \gamma < \alpha$, $\alpha \in \mathbb{N}$ and $-\pi \leq \delta \leq \pi$.
So that

$$\begin{aligned}&\Re e\frac{\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))' + \beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''(1+\theta e^{i\delta})}{(1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}\\ &- \frac{\alpha\theta e^{i\delta}(1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}{(1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}\\ &\geq \gamma\end{aligned} \tag{11}$$

Let

$$\begin{aligned}\mathcal{L}(\xi) &= \xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))' + \beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''(1+\theta e^{i\delta})\\ &-\xi\theta e^{i\delta}(1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'\end{aligned}$$

and

$$\mathcal{N}(\xi) = (1-\beta)\xi^{\alpha} + (1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi) + \beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'.$$

Using the Lemma 2.3 in (11), we have

$$|\mathcal{L}(\xi) + (\xi - \gamma)\mathcal{N}(\xi)| \geq |\mathcal{L}(\xi) - (\xi - \gamma)\mathcal{N}(\xi)|.$$

We have $|\mathcal{L}(\xi) + (\xi - \gamma)\mathcal{N}(\xi)|$

$$= \left|\left(\alpha\xi^{\alpha} - \sum_{k=\alpha+1}^{\infty}\frac{k}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n}d_k\xi^k + \beta\alpha(\alpha-1)\xi^{\alpha} - \sum_{k=\alpha+1}^{\infty}\frac{\beta k(k-1)}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n}d_k\xi^k\right)\right.$$

$$(1+\theta e^{i\delta}) - \alpha\theta e^{i\delta}\left((1-\beta)\xi^{\alpha} + (1-\eta)\beta\xi^{\alpha} - \sum_{k=\alpha+1}^{\infty}\frac{(1-\eta)\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n}d_k\xi^k + \eta\beta\alpha\xi^{\alpha}\right)$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{k\eta\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k) + (\alpha-\gamma)((1-\beta)\xi^\alpha + (1-\eta)\beta\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{(1-\eta)\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k + \eta\beta\alpha\xi^\alpha - \sum_{k=\alpha+1}^{\infty} \frac{k\eta\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k) \Bigg|$$

$$= \Bigg| (\alpha\xi^\alpha - \sum_{k=\alpha+1}^{\infty} \frac{k}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k + \eta\alpha(\alpha-1)\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{\eta k(k-1)}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k)(1+\theta e^{i\delta}) + (\alpha-\alpha\theta e^{i\delta}-\gamma) + ((1-\beta)\xi^\alpha + (1-\eta)\beta\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{(1-\eta)\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k + \eta\beta\alpha\xi^\alpha\xi^\alpha - \sum_{k=\alpha+1}^{\infty} \frac{\eta\beta k}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k) \Bigg|$$

$$= \Big| (\alpha + \eta\xi^2 - \eta\alpha)(1-\theta e^{i\delta})\xi^\alpha + (\alpha-\gamma-\alpha\theta e^{i\delta})(1+\eta\beta\alpha-\eta\beta)\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{(k+\eta k^2-\eta k)(1+\theta e^{i\delta})}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k - \sum_{k=\alpha+1}^{\infty} \frac{\beta(\alpha-\gamma-\alpha\theta e^{i\delta})(1-\eta+\eta k)}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k \Bigg|$$

$$\geq (\alpha+\eta\alpha^2-\eta\alpha)(1+\theta) + (\alpha-\gamma-\alpha\theta)(1+\eta\beta\alpha-\eta\beta)|\xi|^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{(k+\eta k^2-\eta k)(1+\theta) + \beta(\alpha-\gamma-\alpha\theta)(1-\eta+\eta k)}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k|\xi|^k$$

Equivalently, $|\mathcal{L}(\xi) - (\xi+\gamma)\mathcal{N}(\xi)|$

$$= \Bigg| (\alpha\xi^\alpha - \sum_{k=\alpha+1}^{\infty} \frac{k}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k + \eta\alpha(\alpha-1)\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{\eta k(k-1)}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k)(1+\theta e^{i\delta}) - (\alpha\theta e^{i\delta}+\alpha+\gamma) + ((1-\beta)\xi^\alpha + (1-\eta)\beta\xi^\alpha$$

$$- \sum_{k=\alpha+1}^{\infty} \frac{(1-\eta)\beta}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k + \eta\beta\alpha\xi^\alpha - \sum_{k=\alpha+1}^{\infty} \frac{\eta\beta k}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k\xi^k) \Bigg|$$

$$\leq (\alpha+\gamma+\alpha\theta)(1+\eta\beta\alpha-\eta\beta) - (\alpha+\eta\alpha^2-\eta\alpha)(1+\theta)|\xi|^\alpha$$

$$+ \sum_{k=\alpha+1}^{\infty} \frac{((k+\eta k^2-\eta k)(1+\theta) - \beta(\alpha+\gamma+\alpha\theta)(1-\eta+\eta k))}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n} d_k|\xi|^k.$$

Hence

$$|\mathcal{L}(\xi) - (\xi-\gamma)\mathcal{N}(\xi)| - |\mathcal{L}(\xi) - (\xi+\gamma)\mathcal{N}(\xi)|$$

$$\geq (\alpha+\eta\alpha^2-\eta\alpha)(1+\theta) + (\gamma+\alpha\theta)(1+\eta\beta\alpha-\eta\beta)|\xi|^\alpha$$

$$-\sum_{k=\alpha+1}^{\infty}\frac{(1-\eta-\eta k)(k(1+\theta)-\beta(\gamma+\alpha\theta))}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n}d_k\geq 0.$$

Therefore

$$\sum_{k=\alpha+1}^{\infty}\frac{(1-\eta-\eta k)(k(1+\theta)-\beta(\gamma+\alpha\theta))}{[1+(\frac{k}{\alpha}+\zeta-\mu-1)t]^n}d_k\leq (1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\beta\alpha-\eta\beta).$$

Conversely, by inequality (9), we need to show that

$$\begin{aligned}&\Re e\frac{\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'+\beta\xi^2(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))''(1+\theta e^{i\delta})}{(1-\beta)\xi^\alpha+(1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi)+\beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}\\ &-\frac{(\alpha\theta e^{i\delta}+\gamma)(1-\beta)\xi^\alpha+(1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi)+\beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}{(1-\beta)\xi^\alpha+(1-\eta)\beta\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi)+\beta\eta\xi(\mathbf{I}^n(\zeta,\mu,t,\alpha)f(\xi))'}\\ &\geq 0.\end{aligned} \tag{12}$$

Let $0\leq\xi=r<1$, such that $\Re e(-e^{i\delta})\geq-|e^{i\delta}|=-1$ and $r\to 1$, (12) is obtained from (9).

Corollary 3.2 *The function $f(\xi)\in\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$. Then*

$$e_k\leq\frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}. \tag{13}$$

Where $n\in\mathbb{N}_0,\zeta,t\geq 0,\mu\in[0,\zeta]$, $0\leq\beta\leq 1$, $0\leq\eta\leq 1$, $0\leq\gamma<\alpha$, $\theta\in\mathbb{R}$ and $\alpha\in\mathbb{N}$.

Corollary 3.3 *Varying the parameter $n,t,\zeta,\mu,\beta,\eta,\gamma$, θ coefficient inequalities of classes of multivalent functions introduced and studied by the following authors in [5],[6],[11],[12],[13],[17],[20] can be obtained.*

3.2 Radii of starlikeness, convexity and close-to-convexity

Theorem 3.4 *Let $f(\xi)\in\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$. Then $f(\xi)$ is convex of order ρ in the disk $|\xi|<\mathcal{R}_1$, where*

$$\mathcal{R}_1=\inf_k\left(\frac{\alpha(\alpha-\rho)((1+\theta)(k+\eta k^2-\eta k)-\beta(\gamma+\alpha\theta)(1-\eta+\eta k)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{k(k-\rho)((1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}.$$

Where $n\in\mathbb{N}_0,\zeta,t\geq 0,\mu\in[0,\zeta]$, $0\leq\beta\leq 1$, $0\leq\eta\leq 1$, $0\leq\gamma<\alpha$, $\theta\in\mathbb{R}$ and $\alpha\in\mathbb{N}$.
The inequality of the extremal function $f(\xi)$ in (10) is sharp.

Proof It is sufficient to show that

$$\left|1+\frac{\xi f''(\xi)}{f'(\xi)}-\alpha\right| \leq \alpha-\rho, (0 \leq \rho < \alpha),$$

for $|\xi| < \mathcal{R}_1$, we have

$$\left|1+\frac{\xi f''(\xi)}{f'(\xi)}-\alpha\right| \leq \frac{\sum_{k=\alpha+1}^{\infty} k(k-\alpha)e_k|\xi|^{k-\alpha}}{\alpha-\sum_{k=\alpha+1}^{\infty} ke_k|\xi|^{k-\alpha}}.$$

Thus

$$\left|1+\frac{\xi f''(\xi)}{f'(\xi)}-\alpha\right| \leq \alpha-\rho.$$

If

$$\sum_{k=\alpha+1}^{\infty} \frac{k(k-\rho)}{\alpha(\alpha-\rho)} e_k|\xi|^{k-\alpha} \leq 1 \tag{14}$$

Then by Theorem 3.1, equation (14) is equivalent to

$$\frac{k(k-\rho)}{\alpha(\alpha-\rho)} e_k|\xi|^{k-\alpha} \leq \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n (1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}. \tag{15}$$

Hence,

$$|\xi| \leq \left(\frac{\alpha(\alpha-\rho)((1+\theta)(k+\eta k^2-\eta k)-\beta(\gamma+\alpha\theta)(1-\eta+\eta k)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{k(k-\rho)((1+\theta)(\alpha+\eta\xi^2-\eta\alpha)-(\gamma+\theta\xi)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}. \tag{16}$$

$\mathcal{R}_1$ is obtained by letting $|\xi| = \mathcal{R}_1$ and the proof completes.

Theorem 3.5 *Let $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$. Then $f(\xi)$ is starlike of order ρ in the disk $|\xi| < \mathcal{R}_2$, where*

$$\mathcal{R}_2 = \inf_k \left(\frac{(\alpha-\rho)((1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\alpha\theta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{k(k-\rho)((1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}.$$

Where $n \in \mathbb{N}_0, \zeta, t \geq 0, \mu \in [0,\zeta]$, $0 \leq \beta \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \gamma < \alpha$, $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{N}$.
The inequality of the extremal function $f(\xi)$ in (10) is sharp.

Proof It is sufficient to show that

$$\left|\frac{\xi f'(\xi)}{f'(\xi)}-\alpha\right| \leq \alpha-\rho, (0 \leq \rho < \alpha),$$

for $|\xi| < \mathcal{R}_1$, we have

$$\left|\frac{\xi f'(\xi)}{f'(\xi)} - \alpha\right| \leq \frac{\sum_{k=\alpha+1}^{\infty}(k-\alpha)e_k|\xi|^{k-\alpha}}{1-\sum_{k=\alpha+1}^{\infty} e_k|\xi|^{k-\alpha}}.$$

Thus

$$\left|\frac{\xi f'(\xi)}{f'(\xi)} - \alpha\right| \leq \alpha - \rho.$$

If

$$\sum_{k=\alpha+1}^{\infty} \frac{(k-\rho)}{(\alpha-\rho)} e_k|\xi|^{k-\alpha} \leq 1 \tag{17}$$

Then by Theorem 3.1, equation (17) is equivalent to

$$\frac{(k-\rho)}{(\alpha-\rho)} e_k|\xi|^{k-\alpha} \leq \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n (1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}. \tag{18}$$

Hence,

$$|\xi| \leq \left(\frac{(\alpha-\rho)((1+\theta)(k+\eta k^2-\eta k)-\beta(\gamma+\alpha\theta)(1-\eta+\eta k)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(k-\rho)((1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\xi)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}. \tag{19}$$

$\mathcal{R}_2$ is obtained by letting $|\xi| = \mathcal{R}_1$ and the proof completes.

Theorem 3.6 *Let $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$. Then $f(\xi)$ is close to convex of order ρ in the disk $|\xi| < \mathcal{R}_3$, where*

$$\mathcal{R}_3 = \inf_k \left(\frac{(\alpha-\rho)((1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\alpha\theta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{k(k-\rho)((1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}.$$

Where $n \in \mathbb{N}_0, \zeta, t \geq 0, \mu \in [0,\zeta]$, $0 \leq \beta \leq 1$, $0 \leq \eta \leq 1$, $0 \leq \gamma < \alpha$, $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{N}$.
The inequality of the extremal function $f(\xi)$ in (10) is sharp.

Proof It is sufficient to show that

$$\left|\frac{f'(\xi)}{\xi^{\alpha-1}} - \alpha\right| \leq \alpha - \rho, (0 \leq \rho < \alpha),$$

for $|\xi| < \mathcal{R}_1$, we have

$$\left|\frac{f'(\xi)}{\xi^{\alpha-1}} - \alpha\right| \leq \sum_{k=\alpha+1}^{\infty} k e_k|\xi|^{k-\alpha}.$$

Thus

$$\left|\frac{f'(\xi)}{\xi^{\alpha-1}}-\alpha\right|\le\alpha-\rho.$$

If

$$\sum_{k=\alpha+1}^{\infty}\frac{ke_k|\xi|^{k-\alpha}}{\alpha-\rho}\le 1 \tag{20}$$

Then by Theorem 3.1, equation (20) is equivalent to

$$\frac{k}{(\alpha-\rho)}|\xi|^{k-\alpha}\le\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\beta\alpha-\eta\beta)}. \tag{21}$$

Hence,

$$|\xi|\le\left(\frac{(\alpha-\rho)((1+\theta)(k+\eta k^2-\eta k)-\beta(\gamma+\alpha\theta)(1-\eta+\eta k)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{k((1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\xi)(1+\eta\alpha\beta-\eta\beta))}\right)^{\frac{1}{k=\alpha}}. \tag{22}$$

$\mathcal{R}_2$ is obtained by letting $|\xi|=\mathcal{R}_1$ and the proof completes.

3.3 Extreme point

Theorem 3.7 *Let* $f_\alpha(\xi)=\xi^\alpha$ *and*

$$f_k(\xi)=\xi^\alpha-\frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^k. \tag{23}$$

Where $n\in\mathbb{N}_0,\zeta,t\ge 0,\mu\in[0,\zeta]$, $0\le\beta\le 1$, $0\le\eta\le 1$, $0\le\gamma<\alpha$ *and* $\alpha\in\mathbb{N}$.

Then the function $f(\xi)\in\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\lambda,\alpha)$, *if and only if*

$$f(\xi)=\Omega_\alpha\xi^\alpha+\sum_{k=\alpha+1}^{\infty}\Omega_k f_k(\xi), \tag{24}$$

where $\Omega_\alpha\ge 0,\Omega_k\ge 0$ *and* $\Omega_\alpha+\sum_{k=\alpha+1}^{\infty}\Omega_k=1$.

Proof Taking $f(\xi)$ from (24), Then

$$f(\xi)=\Omega_\alpha\xi^\alpha+\sum_{k=\alpha+1}^{\infty}\Omega_k$$

$$(\xi^\alpha-\sum_{k=\alpha+1}^{\infty}\frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^k)$$

$$= \xi^{\alpha} - \sum_{k=\alpha+1}^{\infty} \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^k$$

Hence

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}$$

$$\times \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}$$

$$\sum_{k=\alpha+1}^{\infty} \Omega_k = 1 - \Omega_\alpha \leq 1.$$

Then $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$
The converse of the theorem is true

3.4 Closure Theorem

Theorem 3.8 *Let the function f_c be given as*

$$f_c = \xi^{\alpha} - \sum_{k=\alpha+1}^{\infty} e_{k,c}\xi^k \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha), (e_{k,c} \geq 0), c = 1, 2, \cdots, t. \quad (25)$$

Then the function

$$f_1(\xi) = \xi^{\alpha} - \sum_{k=\alpha+1}^{\infty} m_k\xi^{\alpha} \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$$

whenever

$$m_k = \frac{1}{t}\sum_{c=1}^{t} e_{k,c}$$

Proof Since $f_c \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$, it follows from Theorem 3.1, that

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_{k,c} \leq (1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta).$$

for $c = 1, 2, \cdots, t$.
Hence

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_k$$

$$= \sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_k \left(\frac{1}{t}\sum_{c=1}^{t} e_{k,c}\right)$$
$$= \frac{1}{t}\sum_{c=1}^{t}\left(\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_{k,c}\right)$$
$$\leq (1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta).$$

then $f_1 \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$

Theorem 3.9 *Let the function f_c of the form (25) be in $\mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha), c = 1,2,\cdots,t$. Then the function*

$$f_2(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} \Psi_c f_c(\xi) \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha) \tag{26}$$

whenever

$$\sum_{c=1}^{t} \Psi_c = 1, (\Psi_c \geq 0).$$

Proof From (26), we have

$$f_2(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} \Psi_c f_c(\xi) = \sum_{c=1}^{t} \Psi_c \left(\xi^\alpha - \sum_{k=\alpha+1}^{\infty} e_{k,c}\xi^k\right) = \sum_{k=\alpha+1}^{\infty}\left(\sum_{c=1}^{t}\Psi_c e_{k,c}\right)\xi^k.$$

Since $f_c \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$, it follows from Theorem 3.1, that

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_{k,c} \leq (1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta).$$

for $c = 1,2,\cdots,t$.
Hence

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} \left(\sum_{c=1}^{t}\Psi_c e_{k,c}\right)$$
$$= \sum_{c=1}^{t}\Psi_c\left(\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_{k,c}\right)$$
$$\leq \sum_{c=1}^{t}\Psi_c\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha))}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\right)$$
$$= (1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta).$$

Then $f_2 \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$

3.5 Integral Means inequality

Theorem 3.10 *Let $\varepsilon > 0$ such that $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$, assume*

$$f_c(\xi) = \xi^\alpha - \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi_c, (c \geq \alpha+1),$$

If there exists

$$g(\xi)^{c-\alpha} = \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha}.$$

Then, for $\xi = re^{i\delta}$ and $0 < r < 1$,

$$\int_0^{2\pi} |f(\xi)|^\varepsilon d\delta \leq \int_0^{2\pi} |f_c(\xi)|^\varepsilon d\delta, \varepsilon > 0.$$

Proof we need to show that

$$\int_0^{2\pi} \left|1 - \sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha}\right|^\varepsilon d\delta$$

$$\leq \int_0^{2\pi} \left|1 - \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^{c-\alpha}\right|^\varepsilon d\delta$$

By Lemma 2.5, it is sufficient to show that

$$1 - \sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha} \prec \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^{c-\alpha}$$

Let

$$1 - \sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha}$$

$$= 1 - \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}g(\xi)^{c-\alpha}.$$

We have that

$$g(\xi)^{c-\alpha} = \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha},$$

Then, $g(0) = 0$.
From equation (9)

$$|g(\xi)|^{c-\alpha} = \left|\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha) - (\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha}\right|$$

$$\leq |\xi| \left| \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} \sum_{k=\alpha+1}^{\infty} e_k \right|$$

$$\leq |\xi| < 1.$$

Theorem 3.11 *Let $\varepsilon > 0$ such that $f(\xi) \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$, assume*

$$f_c(\xi) = \xi^\alpha - \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)} \xi_c, (c \geq \alpha+1), \alpha \in \mathbb{N}.$$

Then, for $\xi = re^{i\delta}$ and $0 < r < 1$,

$$\int_0^{2\pi} |f'(\xi)|^\varepsilon d\delta \leq \int_0^{2\pi} |f_c'(\xi)|^\varepsilon d\delta, \varepsilon > 0.$$

Proof It is sufficient to show that

$$1 - \sum_{k=\alpha+1}^{\infty} e_k \frac{k}{\alpha} \xi^{k-\alpha} \prec$$

$$\frac{c((1+\theta)(\alpha+\eta\xi^2-\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta))\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{\alpha(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)} \xi^{c-\alpha}.$$

It follows that

$$|g(\xi)|^{c-\alpha}$$

$$= \left| \frac{\alpha(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)}{c(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} \sum_{k=\alpha+1}^{\infty} \frac{k}{\alpha} e_k \xi^{k-\alpha} \right|$$

$$\leq |\xi| \left| \sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n} e_k \right|$$

$$\leq |\xi| < 1.$$

Theorem 3.12 *Let $k(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} \Xi \xi^k$, $(\xi \in \mathbb{U}); \Xi \geq 0; k \geq \alpha+1; \alpha \in \mathbb{N}$, and $f(\xi) \in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$, for $c \in \mathbb{N}$, then*

$$\frac{\mathcal{Q}_c}{\Xi_c} = \min_{k=\alpha+1} \frac{\mathcal{Q}_c}{\Xi_c},$$

where

$$\mathcal{Q}_c = \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}.$$

Also, for $c \in \mathbb{N}$, the function f_c and k_c be given as

$$f_c(\xi) = \xi^\xi - \frac{((1+\theta)(\alpha+\eta\xi^2-\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta))\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{\alpha(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)} \xi^{c-\alpha}$$

and

$$k_c(\xi) = \xi^\alpha - \Xi_c \xi^c. \tag{27}$$

If there exists an analytic function

$$g(\xi)^{c-\alpha} = \frac{(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n\Xi_c}\sum_{k=\alpha+1}^{\infty}\Xi_k e_k \xi^{k-\alpha}, \tag{28}$$

then, for $\varepsilon > 0$, $\xi = re^{i\delta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} |f(\xi) * k(\xi)|^\varepsilon d\delta \le \int_0^{2\pi} |f(\xi) * k(\xi)|^\varepsilon d\delta, \varepsilon > 0.$$

Proof Since

$$(f * k)(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} e_k \Xi_c \xi^\alpha,$$

from (27), we have

$$(f_c * k_c)(\xi) = \xi^\alpha -$$

$$\frac{((1+\theta)(\alpha+\eta\xi^2-\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta))\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{\alpha(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)\Xi_c}\xi^c$$

we need to prove the Theorem by taking $\varepsilon > 0$, $\xi = re^{i\theta}$ and $(0 < r < 1)$ such that

$$\int_0^{2\pi}\left|1-\sum_{k=\alpha+1}^{\infty}\Xi_k e_k \xi^{k-\alpha}\right|^\varepsilon d\delta$$

$$\le \int_0^{2\pi}\left|1-\frac{(1+\theta)(\alpha+\eta\xi^2-\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n\Xi_c}{(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^{c-\alpha}\right|^\varepsilon d\delta$$

By Lemma 2.5, it is sufficient to show that

$$1-\sum_{k=\alpha+1}^{\infty}\Xi_k e_k \xi^{k-\alpha}$$

$$\prec \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n\Xi_c}{(1-\eta+\eta c)(c(1+\theta)-\beta(\gamma+\theta\alpha)}\xi^{c-\alpha} \tag{29}$$

If (29), holds, then there exist an analytic function $g(\xi)$

$$1-\sum_{k=\alpha+1}^{\infty} e_k \xi^{k-\alpha}$$

$$= 1-\frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}g(\xi)^{c-\alpha}.$$

We have that

$$g(\xi)^{c-\alpha} = \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha},$$

Then, $g(0)=0$.
From (9)

$$|g(\xi)|^{c-\alpha} = \left|\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\xi^{k-\alpha}\right|$$

$$\leq |\xi|\left|\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}\sum_{k=\alpha+1}^{\infty} e_k\right|$$

$\leq |\xi| < 1$.

Theorem 3.13 *Let* $k(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha)$ *and*

$$\Phi = \alpha - \frac{\frac{\rho(1+\beta\alpha)(\alpha+1)(1+\theta)-\eta(\gamma+\alpha\theta)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}}{\alpha+1\left[\frac{(1+\beta\alpha)(\alpha+1)(1+\theta)-\eta(\gamma+\alpha\theta)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}-(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\right]} \tag{30}$$

Then $\mathcal{G}_{k,\rho}(k) \subset \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\lambda,\alpha)$.

Proof Since $f(\xi) \in \mathcal{G}_{k,\rho}(k)$. Then from definition 2.2, we have

$$\sum_{k=\alpha+1}^{\infty} k|e_k-\Xi_k| \leq \rho.$$

it follows that the coefficient inequality below

$$\sum_{k=\alpha+1}^{\infty} |e_k-\Xi_k| \leq \frac{\rho}{\alpha+1}.$$

Since $k(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\lambda,\alpha)$, we have from Theorem 3.1 that

$$\sum_{k=\alpha+1}^{\infty} \Xi_k \leq \frac{(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}.$$

So that

$$\left|\frac{f(\xi)}{k(\xi)}-1\right| \leq \frac{\sum_{k=\alpha+1}^{\infty}|e_k-\Xi_k|}{1-\sum_{k=\alpha+1}^{\infty}\Xi_k}$$

$$\leq \frac{\frac{\rho(1+\beta\alpha)(\alpha+1)(1+\theta)-\eta(\gamma+\alpha\theta)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}}{\alpha+1\left[\frac{(1+\beta\alpha)(\alpha+1)(1+\theta)-\eta(\gamma+\alpha\theta)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n}-(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)\right]}$$

$$= \alpha - \Phi.$$

Therefore, for $0 \leq \Phi < 1$ and from definition (2.2), $f(\xi) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\lambda,\alpha)$.

3.6 Convolution Properties

Theorem 3.14 *Let $f_i(\xi) \in \mathcal{M}_\alpha$, (i =1,2) be defined as*

$$f_i(\xi) = \xi^\alpha - \sum_{k=\alpha+1}^{\infty} e_{k,i}\xi^\alpha, (e_{k,i} \geq 0, i = 1, 2) \tag{31}$$

be in the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \theta, \alpha)$. Then $(f_1(\xi) * f_2(\xi)) \in \mathbf{J}^{n,t}_{\zeta,\mu}(\Delta, \beta, \gamma, \theta, \alpha)$.
where

$$\Delta \leq \frac{(((1+\theta)(D) - (\gamma+\theta\alpha))(A)[B]^2\,[C]^n - (B)(1+\eta\beta\alpha-\eta\beta)}{\eta(\alpha-1)(\gamma+\theta\alpha)(1+\eta k-\eta)\,[C]^n - (\gamma+\theta\alpha)[D](1+\theta) - (\gamma+\theta\alpha)(1+\eta\beta\alpha-\eta\beta)}$$

$A = 1+\eta k-\eta$, $B = k(1+\theta)-\beta(\gamma+\theta\alpha)$, $C = 1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t$, $D = \alpha+\eta\alpha^2-\eta\alpha$
proof The largest Δ need to be find such that

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)} e_{k,1}e_{k,2} \leq 1. \tag{32}$$

Since $f(\xi)_i \in \mathbf{J}^{n,t}_{\zeta,\mu}(\eta, \beta, \gamma, \theta, \alpha), (i = 1, 2)$, then from Theorem 3.1, we have that

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)} e_{k,i} \leq 1, (i = 1, 2). \tag{33}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{k=\alpha+1}^{\infty} \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)} \sqrt{e_{k,1}e_{k,2}} \leq 1. \tag{34}$$

It is obvious to show that

$$\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)} e_{k,1}e_{k,2}$$

$$\leq \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)} \sqrt{e_{k,1}e_{k,2}}$$

So that

$$\sqrt{e_{k,1}e_{k,2}} \leq \frac{(k(1+\theta)-\beta(\gamma+\alpha\theta))[(1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\Delta\alpha-\eta\Delta)]}{(k(1+\theta)-\Delta(\gamma+\alpha\theta))[(1+\theta)(\alpha+\eta\alpha^2-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\beta\alpha-\eta\beta)]} \tag{35}$$

From (34), we have

$$\sqrt{e_{k,1}e_{k,2}} \leq \frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^n(1+\theta)(\alpha+\eta\xi^2\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}. \tag{36}$$

So that, we have

$$\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}$$

$$\leq\frac{(k(1+\theta)-\beta(\gamma+\alpha\theta))[(1+\theta)(\alpha+\eta\alpha^{2}-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\Delta\alpha-\eta\Delta)]}{(k(1+\theta)-\Delta(\gamma+\alpha\theta))[(1+\theta)(\alpha+\eta\alpha^{2}-\eta\alpha)-(\gamma+\theta\alpha)(1+\eta\beta\alpha-\eta\beta)]}.$$

We obtained Δ and the proof completes.

Theorem 3.15 *Let $f_i(\xi)\in\mathcal{M}_\alpha$, (i =1,2) be of the form (29). be in the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\lambda,\alpha)$. Then the function*

$$h(\xi)=\xi^{\alpha}-\sum_{k=\alpha+1}^{\infty}(e_{k,1}^{2}+e_{k,2}^{2})\xi^{\alpha},\tag{37}$$

is in the class $\mathbf{J}^{n,t}_{\zeta,\mu}(\Psi,\beta,\gamma,\theta,\alpha)$.
where

$$\Psi\leq\frac{[((1+\theta)(D)-(\gamma+\theta\alpha))]AB^{2}\,[C]^{n}-2(k(1+\theta)(A)[(1+\theta)(D)-(\gamma+\theta\alpha)(1+\eta\beta(\alpha))]^{2})}{(\eta(\alpha-1)(\gamma+\theta\alpha))[AB]^{2}\,[C]^{n}-2(\gamma+\theta\alpha)(A)[(1+\alpha)(D)-(\gamma+\theta\alpha)(1+\beta\eta(\alpha-1))]^{2}}.$$

$A=1+\eta k-\eta$, $B=k(1+\theta)-\beta(\gamma+\theta\alpha)$, $C=1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t$, $D=\alpha+\eta\alpha^{2}-\eta\alpha$
proof The largest Ψ need to be find such that

$$\sum_{k=\alpha+1}^{\infty}\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}(e_{k,1}^{2}+e_{k,2}^{2})\leq 1.\tag{38}$$

Since $f(\xi)_i\in\mathbf{J}^{n,t}_{\zeta,\mu}(\eta,\beta,\gamma,\theta,\alpha),(i=1,2)$, then from Theorem 3.1, we have that

$$\sum_{k=\alpha+1}^{\infty}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}\right)^{2}e_{k,i}^{2}$$

$$\leq\sum_{k=\alpha+1}^{\infty}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}e_{k,1}\right)^{2}\leq 1.\tag{39}$$

and

$$\sum_{k=\alpha+1}^{\infty}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}\right)^{2}e_{k,2}^{2}$$

$$\leq\sum_{k=\alpha+1}^{\infty}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}e_{k,2}\right)^{2}\leq 1.\tag{40}$$

So that

$$\sum_{k=\alpha+1}^{\infty}\frac{1}{2}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}\right)^{2}(e_{k,1}^{2}+e_{k,2}^{2})\leq 1 \tag{41}$$

Since $f(\xi)\in \mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\lambda,\alpha)$ if and only if

$$\sum_{k=\alpha+1}^{\infty}\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}(e_{k,1}^{2}+e_{k,2}^{2})\leq 1 \tag{42}$$

Then,

$$\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}$$

$$\leq\sum_{k=\alpha+1}^{\infty}\frac{1}{2}\left(\frac{(1-\eta+\eta k)(k(1+\theta)-\beta(\gamma+\theta\alpha)}{\left[1+\left(\frac{k}{\alpha}+\zeta-\mu-1\right)t\right]^{n}(1+\theta)(\alpha+\eta\xi^{2}\eta\alpha)-(\gamma+\theta\alpha)(1-\eta\beta+\alpha\eta\beta)}\right)^{2}.$$

We obtained Ψ and the proof completes.

4 Conclusion

This work has presented a novel integral operator that extends some well-known existing results of the subclasses of multivalent functions. The following properties investigated were coefficient estimate, radii of starlikeness, convexity and close-to-convexity, closure theorems, extreme points, integral means, neighborhood and convolution for the class of multivalent functions denoted as $\mathbf{J}_{\zeta,\mu}^{n,t}(\eta,\beta,\gamma,\theta,\alpha)$.

5 Open Problem

The integral operator is open to researcher for further investigation on some other classes of analytic and multivalent functions .

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